Now we shall show the following:

Proposition 3.3. $\mathbf{R} \in \mathcal{U}(M; N)$.

Proof. If $u \in R, j \geq 1$, then we denote by $u_{\bullet}^{j}$ the j–th power of u in $\mathbf{R}$, i.e.

$$u_{\bullet}^{1} = u, \quad u_{\bullet}^{j+1} = (u_{\bullet}^{j}) \bullet u. \tag{21}$$

(Note that, if $u \in R$, then $u_{\bullet}^{j} \in R$, but it can happen that $u^{j} \in F \setminus R$.)

Assuming (19), by (20) we obtain the equalities (22) in the corresponding cases $\mathcal{U}_l, \mathcal{U}_r$ and $\mathcal{U}$.

$$\text{(22)} \quad \begin{array}{ll} \mathcal{U}_l: & u_{\bullet}^{j} = \begin{cases} u, & \text{if } j = 1 \\ t\ \underline{uj-1}, & \text{if } 2 \leq j \leq m \\ t\ \underline{ur-1}, & \text{if } j = qm + r, 2 \leq r \leq m+1 \end{cases} \\ \mathcal{U}_r: & u_{\bullet}^{j} = t^{i+j}, \text{ if } j \geq 1; \\ \mathcal{U}: & u_{\bullet}^{j} = \begin{cases} u, & \text{if } j = 1 \\ t^{sn+j}, & \text{if } s \geq 1, 1 \leq j \leq (k-s)n+1. \\ t^{j-(k-s)n}, & \text{if } s \geq 1, (k-s)n+2 \leq j \leq kn+1 \end{cases} \end{array}$$

Therefore:

$$\text{(23)} \quad \begin{array}{ll} \mathcal{U}_l: & u_{\bullet}^{m+1} = t^{\langle p+1 \rangle}; \\ \mathcal{U}_r: & u_{\bullet}^{j+1} = u^{i+j+1}, \quad \text{for each } j \in S; \\ \mathcal{U}: & u_{\bullet}^{in+1} = \begin{cases} t^{(i+s)n+1}, & \text{if } i+s \leq k \\ t^{(i+s-k)n+1}, & \text{if } i+s > k, 1 \leq i \leq k. \end{cases} \end{array}$$

If $\eta(u) = t \neq u$ (i.e. $u = t^{sn+1}$, $s \geq 1$) and $i = k$, then in the last case we obtain

$$u_{\bullet}^{m+1} = u.$$

From (20) and (23) we obtain that, for any $v, w \in R$, the following equations hold:

$\mathcal{U}_l$: $(v_{\bullet}^{m+1}) \bullet w = v \bullet w$;

$\mathcal{U}_r$: $v \bullet (w_{\bullet}^{i+1}) = v \bullet w$, for each $i \in S$;

$\mathcal{U}\ :\ (v_{\bullet}^{m+1}) \bullet w = v \bullet w = v \bullet (w_{\bullet}^{n+1})$.

Therefore, we have $\mathbf{R} \in \mathcal{U}_l(m)$, $\mathbf{R} \in \mathcal{U}_r(S)$, $\mathbf{R} \in \mathcal{U}(kn;n)$ in the cases: $\mathcal{U}(M;N) = \mathcal{U}_l(m)$, $\mathcal{U}(M;N) = \mathcal{U}_r(S)$, $\mathcal{U}(M;N) = \mathcal{U}(kn;n)$, respectively. □

The following statement will complete the proof of Theorem 1.

Proposition 3.4. *Let* $\mathbf{G} = (G;\cdot) \in \mathcal{U}(M;N)$. *If* $\lambda : B \to G$ *is a mapping, and* $\varphi : \mathbf{F} \to \mathbf{G}$ *the homomorphism which extends* λ, *then the restriction of* φ *on* R *is a homomorphism from* $\mathbf{R}$ *into* $\mathbf{G}$.

Proof. It suffices to show the equality $\varphi(v \bullet w) = \varphi(v)\varphi(w)$, for each $v, w \in R$ such that $v \bullet w \neq vw$.

Then, in the case $\mathcal{U}(M;N) = \mathcal{U}_l(m)$, we have $v = t^{\langle p+1\rangle}$ for a unique pair (t,p), where $t \in R, p \geq 0, \alpha(t) = t$, and $v \bullet w = tw$. Therefore, we have:

$$\varphi(v \bullet w) = \varphi(tw) = \varphi(t)\varphi(w) = \varphi(t)^{(p+1)}\varphi(w).$$

Then, by (11) we have:

$$\varphi(t)^{(p+1)}\varphi(w) = \varphi(t)^{\langle p+1\rangle}\varphi(w) = \varphi(t^{\langle p+1\rangle})\varphi(w) = \varphi(v)\varphi(w).$$

In the case $\mathcal{U}(M;N) = \mathcal{U}_r(S)$ we have: $w = t^{i+1}$, for a unique pair (t,i), where $t \in R, \beta(t) = t, i \in S$. Therefore,

$$\varphi(v \bullet w) = \varphi(vt) = \varphi(v)\varphi(t) = \varphi(v)\varphi(t)^{i+1} = \varphi(v)\varphi(t^{i+1}) = \varphi(v)\varphi(w).$$

In a similar way, we obtain that $\varphi(v \bullet w) = \varphi(v)\varphi(w)$, in the case $\mathcal{U}$. □

By Propositions 3.2–3.4, $\mathbf{R}$ is a $\mathcal{U}(M;N)$–free groupoid with the unique basis B, i.e. we have completed the proof of Theorem 1.

We say that the formula: $x^{m+1} \cdot y = xy$ ($x \cdot y^{n+1} = xy$) is a *left* (a *right*) equation; a left or a right equation is called *equation.* It is well known that an equation holds in a variety $\mathcal{U}(M;N)$ iff it is satisfied in each $\mathcal{U}(M;N)$–free groupoid. Therefore, the following statement describes the set of equations in a variety $\mathcal{U}(M;N)$.

Proposition 3.5. *Let* $\mathbf{H}$ *be a free groupoid in the variety* $\mathcal{U}(M;N)$. *Then the following statements hold.*

(*i*) *If* $M \neq \emptyset$, $N = \emptyset$, $\gcd(M) = m$, *then a left equation* $x^{m+1}y = xy$ *holds in* $\mathbf{H}$ *iff* $m|n$; *no right equation holds in* $\mathbf{H}$.

(*ii*) *If* $M = \emptyset$, $N \neq \emptyset$, *then the right equation* $xy^{j+1} = xy$ *holds in* $\mathbf{H}$ *iff* $j \in \langle N \rangle$; *no left equation holds in* $\mathbf{H}$.

(*iii*) *If* $M \neq \emptyset$, $N \neq \emptyset$, $\gcd(M) = m$, $n = \gcd(M \cup N)$, *then* $x^{i+1}y = xy$ *iff* $m|i$ *and* $xy^{j+1} = xy$ *iff* $n|j$ *hold in* $\mathbf{H}$.

Proof. Let $\mathbf{R}$ be a $\mathcal{U}(M;N)$–canonical groupoid with the basis B, and $a, b \in B$.

Then:

(*i*) If $M \neq \emptyset$, $N = \emptyset$, $\gcd(M) = m$, then
$(a_\bullet^{i+1}) \bullet b = ab = a \bullet b$ iff $m|i$;
$a \bullet b^{j+1} \neq ab = a \bullet b$ for each $j \geq 1$.

(*ii*) If $M = \emptyset$, $N \neq \emptyset$ and $S = \langle N \rangle$, then:
$a^{i+1} \bullet b = a^{i+1}b \neq ab = a \bullet b$;
and, if $j \geq 1$, then $a \bullet (a_\bullet^{j+1}) = ab = a \bullet b$ iff $j \in S$.

(*iii*) If $M \neq \emptyset$, $N \neq \emptyset$, $\gcd(M) = m$, $n = \gcd(M \cup N)$, $i, j \geq 1$, then:
$(a_\bullet^{i+1}) \bullet b = ab = a \bullet b$ iff $m|i$,
$a \bullet (b_\bullet^{j+1}) = ab = a \bullet b$ iff $n|j$. □

Having in mind the definitions of the transformations ξ, η in each of the cases $\mathcal{U}_l(m), \mathcal{U}_r(S)$ and $\mathcal{U}(kn; n)$, as a corollary of Theorem 1 the following statement can also be obtained.

Proposition 3.6. *If* $\mathbf{H}$ *is a* $\mathcal{U}(M;N)$*-free groupoid with the basis* B, *then there exist retractions* γ *and* δ *of* H *with the following properties:*

(*i*) B *is the set of primes in* $\mathbf{H}$, *and* $B \subseteq \mathrm{im}\gamma \cap \mathrm{im}\delta$;
(If $x \in \mathrm{im}\gamma \cap \mathrm{im}\delta$, then we say that x is a *base* in $\mathbf{H}$)

(*ii*) $(\forall x, y \in H) xy = \gamma(x)\delta(y)$;
$((\gamma(x), \delta(y))$ is the *pair of divisors* of xy in $\mathbf{H}$; i.e. $\gamma(x)$ is the *left* and $\delta(y)$ the *right divisor* of xy.)

(*iii*) *There exists a mapping $x \mapsto |x|$ from H into the set of positive integers with the following properties:*

$$|xy| = |\gamma(x)| + |\delta(y)|,$$

$$\gamma(x) \neq x \iff |\gamma(x)| < |x|; \ \delta(x) \neq x \iff |\delta(x)| < |x|,$$

for any $x, y \in H$.

Proof. If $\mathbf{R}$ is the $\mathcal{U}(M;N)$-canonical groupoid with the basis B, then there exists a unique isomorphism $\varphi : \mathbf{R} \to \mathbf{H}$ such that $\varphi(b) = b$, for each $b \in B$. Defining $\gamma, \delta : H \to H$ by: $\gamma(x) = \xi(\varphi^{-1}(x))$, $\delta(x) = \eta(\varphi^{-1}(x))$, we obtain two retractions γ, δ of H such that (i)– (iii) hold, where the length of $x \in H$ is defined by $|x| = |\varphi^{-1}(x)|$. □

In each of the cases $\mathcal{U}_l(m)$, $\mathcal{U}_r(S)$, $\mathcal{U}(kn;n)$, the results of Proposition 3.6 can be stated more explicitely as follows.

3.6. $\mathcal{U}_l(m)$.

(*i*) $\gamma = \alpha, \delta = 1_H$;

(*ii*) $y \in H$ is a *base in* $\mathbf{H}$ iff $y \in \operatorname{im}\gamma$; for each $x \in H$ there exists a unique $y = \operatorname{bs}(x)$ (the *base of* x) and unique $p = \exp(x) \geq 0$ (the *exponent of* x) such that $x = y^{(p)}$.

(*iii*) $\operatorname{bs}(x)$ is the left (and y the right) divisor of xy.

(*iv*) If b is a base in $\mathbf{H}$, and $1 \leq i < m$, $p \geq 0$, then $c = b\,\underline{b^{(p)}i}$ is also a base in H; $b\,\underline{b(p)i-1}$ is the left and $b^{(p)}$ the right divisor of c; in the same case $b\,\underline{b^{(p)}m-1}$ is the left and $b^{(p)}$ the right divisor of $b^{(p+1)}$.

(*v*) If $x \in H, 1 \leq i \leq j \leq m+1$, then $x^i = x^j \Rightarrow i = j$.

3.6. $\mathcal{U}_r(N)$.

(*i*) $\gamma = 1_H, \delta = \beta$;

(*ii*) y is a base in $\mathbf{H}$ iff $y \in \operatorname{im}\delta$; for each $x \in H$ there exists a unique base y, and a unique $q \in \{0\} \cup \langle N \rangle$, such that $x = y^{q+1}$.

(*iii*) The left divisor of xy is x and its right divisor is $\mathrm{bs}(y)$. Thus,

$$xy = uv \iff x = u,\ \mathrm{bs}(y) = \mathrm{bs}(v).$$

3.6. $\mathcal{U}(kn; n)$.

(*i*) x is a base in **H** iff $x^{m+1} \neq x$; for each $x \in H$ there exists a unique y (the *base of* x) and a unique $i \in \{0, 1, \cdots, k\}$ (the *exponent of* x) such that $x = y^{in+1}$; x is a *left base* in **H** iff $i \neq k$, where i is the exponent of x.

(*ii*) For any $x \in H$, $\delta(x)$ is the base of x, and

$$\gamma(x) = \begin{cases} x, & \text{if } x \text{ is the left base} \\ y, & \text{if } y \text{ is the base of } x, \text{ and } x = y^{m+1} \end{cases}$$

$$\delta(x) = \begin{cases} y, & \text{if } x = y^{in+1},\ 0 \leq i \leq k, \\ x, & \text{otherwise.} \end{cases}$$

(*iii*) $\gamma(x)$ is the left and $\delta(y)$ the right divisor of xy.

4. Free subgroupoids of $\mathcal{U}(M; N)$–free groupoids

We shall describe the set of pairs (M, N) of sets of positive integers such that the variety $\mathcal{U}(M; N)$ is hereditary, i.e. we shall prove Theorem 2.

Proposition 4.1. *For any $m \geq 1$, the class of free objects in the variety $\mathcal{U}_l(m)$ is hereditary.*

Proof. Let **Q** be a subgroupoid of a $\mathcal{U}_l(m)$–free groupoid **H**. We have to show that the set P of prime elements of Q is nonempty, and that **Q** is $\mathcal{U}_l(m)$–free with the basis P. The proof will be given in several steps, where induction on $|x|$, for $x \in Q$, will be used.

1) If $a \in Q \setminus P$ and c is the right divisor of a in **H**, then $c \in Q$.

2) Let $a = b^{(p)} \in Q$, where b is the base of a in **H**. If q is the least non-negative integer such that $b' = b^{(q)} \in Q$, then we say that b' is the base of a in **Q**. Then, if $q \geq 1, b' \in P$.

By 1) and 2) we obtain:

3) $P \neq \emptyset$, and P is the least generating subset of **Q**.

4) If $c, d \in Q$, and b' is the base of c in $\mathbf{Q}$, then we say that (b', d) is *the pair of divisors* of cd in $\mathbf{Q}$. Then: $|d| < |cd|$, and: b' is prime or $|cd| = |b'| + |d|$.

5) Assume that $\mathbf{G} \in \mathcal{U}_l(m)$ and $\lambda : P \to G$ is a given mapping. There is a (unique) homomorphism $\varphi : Q \to G$ such that $\lambda = \varphi_{|P}$ is the restriction of φ on P. Namely, if $x \in Q$ is such that $|x| = \min\{|y| : y \in Q\}$, then $y \in P$, and thus $\varphi(x) = \lambda(x)$ is well defined. Assume that for each $x \in Q$, such that $|x| \leq i$, $\varphi(x) \in G$ is well defined and, moreover, if (y, z) is the pair of divisors of x in $\mathbf{Q}$, then $\varphi(y)$, $\varphi(z)$ are well defined, and $\varphi(x) = \varphi(y)\varphi(z)$.

Let $v \in Q \setminus P$ be such that $|v| = i + 1$, and (t, u) be the pair of divisors of v in $\mathbf{Q}$. Then $\varphi(t)$ and $\varphi(u)$ are well defined, and thus we can define $\varphi(v)$ by $\varphi(v) = \varphi(t)\varphi(u)$. Then $\varphi : \mathbf{Q} \to \mathbf{G}$ is a homomorphism which extends λ. □

Proposition 4.2. *The class of free objects in the variety $\mathcal{U}_r(1)$ is hereditary.*

Proof. This statement is one of the main results of [4], and it is also a corollary of Proposition 4.1. Namely, let $\mathbf{G} = (G, \cdot)$ be a given groupoid, and the groupoid $\mathbf{G}^{\mathbf{op}} = (G, \circ)$ be defined by $x \circ y = yx$. Then, $\mathbf{G} \in \mathcal{U}_r(1) \iff \mathbf{G}^{\mathbf{op}} \in \mathcal{U}_l(1)$, and $\mathbf{H}$ is $\mathcal{U}_r(1)$–free iff $\mathbf{H}^{\mathbf{op}}$ is $\mathcal{U}_l(1)$–free. □

Proposition 4.3. *If N is a nonempty set of positive integers and $1 \notin N$, then the class of free objects in the variety $\mathcal{U}_r(N)$ is not hereditary.*

Proof. Let $n = \min(N)$, and let $\mathbf{H}$ be a $\mathcal{U}_r(N)$–free groupoid with the basis B. Consider the subgroupoid $\mathbf{Q}$ generated by $\{b^n, b^{n+1}\}$, where $b \in B$. Then b^n is the unique prime in $\mathbf{Q}$, and $\{b^n\}$ does not generate $\mathbf{Q}$, which implies that $\mathbf{Q}$ is not free [7]. □

Proposition 4.4. *If $M \neq \emptyset, N \neq \emptyset$, then the class of free objects in the variety $\mathcal{U}(M; N)$ is not hereditary.*

Proof. Let $m = \gcd(M), n = \gcd(M \cup N)$ and let $\mathbf{H}$ be a $\mathcal{U}(M; N)$–free groupoid with the basis B. If $b \in B$, and $\mathbf{Q}$ is the subgroupoid generated by $\{b^{n+1}\}$, then the set of primes in $\mathbf{Q}$ is empty. (Namely, $(b^{n+1})^{m+1} = b^{n+1}$, which implies that b^{n+1} is not a prime in $\mathbf{Q}$.) □

Theorem 2 is a corollary of Propositions 4.1–4.4.

[7] Here, and further on in Section 4, if $\mathbf{H}$ is $\mathcal{U}(M; N)$–free groupoid, and $\mathbf{Q}$ is a subgroupoid of $\mathbf{H}$, we will write "$\mathbf{Q}$ is free" instead of "$\mathbf{Q}$ is $\mathcal{U}(M; N)$–free".

Proposition 4.5. *Let* **H** *be a* $\mathcal{U}(M;N)$*-free groupoid and* **Q** *a subgroupoid of* **H**, *such that:*

$$(\forall x \in H)\big(x \in Q \Rightarrow \mathrm{bs}(x) \in Q\big). \tag{24}$$

Then **Q** *is free.*

Proof. From (24) it follows that if $a \in QQ$ and (c,d) is the pair of divisors of a in **H**, then $c,d \in Q$, and moreover the following equation holds:

$$|a| = |c| + |d|.$$

This implies that the set P of primes in **Q** is nonempty and generates **Q**. In the same way as 5) in the proof of Proposition 4.1, one can show that **Q** is free with the basis P. □

In the next three statements we describe free subgroupoids of $\mathcal{U}(M;N)$-free groupoids when the class of $\mathcal{U}(M;N)$-free groupoids is not hereditary.

Proposition 4.6. *Let* **H** *be a* $\mathcal{U}(M;N)$*-free groupoid, where* $M \neq \emptyset, N \neq \emptyset$, *and* **Q** *be a subgroupoid of* **H**. *If* **Q** *does not satisfy* (24), *then* **Q** *is not free.*

Proof. Let $m = \gcd(M), n = \gcd(M \cup N), m = kn$, and let $a \in Q$ be such that $b \notin Q$, where b is the base of a in **H**. Then, there exists an $i \in \{1, 2, \cdots, k\}$, such that $a = b^{in+1}$. Then:

$$a^2 = b^{in+2},\ a^3 = b^{in+3}, \cdots,\ a^{(k-i)n+1} = b^{kn+1} = b^{m+1},$$

$$a^{(k-i)+2} = b^2,\ a^{(k-i)n+3} = b^3, \cdots, a^{(k-i)n+n} = b^n,$$

$$a^{(k-i+1)n+1} = b^{n+1}, \cdots, a^{(k-i+2)n+1} = b^{2n+1}, \cdots, a^{kn} = b^{in},\ a^{kn+1} = a$$

are elements of Q. Thus, $b^2, b^3, \cdots, b^{m+1} \in Q$, but $b \notin Q$. From the equality $(b^{m+1})^{m+1} = b^{m+1}$ it follows that b^{m+1} is not a base, and if **Q** were free, a base c in **Q** and $j \in \{1, 2, \cdots, k\}$ would exist, such that $b^{m+1} = c^{in+1}$, which would imply $i = k, c = b \in Q$, i.e. we would obtain a contradiction. □

Proposition 4.7. *Let* **Q** *be a subgroupoid of a* $\mathcal{U}_r(N)$*-free groupoid, where* $n = \min(N) \in N$, *and let, for* $x \in Q$, $j(x)$ *be defined as follows:*

$$j(x) = \min\{s : (\mathrm{bs}(x))^{s+1} \in Q, s \geq 0\}. \tag{25}$$

Then, **Q** *is free iff*

$$(\forall x \in Q)\ n | j(x). \tag{26}$$