Proof. Assuming the condition (26), in the same way as in the proof of Proposition 4.1, one can show that $\mathbf{Q}$ is free.

Thus, we can assume that there exists an $a \in Q$, such that n is not a divisor of $j = j(a)$. Then, b^{j+1} is a prime in $\mathbf{Q}$, where $b = \text{bs}(a)$ is the base of a; moreover, then $b^{j+\nu} \in Q$, for any $\nu \geq 1$. Denote by $\mathbf{T}$ the subgroupoid of $\mathbf{Q}$ generated by the set P of primes in $\mathbf{Q}$. Thus: $T = \bigcup\{P_\nu : \nu \geq 1\}$, where $P_1 = P$, $P_{\nu+1} = P_\nu \cup \{xy : x, y \in P_\nu\}$. Then, if $i \geq 1$ is such that $in > j$, $b^{in+1} \in Q \setminus T$, and therefore the set P of primes in $\mathbf{Q}$ does not generate $\mathbf{Q}$. □

Proposition 4.8. *Let* $\mathbf{H}$ *be a* $\mathcal{U}_r(N)$*-free groupoid, where* $\gcd(N) \notin N$*. A subgroupoid* $\mathbf{Q}$ *of* $\mathbf{H}$ *is free iff* $\mathbf{Q}$ *satisfies the condition* (24).

Proof. Denote by S the additive groupoid of positive integers generated by N. Then $n = \gcd(S) = \gcd(N) \notin S$. If $\mathbf{Q}$ satisfies the condition (24), then, by Proposition 4.5, $\mathbf{Q}$ is $\mathcal{U}_r(N)$-free.

Assume that there exists an $a \in Q$, such that $b \notin Q$, where b is the base of a in $\mathbf{H}$. Therefore, there exists an $s \in S$, such that $a = b^{s+1}$. Let

$$i = \min\{\nu : b^{\nu+1} \in Q, \nu \geq 1\}. \tag{27}$$

Then $i \geq 1$, and $b^{i+1} \in Q$; moreover b^{i+1} is prime in $\mathbf{Q}$. If $i \notin S$, then one can show that $\mathbf{Q}$ is not $\mathcal{U}_r(N)$-free in the same way as in Proposition 4.7, in the case "n is not a divisor of j".

Thus we can assume that $i \in S$. Then $b^{i+\nu} \in Q$, for any $\nu \geq 1$.

Let s be the least element of S such that $s + \nu n \in S$, for any $\nu \geq 1$. (See [6, Lemma 1.6.iii] or [7].) Then $j = s + (i - n) \in S$, but $j - i = s - n \notin S$. Let k be the least element of S such that $i < k \leq j$, and $j - k \in S \cup \{0\}$. Then: $i + j = k + i + \alpha$, where $\alpha \in S \cup \{0\}$.

Assume that $\mathbf{Q}$ is free. Then b^{i+1} and b^{k+1} are different bases in $\mathbf{Q}$, but

$$(b^{i+1})^{j+1} = b^{i+j+1} = b^{i+k+\alpha+1} = (b^{k+1})^{i+\alpha+1},$$

which is impossible. □

5. Ranks of free subgroupoids of $\mathcal{U}(M;N)$–free groupoids

We shall first consider $\mathcal{U}(M;N)$-free groupoids with one element basis $B = \{b\}$ in the cases $\mathcal{U}_l(1)$, $\mathcal{U}_r(1)$ and $\mathcal{U}(1;1)$ and then prove Theorem 3.

Proposition 5.1. *If Z^+ is the set of positive integers, then the groupoid $(Z^+, \bullet)$ defined by $i \bullet j = i + 1$ is $\mathcal{U}_l(1)$-free groupoid with the basis $\{1\}$. If $\mathbf{Q}$ is a subgroupoid of $(Z^+, \bullet)$ and m is the least element of Q, then $\mathbf{Q}$ is a $\mathcal{U}_l(1)$-free groupoid with the basis $\{m\}$. The groupoid $(Z^+, \bullet)^{op}$ is $\mathcal{U}_r(1)$-free with the basis $\{1\}$.*

Proposition 5.2. *The groupoid $\mathbf{H} = (\{1, 2, \cdots, m, m+1\}, \bullet)$, defined by*

$$i \bullet j = \begin{cases} i+1, & \text{for } i \leq m \\ 2, & \text{for } i = m+1 \end{cases}$$

is $\mathcal{U}(1;1)$-free groupoid with the basis $\{1\}$. If Q is a proper subgroupoid of $\mathbf{H}$, then $\mathbf{Q}$ is not $\mathcal{U}(1;1)$-free.

Proof. $Q = \{2, 3, \cdots, m, m+1\}$ is the unique proper subgroupoid in $\mathbf{H}$. The set of primes in $\mathbf{Q}$ is empty, and thus, $\mathbf{Q}$ is not $\mathcal{U}(1;1)$-free. □

Proposition 5.3. *Let $\mathbf{H}$ be a $\mathcal{U}(M;N)$-free groupoid with the basis $\{b\}$, where*

(28) $$\mathcal{U}(M;N) \notin \{\mathcal{U}_l(1), \mathcal{U}_r(1)\} \cup \{\mathcal{U}(m;1) : m \geq 1\},$$

and let $A = \{a_i : i \geq 1\} \subseteq H$ be defined as follows:

(29) $$a_1 = b^2, \quad a_{i+1} = ba_i.$$

If $\mathbf{Q}$ is a subgroupoid of $\mathbf{H}$ generated by A, then $\mathbf{Q}$ is $\mathcal{U}(M;N)$-free with the basis A, and $a_i = a_j \Rightarrow i = j$.

Proof. Assuming that $\mathbf{H} = \mathbf{R}$ is the $\mathcal{U}(M;N)$-canonical groupoid with the basis $\{b\}$, we obtain that $a_i = a_j \Rightarrow i = j$, i.e. A is an infinite subset of H. Moreover, for each $i \geq 1$, b is the left divisor for a_i, and $b \notin Q$. This implies that a_i is prime in $\mathbf{Q}$.

It remains to show that $\mathbf{Q}$ is $\mathcal{U}(M;N)$-free with the basis A.

If $\mathcal{U}(M;N) = \mathcal{U}_l(m)$, by Proposition 4.1 we obtain that $\mathbf{Q}$ is $\mathcal{U}(M;N)$-free with the basis A.

Assume now that $\mathcal{U}(M;N) = \mathcal{U}_r(S), 1 \notin S = \langle N \rangle$, and that $d = c^{i+1} \in Q$, where $i \in S$, and c is a base in $\mathbf{H}$. Then $d \notin A$, and therefore $c^i \in Q$. Continuing in such a way, we would obtain $c \in Q$. Thus by Proposition 4.5, (24) is satisfied, and thus $\mathbf{Q}$ is $\mathcal{U}_r(S)$-free with the basis A.

Finally, let $\mathcal{U}(M;N) = \mathcal{U}(kn;n), n \geq 2$. The fact that A generates $\mathbf{Q}$ implies:

$$Q = \bigcup\{A_i : i \geq 1\}, \text{ where } A_1 = A,\ A_{i+1} = A_i \cup \{xy : x, y \in A_i\}.$$

Assume that $d = c^{in+1} \in Q$, where $1 \leq i \leq k$, and c is a base in $\mathbf{H}$. Let s be the least positive integer, such that $d \in A_{s+1} \setminus A_s$. Such an s exists as $d \notin A$. Thus, there exist $d', d'' \in A_s$, such that $c^{in+1} = c^{in}c = d = d'd''$, and therefore $c^{in} = d'$, $d'' = c^{jn+1}$ for some $0 \leq j \leq k$. So, $c^{in} \in Q \setminus A$; then, by the same argument, $c^{in-1} \in Q$, etc., and by an obvious induction we obtain that $c \in Q$. Therefore, $\mathbf{Q}$ is free. □

We note that in the case $\mathcal{U}(M;N) = \mathcal{U}_l(m)$, $\mathbf{Q}$ satisfies the relation (24). Also:

$\mathcal{U}(M;N) = \mathcal{U}_l(1) \Rightarrow A = \{b^2\}$, and $\mathbf{Q}$ is $\mathcal{U}_l(1)$–free with the basis A;

$\mathcal{U}(M;N) = \mathcal{U}_r(1) \Rightarrow Q = A$, and $\mathbf{Q}$ is $\mathcal{U}_r(1)$–free with the basis $\{b^2\}$;

$\mathcal{U}(M;N) = \mathcal{U}(m;1) \Rightarrow A = \{b^2\}, Q = \{b^2, b^3, \cdots, b^{m+1}\}$, and $\mathbf{Q}$ is not $\mathcal{U}(m;1)$–free.

The proof of the following statement is the same as the proof of Proposition 5.3, and moreover, the assumption (28) is not necessary.

Proposition 5.4. *Let* $\mathbf{H}$ *be a* $\mathcal{U}(M;N)$*-free groupoid with the basis* $\{a, b\}$, $a \neq b$, *and let* $C = \{c_i : i \geq 1\}$ *be defined as follows:*

$$c_1 = ab, \quad c_{i+1} = ac_i. \tag{30}$$

Then $c_i = c_j \Rightarrow i = j$, *and the subgroupoid* $\mathbf{Q}$ *of* $\mathbf{H}$ *generated by* C *is* $\mathcal{U}(M;N)$*-free with the basis* C.

This completes the proof of Theorem 3.

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