NOVI SAD J. MATH.
VOL. 29, NO. 2 (1999), 131–147
PROC. VIII INT. CONF.
"ALGEBRA & LOGIC" (NOVI SAD, 1998)

FREE GROUPOIDS WITH AXIOMS OF THE FORM $x^{m+1}y = xy$ AND/OR $xy^{n+1} = xy$

Ǵorǵi Čupona
Macedonian Academy of Sciences and Arts
91000 Skopje, Macedonia

Naum Celakoski
Faculty of Mechanical Engineering, University of Skopje
91000 Skopje, Macedonia

Biljana Janeva
Institute of Informatics, Faculty of Sciences and Mathematics
University of Skopje, 91000 Skopje, Macedonia

Abstract

The main result of the paper is a canonical description of free objects in the variety $\mathcal{U}(M;N)$ of groupoids with the following axioms:

$$\{x^{m+1} \cdot y = xy|\ m \in M\} \cup \{x \cdot y^{n+1} = xy|\ n \in N\},$$

where M and N are sets of positive integers, such that $M \cup N \neq \emptyset$. Applying the obtained description, corresponding characterization of free subgroupoids of a $\mathcal{U}(M;N)$–free groupoid is given.

AMS Math. Subject Classification (1991): 20N02, 08A30, 08B20
Key words and phrases: groupoid, variety, free groupoid

1. Main results

Throughout the paper $\mathbf{F} = (F; \cdot)$ denotes the absolutely free groupoid (i.e. free groupoid in the variety of all groupoids) with a given basis B. Therefore, $\mathbf{F}$ is injective[1] and B is the set of primes[2] in F. Moreover, each subgroupoid of $\mathbf{F}$ is free and there exist subgroupoids of $\mathbf{F}$ with infinite basis (see [1], I.1)

There exist $\frac{(2k-2)!}{(k-1)!k!}$ k–th groupoid powers[3] $x \mapsto x^k$. In this paper x^k is defined by

$$x^1 = x, \quad x^{k+1} = x^k x,$$

and this is the meaning of the groupoid power in the axioms of $\mathcal{U}(M; N)$.

If $\xi, \eta : F \to F$ are two transformations on F, then we denote by $\mathbf{F}(\xi, \eta)$ the groupoid $(F, \bullet)$ defined by $x \bullet y = \xi(x)\eta(y)$. We say that the pair ξ, η of transformations on F is *compatible* with $\mathbf{F}$ iff the following two conditions are satisfied:

1) $(\forall b \in B)\ \xi(b) = b = \eta(b)$

2) The least subset R of F with the following property:

(1) $$B \subseteq R \ \& \ (\forall t, u \in R)\big(\xi(t) = t, \eta(u) = u \Rightarrow tu \in R\big)$$

is a subgroupoid of $\mathbf{F}(\xi, \eta)$.

Here we introduce several notations.

The varieties $\mathcal{U}(M; \emptyset)$, $\mathcal{U}(\emptyset; N)$, $\mathcal{U}(M; N)$, where $M \neq \emptyset$ and $N \neq \emptyset$, are said to be *left, right* and *two–sided*, respectively. The variety $\mathcal{U}(M; \emptyset)$ will be also denoted by $\mathcal{U}_l(M)$, and $\mathcal{U}(\emptyset; N)$ by $\mathcal{U}_r(N)$. Further,

$$\mathcal{U}(m_1, m_2, \cdots; n_1, n_2, \cdots)$$

will be an abbreviation for $\mathcal{U}(\{m_1, m_2, \cdots\}; \{n_1, n_2, \cdots\})$

We state below the main results of the paper.

Theorem 1. *If B is a nonempty set and M, N are sets of positive integers such that $M \cup N \neq \emptyset$, then there exists a pair (ξ, η) of transformations on F compatible with* $\mathbf{F}$ *with the following properties:*

[1] A groupoid $\mathbf{G}$ is *injective* iff $(\forall x, y, u, v \in G)(xy = uv \Rightarrow x = u \ \& \ y = v)$

[2] an element $a \in G$ is *prime* in $\mathbf{G}$ iff $a \in G \setminus GG$.

[3] see [3], III.2, Ex.2, p.125 or [8], pp.39–40

(*i*) *The restrictions of* ξ *and* η *on* R *are retractions of* R.

(*ii*) *The corresponding groupoid* $\mathbf{R}$ *is a* $\mathcal{U}(M;N)$*-free groupoid with a unique basis* B, B *being the set of primes in* $\mathbf{R}$.

We say that $\mathbf{R}$ is the $\mathcal{U}(M;N)$*-canonical groupoid* with the basis B.

Theorem 2. *The class of free objects in a variety* $\mathcal{U}(M;N)$ *is hereditary iff*

$$(M \neq \emptyset,\ N = \emptyset) \text{ or } (M = \emptyset,\ 1 \in N).$$

Theorem 3. *Let* $\mathbf{H}$ *be a* $\mathcal{U}(M;N)$*-free groupoid with the basis* B. *If* B *contains at least two distinct elements or* $\mathcal{U}(M;N) \notin \{\mathcal{U}_l(1), \mathcal{U}_r(1)\} \cup \{\mathcal{U}(m;1) : m \geq 1\}$, *then there exists* $\mathcal{U}(M;N)$*-free subgroupoid of* $\mathbf{H}$ *with infinite basis.*

In Section 2 we state some preliminary results, and in Section $i+2$ we give the proof of Theorem i. Moreover, in Section 4 we describe the family of free subgroupoids of a $\mathcal{U}(M;N)$–free groupoid in the case when the class of $\mathcal{U}(M;N)$-free groupoids is not hereditary.

2. Preliminaries

Here we state some properties of the groupoid $\mathbf{F}$ and one of the main results of [6]. Let $x \mapsto |x|$ be the homomorphism of $\mathbf{F}$ into the additive groupoid of positive integers which extends the mapping $B \to \{1\}$. In other words, we have:

$$(\forall b \in B)\ |b| = 1,$$

(2)

$$(\forall x, y \in F)\ |xy| = |x| + |y|.$$

(We say that $|t|$ is the *length* of t in $\mathbf{F}$.)

Below we assume that m is a given positive integer, p, q arbitrary non-negative integers, and $i, j, k, \cdots$ arbitrary positive integers. We define two kinds of groupoid powers $x \mapsto x^{(p)}$, $x \mapsto x^{\langle p \rangle}$ as follows:

(3) $$x^{(0)} = x^{\langle 0 \rangle} = x,\ x^{(p+1)} = (x^{(p)})^{m+1};\ x^{\langle p+1 \rangle} = x\ \underline{x^{\langle p \rangle} m},$$

where the right-hand side of the last equation has the following meaning:

(4) $$x\,\underline{y0} = x,\ x\,\underline{yp+1} = (x\,\underline{yp})y.$$

By induction on the length of elements of F we obtain that, for any $t, u \in F,\ p, q \geq 0,\ i, j \geq 1$, the following relations hold:

(5) $$|t^{i}| = i|t|;\ |t^{(p)}| = (m+1)^{p}|t|;\ |t^{\langle p\rangle}| = |t|\sum_{q=0}^{p} m^{q};$$

(6) $$t^{i+1} = u^{j+1} \Rightarrow t = u,\ i = j;$$

(7) $$t^{(p)} = u^{(p+q)} \iff t = u^{(q)};$$
$$(t^{(p)})^{(q)} = t^{(p+q)};$$

(8) $$1 \leq i < m \Rightarrow \left(t^{i+1} \neq u^{(p+1)}\ \&\ t\,\underline{t^{\langle p\rangle}i} \neq u^{\langle q+1\rangle}\right);$$

(9) $$t^{\langle p+1\rangle} = u^{\langle q+1\rangle} \iff t = u,\ p = q.$$

One of the main results in [6] is the following

Theorem 2.1. *If M and N are nonempty sets of positive integers, then:*

(*i*) $\mathcal{U}(M;\emptyset) = \mathcal{U}(\gcd(M);\emptyset)$;[4]

(*ii*) $\mathcal{U}(\emptyset;N) = \mathcal{U}(\emptyset;\langle N\rangle)$; [5]

(*iii*) $\mathcal{U}(M;N) = \mathcal{U}(\gcd(M);\gcd(M \cup N))$.

Considering Theorem 2.1 we shall examine three types of $\mathcal{U}(M;N)$ varieties with corresponding canonical sets of axioms, i.e. $\mathcal{U}(\emptyset;S)$,$\mathcal{U}(m;\emptyset)$ and $\mathcal{U}(m;n)$ which will be denoted as $\mathcal{U}_r(S)$,$\mathcal{U}_l(m)$ and $\mathcal{U}(m;n)$, respectively. Here S is the additive groupoid of positive integers generated by N, $m = \gcd(M)$ and $n = \gcd(M \cup N)$in the case when both $M \neq \emptyset$ and $N \neq \emptyset$.

[4] $\gcd(M)$ denotes the greatest common divisor of M.

[5] $\langle N\rangle$ is the subgroupoid of the additive groupoid of positive integers generated by N.

We shall also use the following relations:[6]

$$\mathcal{U}_l(m) \models x^{(p)}y = xy; \tag{10}$$

$$\mathcal{U}_l(m) \models x^{(p)} = x^{\langle p \rangle}; \tag{11}$$

$$\mathcal{U}_l(m) \models x^{pm+i+1} = x^{i+1}; \tag{12}$$

$$\mathcal{U}_r(i) \models (x^{i+1})^{j+1} = x^{i+j+1}; \tag{13}$$

$$\mathcal{U}(kn; n) \models (x^{in+1})^{kn+1} = x^{in+1}. \tag{14}$$

3. $\mathcal{U}(M; N)$–canonical groupoids

We assume below that m is a positive integer, and S is an additive groupoid of positive integers.

Define two transformations $\alpha, \beta : F \to F$, as follows:

$$\alpha(u) = \begin{cases} t, & \text{if } u = t^{\langle p+1 \rangle}, p \geq 0 \\ u, & \text{otherwise} \end{cases} \tag{15}$$

$$\beta(u) = \begin{cases} t, & \text{if } u = t^{i+1}, i \in S \\ u, & \text{otherwise} \end{cases} \tag{16}$$

By (9) and (6), α and β are well defined.

Assume now that M and N are sets of positive integers such that $M \cup N \neq \emptyset$. Using α and β, we define two transformations ξ, $\eta : F \to F$ for each of the following cases $\mathcal{U}_l, \mathcal{U}_r, \mathcal{U}$:

$\mathcal{U}_l$: If $M \neq \emptyset$, $N = \emptyset$, $m = \gcd(M)$, then $\xi = \alpha$ and $\eta = 1_F$;

$\mathcal{U}_r$: If $M = \emptyset$, $N \neq \emptyset$, $S = \langle N \rangle$, then $\xi = 1_F$ and $\eta = \beta$;

$\mathcal{U}$: If $M \neq \emptyset$, $N \neq \emptyset$, $m = \gcd(M)$, $n = \gcd(M \cup N)$, $S = \{in : i \geq 1\}$, then $\xi = \alpha$ and $\eta = \beta$.

[6] $\mathcal{V} \models \tau_1 = \tau_2$ means: the equation $\tau_1 = \tau_2$ is true in the variety $\mathcal{V}$.

Clearly, in each of the cases: $\mathcal{U}_l$, $\mathcal{U}_r$, $\mathcal{U}$ the condition 1) of Section 1 (for the pair (ξ, η) to be compatible with $\mathbf{F}$) is satisfied. Moreover, according to the condition 2), the corresponding subset R of F is defined as follows: $B \subseteq R$, and

$$\begin{array}{lll} & \mathcal{U}_l: & (\forall v, w \in F)(vw \in R \iff v, w \in R \,\&\, \alpha(v) = v) \\ (17) & \mathcal{U}_r: & (\forall v, w \in F)(vw \in R \iff v, w \in R \,\&\, \beta(w) = w) \\ & \mathcal{U}: & (\forall v, w \in F)(vw \in R \iff v, w \in R \,\&\, \alpha(v) = v \,\&\, \beta(w) = w). \end{array}$$

From (15), (16) and (17), we obtain the following relations:

$$\begin{array}{lll} & \mathcal{U}_l: & v = u^{\langle p+1 \rangle} \Rightarrow (v \in R \iff u \in R \,\&\, \alpha(u) = u); \\ (18) & \mathcal{U}_r: & v = u^{i+1} \Rightarrow (v \in R \iff u \in R \,\&\, \beta(u) = u); \\ & \mathcal{U}: & v = u^{\langle p+1 \rangle} \Rightarrow (v \in R \iff u \in R, p = 0 \,\&\, \alpha(u) = u); \\ & & i \geq 1, v = u^{in+1} \Rightarrow (v \in R \iff u \in R, i \leq k \,\&\beta(u) = u), \\ & & \text{where } kn = m. \end{array}$$

From (18), we obtain:

Proposition 3.1. *The restrictions of ξ and η on R are retractions of R.*

From the definition of the groupoid $\mathbf{F}(\xi, \eta)$ and Proposition 3.1 it follows:

Proposition 3.2. $\mathbf{R} = (R, \bullet)$ *is a subgroupoid of* $\mathbf{F}(\xi, \eta)$, *and B is the least generating subset of R.*

From (18), the definitions of the pair (ξ, η) and Proposition 3.1 it follows that for each $u \in R$, there exists a unique $t \in R$ and a unique:
$p \geq 0$, in the case $\mathcal{U}_l$; $i \in S \cup \{0\}$, in the case $\mathcal{U}_r$; $s:\ 0 \leq s \leq k$, in the case $\mathcal{U}$, such that:

$$(19) \qquad \mathcal{U}_l: u = t^{\langle p \rangle},\ \alpha(t) = t;\ \mathcal{U}_r: u = t^{i+1},\ \beta(t) = t; \qquad \mathcal{U}: u = t^{sn+1},\ \beta(t) = t.$$

If $v, w \in R$, then $v \bullet w$ can be expressed more explicitely as follows:

$$\begin{array}{lll} & \mathcal{U}_l: & v \bullet w = tw, \text{ where } v = t^{\langle p \rangle}, p \geq 0, \alpha(t) = t; \\ (20) & \mathcal{U}_r: & v \bullet w = vu, \text{ where } w = u^{i+1}, i \in S \cup \{0\}, \beta(u) = u; \\ & \mathcal{U}: & v \bullet w = \begin{cases} vu, & \text{if } \alpha(v) = v,\ w = u^{in+1}, 1 \leq i \leq k \\ tw, & \text{if } v = t^{m+1}, \beta(w) = w \\ tu, & \text{if } v = t^{m+1}, w = u^{in+1}, 1 \leq i \leq k \\ vw, & \text{if } \alpha(v) = v, \beta(w) = w. \end{cases} \end{array}$$