Математички Билтен
23 (XLIX)
1999 (11-18)
Скопје, Македонија

ISSN 0351-336X

CANONICAL GROUPOIDS WITH $x^m \cdot y^n = xy$

Ǵ. Čupona*, N. Celakoski** and B. Janeva***

Abstract

We give a convinient description of free objects in the variety of groupoids with the axiom $x^m \cdot y^n = xy$.

1. Main Results

First we state necessary preliminaries.

Among all the possible $\frac{(2k-2)!}{k!(k-1)!}$ k-th groupoid powers, here x^k is defined as follows:

$$x^1 = x\,, \quad x^{k+1} = x^k \cdot x\,, \tag{1.1}$$

and this is the meaning of the powers in the axiom

$$x^m \cdot y^n = xy \tag{1.2}$$

of the variety of groupoids $\mathcal{U}^{(m,n)}$.

Throughout the paper, we assume that $\boldsymbol{F} = (F, \cdot)$ is an absolutely free groupoid with a given basis B. A mapping $x \mapsto P(x)$, from F into the family of finite nonempty subsets of F is defined as follows:

$$P(b) = \{b\}, \quad P(tu) = \{tu\} \cup P(t) \cup P(u)\,, \tag{1.3}$$

for any $b \in B$, t, $u \in F$. (We say that $P(u)$ is a *part* of u.)

We say that a groupoid $\boldsymbol{R} = (R, *)$ is a $\mathcal{U}^{(m,n)}$-*canonical groupoid* iff the following conditions hold:

a) $B \subseteq R \subseteq F$; b) $(\forall t, u \in F)\ (tu \in R \Rightarrow t, u \in R \;\&\; t * u = tu)$;

c) $\boldsymbol{R}$ is free in $\mathcal{U}^{(mn)}$ with the basis B.

The following statement is a special case of the main result in the paper [5, Theorem 1].

Theorem 1. *Assume that*

$$(m,n) \in \{(i,j) : (i = 1 \;\; or \;\; j = 1) \;\&\; i, j \in \mathbb{N}\} \cup \{(2,2)\}\,.$$[1]

[1] $\mathbb{N}$ is the set of positive integers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of nonnegative integers.

Then, there exist two transformations ξ, η on F such that the structure $\boldsymbol{R} = (R, \bullet)$ *defined by:*

$$B \subseteq R \;\&\; (\forall t,\, u \in R)(tu \in R \Leftrightarrow \xi(t) = t,\, \eta(u) = u)\,,$$
$$(\forall t,\, u \in R)t \bullet u = \xi(t)\,\eta(u)\,.$$

is a $\mathcal{U}^{(m,n)}$-canonical groupoid.

Now, we shall state the main results in this paper.

Theorem 2. *If $m, n \in \mathbb{N}$ are such that $m, n \geq 2$ and $m + n \geq 5$, then there exist mappings $\xi, \eta\colon F \times \mathbb{N}_0 \to F$ such that a $\mathcal{U}^{(m,n)}$-canonical groupoid* $\boldsymbol{R} = (R, \bullet)$ *is defined as follows:*

$$R = \{v \in F\colon (\forall t,\, u \in F, p, q \geq 0)\xi(t, p+1) \cdot \eta(u, q+1) \notin P(v)\}$$

$$(\forall v,\, w \in R)v \bullet w = \begin{cases} vw, & \text{if } vw \in R \\ \xi(t,p) \bullet \eta(u,q), & \text{if } vw = \xi(t, p+1) \cdot \eta(u, q+1)\,. \end{cases}$$

Theorem 3. *The class of free objects in $\mathcal{U}^{(m,n)}$ is hereditary iff $n = 1$ or $(m = 1,\ n = 2)$.*

2. Some Properties of F and $\mathcal{U}^{(m,n)}$

In the proof of Theorem 2 we shall use another three kinds of groupoid powers: $x^{(k,p)}$, $x^{\langle p\rangle}$, $x^{[p]}$. We define them below, assuming that k, n, m, p, q are integers such that $k \geq 1$, $n \geq 3$, $2 \leq m < n$, $p, q \geq 0$:

$$x^{(k,0)} = x,\ x^{(k,p+1)} = (x^{(k,p)})^k\,,$$
$$x^{\langle 0\rangle} = x,\ x^{\langle p+1\rangle} = x^2\ \underline{x^{\langle p\rangle} n-2}\,,$$
$$x^{[0]} = x,\ x^{[1]} = x^n,\ x^{[p+2]} = x^{[p+1]} x^{[p]}\ \underline{x^{[p+1]} n - m - 1}\,,$$

where

$$xyz = (xy)z,\quad xy\,\underline{z0} = xy,\quad xy\,\underline{zp+1} = (xy\,\underline{zp})z\,.$$

It is clear that

$$(x^{(k,p)})^{(k,q)} = x^{(k,p+q)}\,,$$

and it is easy to show that the following identities hold in $\mathcal{U}^{(m,n)}$:

$$x^{(m,p)} y^{(n,p)} = xy, \qquad (\text{for } m, n \geq 1,\ p \geq 0)\,, \tag{2.1}$$
$$x^{\langle p\rangle} = x^{(n,p)}, \quad (\text{for } m = n \geq 3,\ p \geq 0)\,, \tag{2.2}$$
$$x^{[p]} = x^{(n,p)}, \quad (\text{for } 2 \leq m < n,\ p \geq 0)\,. \tag{2.3}$$

Some properties of the groupoid $\boldsymbol{F}$ shall be stated below. First, note that $\boldsymbol{F}$ is injective[2] groupoid, and that the basis B of $\boldsymbol{F}$ consists of the set of primes[3] in $\boldsymbol{F}$. As a consequence, we obtain that the mapping $x \mapsto P(x)$, is well defined by (1.3).

---

[2] A grupoid $\boldsymbol{G}$ is *injective* iff $(\forall x, y, u, v \in G)\,(xy = uv \Rightarrow x = u,\ y = v)$.

[3] An element $a \in G$ is *prime* in $\boldsymbol{G}$ iff $(\forall x,\ y \in G)a \neq xy$.

We denote by $|\,|: x \mapsto |x|$ the homomorphism from $\boldsymbol{F}$ into the additive groupoid of positive integers which is an extension of the mapping $B \to \{1\}$. Thus:

$$|b| = 1,\ |tu| = |t| + |u|\,, \tag{2.4}$$

for any $b \in B,\ t,\ u \in F$. (We say that $|t|$ is the *lenght* of t.)

By [5, (2.4)], the definitions of groupoid powers and (2.4) it follows that, for $i \geq 1,\ k \geq 2,\ n \geq 3,\ 2 \leq m < n,\ p, q \geq 0$, the following relations hold in $\boldsymbol{F}$:

$$\begin{gathered} |x^{i+1}| > |x^i|,\ |x^{(k,p+1)}| > |x^{(k,p)}|\,, \\ |x^{\langle p+1\rangle}| > |x^{\langle p\rangle}|,\ |x^{[p+1]}| > |x^{[p]}|\,. \end{gathered} \tag{2.5}$$

By the injectivity of $\boldsymbol{F}$ and (2.5) we obtain the following properties:

$$t^{i+1} = u^{j+1} \Rightarrow i = j,\ t = u\,; \tag{2.6}$$

$$t^{(k,p+q)} = u^{(k,p)} \Rightarrow u = t^{(k,q)}\,; \tag{2.7}$$

$$t^{(k+i,p)} = u^{(k,q)} \Rightarrow p = q = 0,\ t = u\,; \tag{2.8}$$

$$t^{\langle p+1\rangle} = u^{\langle q+1\rangle} \Rightarrow t = u,\ p = q\,; \tag{2.9}$$

$$t^{[p+1]} = u^{[q+1]} \Rightarrow t = u,\ p = q\,; \tag{2.10}$$

$$t^{(m,p)} = u^{[q]} \Rightarrow p = q = 0\,; \tag{2.11}$$

where $i, j \geq 1,\ k \geq 2,\ p, q \geq 0\ t, u \in F$.

By (2.7) it follows that for a given $u \in F$ and $k \geq 1$, there exists at most one pair (t, p), such that $u = t^{(k,p+1)}$. If such a pair (t, p) exists, then we write $(u)_k = p,\ t = u^{(k,-)}$, and if $(u)_k = 0$, then $t = u = u^{(k,-)}$. If k is fixed, we shall often write $x^{(p)},\ (x),\ x^{(-)}$ instead of $x^{(k,p)},\ (x)_k,\ x^{(k,-)}$-respectively. In the same sence, by (2.9) and (2.10), for a given $u \in F$ one defines $\langle u\rangle,\ [u]$-respectively.

3. Proofs of Theorems

As we mentioned in Section 1, Theorem 1 is a special case of Theorem 1 in [5]. Therefore we shall merely define the corresponding transformations ξ and η, without entering the proof in details.

Case $\underline{m = n = 1}$. $\xi(u) = \eta(u) = u$, for every $u \in F$,

Therefore $\boldsymbol{F}$ is the $\mathcal{U}^{(1,1)}$-canonical groupoid.

Case $\underline{m = 1,\ n \geq 2}$. $\xi(u) = u$, for every $u \in F$,

$$\eta(u) = \begin{cases} u\,, & \text{if } u \neq t^k\,,\ \ t \in F,\ k \geq 2\,, \\ t\,, & \text{if } u = t^k\,,\ \ k \geq 2,\ t \in F\,. \end{cases}$$

Case $\underline{m \geq 2\,,\ n = 1}$. First, for $x \in F,\ p \geq 0$, define $f(x, p)$ by:

$$f(x, 0) = x\,,\ \ f(x, p+1) = x\,\underline{f(x, p)m}\,.$$

Then: $\eta(u) = u$, for every $u \in F$, and

$$\xi(u) = \begin{cases} u\,, & \text{if } (\forall t \in F\,,\, p \geq 0) u \neq f(t, p+1)\,, \\ t\,, & \text{if } u = f(t, p+1)\,. \end{cases}$$

Case $m = n = 2$.

$$\xi(u) = \eta(u) = \begin{cases} u\,, & \text{if } (\forall t \in F) u \neq t^2\,, \\ t\,, & \text{if } u = t^2\,. \end{cases}$$

We begin the proof of Theorem 1, assuming that $m \geq 2$, $n \geq 2$, $m+n \geq 5$. First of all, the identity (2.1) suggests to define ξ and η by:

$$\xi(t,p) = t^{(m,p)}\,, \quad \eta(t,p) = t^{(n,p)}\,, \tag{3.1}$$

having in mind the definition of $x^{(k,p)}$, given in Section 2.

The next proposition implies that the definition (3.1) is "successful" only for $m > n \geq 2$.

Proposition 3.1. *Let $m+n \geq 5$, $m \geq 2$, $n \geq 2$, and $R_{(m,n)} (= R)$ be the subset of F defined by:*

$$R = \{v \in F \colon (\forall t,\, u \in F)\big((t)_m (u)_n \neq 0 \Rightarrow tu \notin P(v)\big)\}\,. \tag{3.2}$$

For v, $w \in R$, let $v \bullet w$ be defined by:

$$v \bullet w = \begin{cases} vw\,, & \text{if } vw \in R\,, \\ t^{(m,p)} \bullet u^{(n,q)}\,, & \text{if } v = t^{(m,p+1)},\ w = u^{(n,q+1)}\,. \end{cases} \tag{3.3}$$

Then

(*i*) *$\boldsymbol{R} = (R, \bullet)$ is a grupoid, B coincides with the set of primes in $\boldsymbol{R}$, and it is the least generating set for $\boldsymbol{R}$.*

(*ii*) *If $\boldsymbol{G} \in \mathcal{U}^{(m,n)}$, and $\lambda \colon B \to G$ is a mapping, then there exists a unique homomorphism $\varphi \colon \boldsymbol{R} \to \boldsymbol{G}$ which extends λ.*

(*iii*) *$\boldsymbol{R} \in \mathcal{U}^{(m,n)}$ iff $m > n \geq 2$.*

(*iv*) *If $m > n \geq 2$, then $\boldsymbol{R}$ is $\mathcal{U}^{(m,n)}$-free groupoid with a (unique) basis B.*

Proof. (*i*) From (3.2), (3.3) and (2.7) it follows that $\bullet$ is a well-defined operation on R. The assertion for B follows from the fact that B is the set of primes in $\boldsymbol{F}$ and it is the least generating set for $\boldsymbol{F}$.

(*ii*) Let $\psi \colon \boldsymbol{F} \to \boldsymbol{G}$ be the homomorphism which extends λ, and let $\varphi \colon R \to G$ be the restriction of ψ on R. By (3.3), $\boldsymbol{G} \in \mathcal{U}^{(m,n)}$ and (2.1), it follows that $\varphi \colon \boldsymbol{R} \to \boldsymbol{G}$ is a homomorphism.

(*iii*) If $t \in R$, $i \geq 1$, then we denote by $t^i_\bullet$ the i-th power of t in $\boldsymbol{R}$. i.e. $t^1_\bullet = t$, $t^{i+1}_\bullet = (t^i_\bullet) \bullet t$.

(*iii*.1) Let $m > n \geq 2$. Then: $t^i_\bullet = t^i$, for each $i \colon 1 \leq i \leq m$. (If $a \in B$, then $a^i_\bullet = a^i$, for every $i \geq 1$.) Hence, for t, $u \in R$, we have:

$$(t^m_\bullet) \bullet (u^n_\bullet) = t^m \bullet u^n = t^{(m,1)} \bullet u^{(n,1)} = t^{(m,0)} \bullet u^{(m,0)} = t \bullet u\,,$$

i.e. $\boldsymbol{R} \in \mathcal{U}^{(m,n)}$.

(*iii*.2) If $2 \leq m = n$, and $a \in B$, then we have: