$$((a^n)^n_\bullet) \bullet a^n = (a^2\, \underline{a^n n-2}) \bullet a^n = a^2\, \underline{a^n n-1} \neq a^{n+1},$$

and thus, $\boldsymbol{R} \notin \mathcal{U}^{(m,n)}$.

(iii.3) Let $2 \leq m < n$, and $a \in B$. Then $(a^n)^n_\bullet = a^{n+1}\, \underline{a^n n-m-1}$, and therefore

$$(a^m_\bullet) \bullet (a^n)^n_\bullet = a^m(a^{n+1}\, \underline{a^n n-m-1}) \neq a \bullet a^n = aa^n.$$

This completes the proof of part (iii).

As a consequence of (i), (ii) and (iii) one obtains that (iv) is true. □

The following property proves Theorem 2 in the case $m = n \geq 3$.

Proposition 3.2. *Let $n \geq 3$ and let $S_n(= S)$ be the subset of F defined by:*

$$S = \{v \in F\colon (\forall t,\ u \in F)(\langle t\rangle\langle u\rangle > 0 \Rightarrow tu \notin P(v))\}. \qquad (3.4)$$

For $v, w \in S$ define $v \bullet w$ by:

$$v \bullet w = \begin{cases} vw, & \text{if } vw \in S \\ t^{\langle p\rangle} \bullet u^{\langle q\rangle}, & \text{if } vw = t^{\langle p+1\rangle} u^{\langle q+1\rangle}. \end{cases} \qquad (3.5)$$

Then $\boldsymbol{S} = (S, \bullet)$ is a $\mathcal{U}^{(n,n)}$-free groupoid with the unique basis B.

Proof.

1) Since $|t^{\langle p+1\rangle}| > |t^{\langle p\rangle}|$, we obtain $\boldsymbol{S} = (S, \bullet)$ is a well-defined groupoid. It is clear that B coincides with the set of primes in $\boldsymbol{S}$ and it generates $\boldsymbol{S}$.

2) Now we shall prove that $\boldsymbol{S} \in \mathcal{U}^{(n,n)}$.

Let $v,\ w \in S$ be such that $0 \leq p \leq \langle v\rangle$, $0 \leq q \leq \langle w\rangle$, $v = t^{\langle p\rangle}$, $w = u^{\langle q\rangle}$. Then

$$v^2_\bullet = t^2\,,\ v^3_\bullet = t^2 t^{\langle p\rangle}, \ldots, v^n_\bullet = t^2\, \underline{t^{\langle p\rangle} n-2} = t^{\langle p+1\rangle}.$$

In the same way we obtain that: $w^n_\bullet = u^{\langle q+1\rangle}$. Therefore:

$$(v^n_\bullet) \bullet (w^n_\bullet) = t^{\langle p+1\rangle} \bullet u^{\langle q+1\rangle} = t^{\langle p\rangle} \bullet u^{\langle q\rangle} = v \bullet w,$$

i.e. we obtain that $\boldsymbol{S} \in \mathcal{U}^{(n,n)}$.[4]

3) Let $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}^{(n,n)}$, $\lambda\colon B \to G$ be a given mapping and $\psi\colon \boldsymbol{F} \to \boldsymbol{G}$ the homomorphism which extends λ. Then (using the fact that the identity (2.2), i.e. $x^{(p)} = x^{\langle p\rangle}$ holds in $\boldsymbol{G}$) we obtain that the restriction φ of ψ on S, i.e. $\varphi\colon \boldsymbol{S} \to \boldsymbol{G}$, is a homomorphism.

From 1), 2) and 3) it follows that $\boldsymbol{S}$ is a $\mathcal{U}^{(n,n)}$-free groupoid with the basis B, and that B coincides with the set of primes in $\boldsymbol{S}$. □

It remains the case $2 \leq m < n$. Bellow we shall write $x^{(p)}$, (x) instead od $x^{(m,p)}$, $(x)_m$-respectively.

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[4] We note that, if $\boldsymbol{G} \in \mathcal{U}^{(2,2)}$, then the identity $x^2 = x^k$ is true in $\boldsymbol{G}$ for every $k \geq 2$ and every groupoid k-th power among all possible groupoid k-th powers.

Proposition 3.3. *Let $2 \leq m < n$ and let the structure $\boldsymbol{T} = (T, \bullet)$ be defined as follows:*

$$T = \{v \in F : (\forall t,\ u \in F, p, q \geq 0)\big((t)[u] \neq 0 \Rightarrow tu \notin P(v)\big)\,,$$

and for $v, w \in F$:

$$v \bullet w = \begin{cases} vw\,, & \text{if } vw \in T \\ t^{(p)} \bullet u^{[q]}\,, & \text{if } v = t^{(p+1)},\ w = u^{[q+1]}\,. \end{cases}$$

Then $\boldsymbol{T}$ is a $\mathcal{U}^{(m,n)}$-free groupoid with the unique basis B, where B coincides with the set of primes in $\boldsymbol{T}$.

Proof.

1) By the same reasoning as for the proof of the Proposition 3.2, $\boldsymbol{T}$ is a well defined groupoid, B is the least generating set of $\boldsymbol{T}$, and it coincides with the set of primes in $\boldsymbol{T}$.
2) By (2.11), $(v)[v] = 0$ for every $v \in F$, and this implies that $v_\bullet^i = v^i$, for every $v \in T$ and $1 \leq i \leq m$. Moreover, for $[v] = 0$, we have:
$$v_\bullet^{m+1} = v^m \bullet v = v^{m+1}, \ldots, v_\bullet^n = v^n\,.$$
In the case $[v] = p+1$, $v = t^{[p+1]}$, we have:
$$v_\bullet^{m+1} = v^m \bullet t^{[p+1]} = vt^{[p]} = t^{[p+1]}t^{[p]}\,,$$
$$v_\bullet^n = t^{[p+1]}t^{[p]}\, \underline{t^{[p+1]} n - m - 1} = t^{[p+2]}\,.$$
Therefore:
$$(v_\bullet^m) \bullet (w_\bullet^n) = v^m \bullet u^{[q+1]} = v \bullet u^{[q]} = v \bullet w\,,$$
where $[w] = q$, $w = u^{[q]}$.
This shows that $\boldsymbol{T} \in \mathcal{U}^{(m,n)}$.
3) In the same way as in the proof of Proposition 3.2 one shows that $\boldsymbol{T}$ is $\mathcal{U}^{(m,n)}$-free with the (unique) basis B. □

Proposition 3.1-3.3 complete the proof of Theorem 2. Here, we define ξ and η by:

$$\xi(x,p) = x^{(m,p)},\quad \eta(x,p) = x^{(n,p)}, \qquad \text{for } m > n \geq 2\,,$$
$$\xi(x,p) = \eta(x,p) = x^{(p)}, \quad \text{for } m = n \geq 3\,,$$
$$\xi(x,p) = x^{(m,p)},\quad \eta(x,p) = x^{[p]}, \qquad \text{for } 2 \leq m < n\,.$$

It remains to prove Theorem 3. We note that the part of the assertion in Theorem 3 for $(m,n) \in \{(i,j) : i = 1 \text{ or } j = 1\} \cup \{(2,2)\}$ is a special case of [5, Theorem 2], and therefore we shall prove the following:

Proposition 3.4. *If $m, n \geq 2$, $m+n \geq 5$, then the class of $\mathcal{U}^{(m,n)}$-free groupoids is not hereditary.*

Using the fact that, if $\boldsymbol{Q}$ is a free object in $\mathcal{U}^{(m,n)}$ then the set of primes in $\boldsymbol{Q}$ is the basis of $\boldsymbol{Q}$, we obtain that Proposition 3.4 is a consequence of the following:

Proposition 3.5. *Let $m \geq 2$, $n \geq 2$, $\boldsymbol{G} \in \mathcal{U}^{(m,n)}$ and $a \in G$. If $\boldsymbol{Q}$ is the subgroupoid of $\boldsymbol{G}$ generated by $\{a^m, a^n\}$, then there are no primes in $\boldsymbol{Q}$.*

Proof. Let us first define a sequence $\{f_i : i \geq 2\}$ of transformations on G. Namely,

$$f_2(x) = a^m a^n\,, \quad f_{k+1}(x) = \big(f_k(x)\big)^m x^n\,.$$

Then, $a^k = f_k(a) = \big(f_{k-1}(a)\big)^m a^n$ is not prime for any $k \geq 2$. Thus, neither a^m, nor a^n is a prime in $\boldsymbol{Q}$. Therefore there are no primes in $\boldsymbol{Q}$. □

4. Relations between $\mathcal{U}^{(m,n)}$, $\mathcal{U}^{(m,1)}$, $\mathcal{U}^{(1,n)}$

By the definition of the variety $\mathcal{U}^{(m,n)}$ it is clear that

$$\mathcal{U}^{(m,1)} \cap \mathcal{U}^{(1,n)} \subseteq \mathcal{U}^{(m,n)}\,. \tag{4.1}$$

It is natural to seak for the answer of the question: what are the cases for which equality holds in (4.1)? The answer is given in the following.

Proposition 4.1. *The equality holds in (4.1) in the following three cases only:*

$$a)\ m = 1, \quad b)\ n = 1, \quad c)\ m = n = 2.$$

Proof. It is clear that the equality holds in (4.1) for each of the cases a), b), and it is easily shown that the equality holds in the case c), as well (see, for example, [3, 1.1 and Remark]). In any other case, the equality $\mathcal{U}^{(m,1)} \cap \mathcal{U}^{(1,n)} = \mathcal{U}^{(m,1)} \cap \mathcal{U}^{(1,d+1)}$ holds, where $d = \gcd(m-1, n-1)$ (the greatest common divisor of $m-1$ and $n-1$). In this case it is easy to show that $\mathcal{U}^{(m,n)}$-canonical groupoid does not belong to the variety $\mathcal{U}^{(m,1)}$. □

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КАНОНИЧНИ ГРУПОИДИ СО $x^m y^n = xy$

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Резиме

Во овој труд се разгледуваат многуобразијата групоиди, определени со аксиома од обликот $x^m y^n = xy$, кои што се означени со $\mathcal{U}^{(m,n)}$. Добиен е погоден опис на слободен групоид со база B, а имено групоид $\mathbf{R} = (R, \star)$ којшто ги задоволува следниве услови:

а) $B \subseteq R \subseteq F$; б) $(\forall t, u \in F)(tu \in R \Rightarrow t, u \in R \& t \star u = tu)$;

в) $\mathbf{R}$ е слободен во $\mathcal{U}^{(m,n)}$ со база B;

$\mathbf{R}$ се нарекува $\mathcal{U}^{(m,n)}$–каноничен групоид. Со $\mathbf{F} = (F, \cdot)$ се означува апсолутно слободниот групоид со база B.

Докажани се следниве теореми:

Теорема 1. *Нека*

$$(m, n) \in \{(i, j) : (i = 1 \text{ or } j = 1) \& i, j \in \mathbb{N}\} \cup \{(2, 2)\}.$$

Тогаш постојат две трансформации ξ, η на F такви што $\mathbf{R} = (R, \bullet)$ определен со:

$$B \subseteq R \ \& \ (\forall t, u \in R)(tu \in R \iff \xi(t) = t, \ \eta(u) = u),$$
$$(\forall t, u \in R) t \bullet u = \xi(t)\, \eta(u).$$

е $\mathcal{U}^{(m,n)}$–каноничен групоид.

Теорема 2. *Ако $m, n \in \mathbb{N}$ се такви што $m, n \geq 2$ и $m + n \geq 5$, тогаш постојат пресликувања $\xi, \eta : F \times \mathbb{N}_0 \to F$ такви што $\mathcal{U}^{(m,n)}$–каноничниот групоид $\mathbf{R} = (R, \bullet)$ е дефиниран со:*

$$R = \{v \in F : (\forall t, u \in F, p, q \geq 0) \xi(t, p + 1) \cdot \eta(u, q + 1) \notin P(v)\}$$

$$(\forall v, w \in R) v \bullet w = \begin{cases} vw, & \text{ако } vw \in R \\ \xi(t, p) \bullet \eta(u, q), & \text{ако } vw = \xi(t, p + 1) \cdot \eta(u, q + 1). \end{cases}$$

Покрај тоа дадена е карактеризација на многуобразијата $\mathcal{U}^{(m,n)}$, такви што класата слободни објекти е наследна. Имено докажана е:

Теорема 3. *Класата слободни објекти во $\mathcal{U}^{(m,n)}$ е наследна акко $n = 1$ или $(m = 1, n = 2)$.*

На крајот, покажано е дека равенството

$$\mathcal{U}^{(m,1)} \cap \mathcal{U}^{(1,n)} = \mathcal{U}^{(m,n)}$$

важи акко $m = 1$ или $n = 1$ или $m = n = 2$.

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