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INJECTIVE VECTOR VALUED SEMIGROUPS

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Abstract

In this paper we introduce the notion of injective vector valued semigroups. Then we show that the class of injective vector valued semigroups is larger than the class of free vector valued semigroups, and give a necessary and sufficient condition for injective vector valued semigroups to be free vector valued semigroups. All this gives a way for checking if a given vector valued semigroup is free.

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1. Preliminary notions

We start with several notions and definitions.

The set of positive integers is $\mathbf{N} = \{1, 2, 3, \ldots\}$, and the set of the first m positive integers is $\mathbf{N}_m = \{1, 2, 3, \ldots, m\}$. Moreover, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and $\mathbf{N}_{k,0} = \mathbf{N}_k \cup \{0\}$.

For a given set $Q \neq \emptyset$, and $t \in \mathbf{N}$, let Q^t be the cartesian product of t copies of Q. If $\mathbf{x} = (a_1, a_2, \ldots, a_t) \in Q^t$, then, for short, we write

$\mathbf{x} = a_1^t$, and moreover we identify $\mathbf{x}$ with the word $a_1a_2\dots a_t$. For such an $\mathbf{x}$ we say that its length $|\mathbf{x}|$ is t, and its contents is the set $cn(\mathbf{x}) = \{a|a = a_i$ for some a_i in the word $\mathbf{x} = a_1a_2\dots a_t\}$. Let Q^+ be the union of all the cartesian products Q^t, for $t \in \mathbf{N}$.

For the rest of the paper, let $n, m, k \in \mathbf{N}$, $n = m + k$ be given.

For a set $Q \neq \emptyset$, let $Q^{m,k} = \{\mathbf{x}|\mathbf{x} \in Q^+, |\mathbf{x}| = m + sk, s \in \mathbf{N}\}$.

We recall the following definitions, given in [1, 2]. A pair $\mathbf{Q} = (Q, f)$ for a map $f : Q^n \to Q^m$ is called an *(n,m)-groupoid*, i.e. a *vector valued groupoid*, written in short as VVG, and the map f is called (n,m)-operation. A VVG, i.e. an (n,m)-groupoid $\mathbf{Q} = (Q, f)$ is called (n,m)-semigroup, i.e. *vector valued semigroup*, written in short VVS, if $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}=\mathbf{uvw}\in Q^{n+k}, \mathbf{y}, \mathbf{v} \in Q^n$, i.e. if the associative law holds for the (n,m)-operation f. Because of the associative law, the operation f can be extended to an operation, denoted by the same letter, $f : Q^{m,k} \to Q^m$ such that for each $\mathbf{xyz}\in Q^{m,k}$, and $\mathbf{y} \in Q^{m,k}, f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{xyz})$. The notion of (n,m)-semigroup is introduced in [5] and is examined in more details in [1], while a convenient construction of free (n,m)-semigroups is given in [4]. It is obvious that a (2,1)-semigroup is a usual semigroup, and that an $(n,1)$-semigroup is an n-semigroup.

A VVG (Q, f) can be considered as an algebra with m n-ary operations $f_j : Q^n \to Q$, where $j \in \mathbf{N}_m$. These operations can be extended to an infinite family of operations $f_{j,s} : Q^{m+sk} \to Q$ for $s \in \mathbf{N}$, where for a given s, there are more than one operation $f_{j,s}$. In the semigroup case, for each $s \in \mathbf{N}$, there is only one operation $f_{j,s} : Q^{m+sk} \to Q$, whose union is a map $f_j : Q^{m,k} \to Q^m$. The above discussion shows that all the notions, (such as generating set, (n,m)-subsemigroup, homomorphism, a free (n,m)-semigroup), in a variety of algebras hold in the variety of (n,m)-semigroups.

Let $\mathbf{Q} = (Q, f)$ be a VVS, and let $\mathbf{x} \in Q^{m,k}$. We say that $\mathbf{x}$ is: *reducible*, if $\mathbf{x} = \mathbf{x}'\mathbf{y}\mathbf{x}''$, where $\mathbf{y} = f(\mathbf{z})$ for some $\mathbf{z} \in Q^{m,k}$, and *irreducible*, otherwise. Usually, we denote the set of all the irreducible words by $R(\mathbf{Q})$.

For a given VVS $\mathbf{Q} = (Q, f)$, let $P(\mathbf{Q}) = Q \setminus \cup\, cnf(\mathbf{x})$, where the union is over all $\mathbf{x} \in Q^{m,k}$. The elements of $P(\mathbf{Q})$ are called *prime elements* of Q.

2. Free vector valued semigroups

Below we will give a construction of a canonical form of a free VVS, slightly different from the construction given in [4]. Let $B \neq \emptyset$, and for each

$j \in \mathbf{N}_m$, let ρ^j be a symbol interpreted as an n-ary functional symbol. Let $\mathbf{F} = (F; \rho^1, \rho^2, \dots, \rho^m)$ be the free algebra with the basis B, of type $\Omega = \{\rho_r^j | j \in \mathbf{N}_m, r \geq 1\}$, where $\rho_r^j \in \Omega_{m+rk}$. So, the elements of F are all the elements of B, and all the elements of the form $\rho_r^j(\mathbf{x})$, for $\mathbf{x} \in F^{m+rk}, r \geq 1$. By choosing different letters, if necessary, for the elements of B, we will have that no element of B is of the form $\rho_r^j(\mathbf{x})$. From now on we are not going to write the lower index. Two elements $\rho^j(\mathbf{x}), \rho^t(\mathbf{y}) \in F$ are equal if and only if $j = t$, and $\mathbf{x}=\mathbf{y}$. For each $u \in F$ we define the norm of u, denoted by $|u|$, to be 1 for $u \in B$, and by induction $|\rho^j(u_1^{m+rk})| = \sum_{t=1}^{m+rk} |u_t|$. Thus, $|\rho^j(u_1^{m+rk})|$ is the number of appearances of elements from B in $\rho^j(u_1^{m+rk})$. We define also the norm of an element $\mathbf{x} \in F^+, \mathbf{x} = u_1^s$, to be $|u_1^s| = \sum_{t=1}^{s} |u_t|$. We say that an element $u \in F$ is *reducible* if it has the form $u = \rho^j(\mathbf{xyz})$, such that $\mathbf{xyz} \in F^{m,k}, \mathbf{y} \in F^m, \mathbf{y} = y_1^m$, and $y_j = \rho^j(\mathbf{w})$ for some $\mathbf{w}$. Otherwise, we say that $u \in F$ is *irreducible*. Let $F(B)$ be the set of all the irreducible elements in F. For an element $\mathbf{x} = u_1^{m+rk} \in F^+$, we say that is *irreducible* if all the u_t are in $F(B)$, and $\rho^1(u_1^{m+rk})$ is in $F(B)$, which is equivalent to $\rho^j(u_1^{m+rk}) \in F(B)$, for each $j \in \mathbf{N}_m$. For each $j \in \mathbf{N}_m$, we define a map $f_j : F(B)^{m,k} \to F(B)$ as follows:

If $\mathbf{u} \in F(B)^{m,k}$ is such that $\rho^1(\mathbf{u}) \in F(B)$, then $f_j(\mathbf{u}) = \rho^j(\mathbf{u})$.

Let $\mathbf{u} = u_1^{m+rk} \in F(B)^{m,k}$ be such that $\rho^1(u_1^{m+rk}) \notin F(B)$. Then, $\rho^t(u_1^{m+rk}) \notin F(B)$ for every $t \in \mathbf{N}_m$, and there is $p, 0 \leq p \leq rk$, such that $u_{p+i} = \rho^i(\mathbf{v})$, for some $\mathbf{v} \in F(B)^{m,k}$ and each $i \in \mathbf{N}_m$. Let p be the smallest such number. Then, by induction on the norm of elements in $F(B)^+$, we define:

$$f_j(\mathbf{u}) = f_j(u_1^{m+rk}) = f_j(u_1^p \mathbf{v} u_{p+m+1}^{m+rk}).$$

Next, we define $f : F(B)^{m,k} \to F(B)^m$, by $f(\mathbf{x}) = f_1(\mathbf{x}) f_2(\mathbf{x}) \dots f_m(\mathbf{x})$. The proof of the following theorem, is by induction on the norm and using the above definition.

Theorem 2.1. $\mathbf{F}(B) = (F(B), f)$ *is a free* (n, m)*-semigroup with basis* B.

Further, we will state several properties of $\mathbf{F}(B)$, whose proof follows from the definition of $\mathbf{F}(B)$ using induction on the norm. These properties led us to the notion of injective VVS.

Theorem 2.2.

(1) The set of prime elements in $\mathbf{F}(B)$ *is* B, *i.e.* $P(\mathbf{F}(B)) = B$;

(2) *The set* $R = R(\mathbf{F}(B))$ *of all the irreducible elements in* $\mathbf{F}(B)$ *consists of all* $\mathbf{x} \in F(B)^{m,k}$ *such that* $\rho^1(\mathbf{x}) \in F(B)$;

(3) *If* $\mathbf{x} \in R$, *then* $\rho^j(\mathbf{x}) \in F(B)$ *for each* $j \in \mathbf{N}_m$;

(4) *For each* $\mathbf{x} \in F(B)^{m,k}$ *there exists* $\mathbf{y} \in R$, *such that* $f(\mathbf{x}) = f(\mathbf{y})$, *and for each* $j \in \mathbf{N}_m, f_j(\mathbf{x}) = \rho^j(\mathbf{y})$;

(5) *If* $\mathbf{x} \in R = R(\mathbf{F}(B))$ *then for each* $j \in \mathbf{N}_m, |\rho^j(\mathbf{x})| = |\mathbf{x}|$;

(6) *If* $f_j(\mathbf{x}) = f_i(\mathbf{y})$ *for some* $\mathbf{x}, \mathbf{y} \in F(B)^{m,k}$, *then* $i = j$ *and for each* $t \in \mathbf{N}_m, f_t(\mathbf{x}) = f_t(\mathbf{y})$.

(7) *If* $\mathbf{x}, \mathbf{y} \in R$, *then* $f(\mathbf{x}) = f(\mathbf{y})$ *implies that* $\mathbf{x}=\mathbf{y}$.

(8) $F(B) = B \cup im\ f_1 \cup im\ f_2 \cup \ldots \cup im\ f_m$, *where the union is disjoint.*

At the end of this part we note that in the case $n = 2, m = 1, k = 1$: $\mathbf{F}(B) = B \cup \{\rho(\mathbf{x}) | \mathbf{x} \in B^+ \setminus B\}$, i.e. after replacing $\rho(\mathbf{x})$ by $\mathbf{x}$, we obtain the well known description of the free semigroup $(B^+, \cdot)$ with the basis B.

3. Injective vector valued semigroups

The above properties of free VVS led to a class of VVS called injective VVS. Let $\mathbf{Q} = (Q, f)$ be an (n, m)-semigroup, $n = m + k$ and let $R(\mathbf{Q})$ be its set of irreducible words. We call $\mathbf{Q}$ *injective* VVS if:

(Inj.1.) If $f_j(\mathbf{a}) = f_i(\mathbf{b})$ for some $i, j \in \mathbf{N}_m$ and $\mathbf{a}, \mathbf{b} \in Q^{m,k}$, then $i = j$ and $f(\mathbf{a}) = f(\mathbf{b})$,

(Inj.2.) For each $j \in \mathbf{N}_m$, if $\mathbf{x}, \mathbf{y} \in R(\mathbf{Q})$, and $f_j(\mathbf{x}) = f_j(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$, i.e. the restriction of each f_j on $R(\mathbf{Q})$ is an injection.

The conditions *(Inj.1.)* and *(Inj.2.)* are equivalent to the conditions *(Inj.1.)* and *(Inj.2'.).*, where:

(Inj.2'.) If $\mathbf{x}, \mathbf{y} \in R(\mathbf{Q})$, and $f(\mathbf{x}) = f(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$, i.e. the restriction of f on $R(\mathbf{Q})$ is an injection.

Theorem 2.2 implies

Proposition 3.1. *A free VVS is an injective VVS.*

For a binary, i.e. a (2,1)-semigroup, $(Q,\cdot)$, an element is prime if it is not a product of two elements, and a word of elements, i.e. $\mathbf{x} \in Q^+$ is irreducible if it is a word of prime elements. A semigroup $(Q,\cdot)$ is injective if and only if for irreducible $a_1^k, b_1^t \in Q^+, a_1 \cdot a_2 \cdot \ldots \cdot a_k = b_1 \cdot b_2 \cdot \ldots \cdot b_t$ implies $k = t$ and $a_1^k = b_1^k$ in Q^+.

It is easy to check that a semigroup is free if and only if it is injective and is generated by the set of prime elements.

The following main theorem in this paper is a generalization of the previous fact.

Theorem 3.2. *A VVS is free if and only if it is injective and is generated by the set of prime elements.*

Proof. Proposition 3.1 and Theorem 2.2, imply that a free VVS is injective and is generated by the set of prime elements.

Conversely, let $\mathbf{H} = (H, h)$ be an injective (n,m)-semigroup generated by its set of prime elements $P(\mathbf{H})$=P. Let $R(\mathbf{H}) = R \subseteq H^{m,k}$ be the set of irreducible words. Let $B = P$, and let $\mathbf{F}(B) = (F(B), f)$ be the free (n,m)-semigroup as constructed in Theorem 2.1. Let $\xi : B \to H$ be defined by $\xi(b) = b$, and let $\varphi : F(B) \to H$ be the homomorphism extending ξ. We extend the map φ to a map $\psi : F(B)^{m,k} \to H^{m,k}$, by: $\psi(x_1^{m+sk}) = \varphi(x_1)\varphi(x_2)\ldots\varphi(x_{m+sk})$. Then, for each $a = \rho^j(\mathbf{u}) \in F(B), \varphi(a) = h_j(\psi(\mathbf{u}))$. Since φ is an extension of ξ, which is in fact the identity map from $B = P$ to P, and since $\mathbf{H}$ is generated by $P = B$, it follows that the map φ is a surjection. In order to show that $\mathbf{H} = (H, h)$ is a free VVS, it is enough to show that the map φ is an injection.

First, by induction on the norm of the elements in $F(B)$ and $F(B)^{m,k}$ we will show that for each $t \in \mathbf{N}$:

(a) If $\mathbf{x} \in R = R(\mathbf{F}(B))$ and $|\mathbf{x}| \leq t$, then $\psi(\mathbf{x}) \in R(\mathbf{H})$; and

(b) If $u, v \in F(B)$ and $|u|, |v| \leq t$, then $\varphi(u) = \varphi(v)$ implies $u = v$.

For $t = 1$, the condition is trivially true, since the elements in R have norm bigger than 1. The condition (b) is true by the definition of φ, which is an extension of the identity map ξ.

Next, let (a) and (b) be true for each $t \leq q - 1$.

Prove first (a) for $t = q$.

Let $\mathbf{x} \in R = R(\mathbf{F}(B)), |\mathbf{x}| = q, \mathbf{x} = x_1^{m+sk}$, and let $\psi(\mathbf{x}) \notin R(\mathbf{H})$. Then, there is $0 \leq p \leq sk$, and $\mathbf{c} \in H^{m,k}$, such that $\varphi(x_{p+i}) = h_i(\mathbf{c})$, for each

$i \in \mathbf{N}_m$, i.e. $\psi(\mathbf{x}) = \psi(x_1^p)h(\mathbf{c})\psi(x_{p+m+1}^{m+sk})$. Since $\varphi(x_{p+i}) = h_i(\mathbf{c})$, for each $i \in \mathbf{N}_m$, it follows that $\varphi(x_{p+i}) \notin P = B$, and so, $x_{p+i} \notin B$ for each $i \in \mathbf{N}_m$. This implies that for each $i \in \mathbf{N}_m$, there is $\mathbf{v}_i \in R = R(\mathbf{F}(B))$, such that $x_{p+i} = \rho^j(\mathbf{v}_i)$, for some $j \in \mathbf{N}_m$. Then, $h_i(\mathbf{c}) = \varphi(x_{p+i}) = h_j(\psi(\mathbf{v}_i))$ and *(Inj.1.)* for $\mathbf{H}$, imply that j=i. Hence, for each $i \in \mathbf{N}_m$, $x_{p+i} = \rho^i(\mathbf{v}_i) \in F(B)$, and by Theorem 2.2.(5), $|\mathbf{v}_i| = |x_{p+i}| < |\mathbf{x}|$, i.e. $|\mathbf{v}_i| \leq q-1$.

Then, $h_i(\mathbf{c}) = \varphi(x_{p+i}) = \varphi(\rho^i(\mathbf{v}_i)) = h_i(\psi(\mathbf{v}_i))$, and *(Inj.1.)* imply that for each $i \in \mathbf{N}_m$, $h(\mathbf{c}) = h(\psi(\mathbf{v}_i))$. Further, since $|\mathbf{v}_i| \leq q-1$ and $\mathbf{v}_i \in R = R(\mathbf{F}(B))$, the induction hypothesis for (a), implies that $\psi(\mathbf{v}_i) \in R(\mathbf{H})$, for each $i \in \mathbf{N}_m$. Now, *(Inj.2.)* together with the equalities

$$h(\mathbf{c}) = h(\psi(\mathbf{v}_1)) = h(\psi(\mathbf{v}_2)) \ldots = h(\psi(\mathbf{v}_m)),$$

implies that $\psi(\mathbf{v}_1) = \psi(\mathbf{v}_2) \ldots = \psi(\mathbf{v}_m)$. Then, for $x_{p+1} = \rho^1(\mathbf{v}_1)$ and $y = \rho^1(\mathbf{v}_j)$ in $F(B)$, we have, $|x_{p+1}| \leq q-1$ and $|y| = |\mathbf{v}_j| \leq q-1$, which by the inductive hypothesis for (b) and the fact that $\varphi(x_{p+1}) = h_1(\psi(\mathbf{v}_1) = h_1(\psi(\mathbf{v}_j)) = \varphi(y)$, implies that $x_{p+1} = y$, i.e. $\rho^1(\mathbf{v}_1) = \rho^1(\mathbf{v}_j)$. The last equality shows that for each $j \in \mathbf{N}_m, \mathbf{v}_1 = \mathbf{v}_j = \mathbf{v} \in R(\mathbf{F}(B))$. This implies that $x_1^m = f(\mathbf{v})$, i.e. that $\mathbf{x} = x_1^p f(\mathbf{x}) x_{p+m+1}^{m+sk}$ is not irreducible, i.e. that $\mathbf{x} \notin R(\mathbf{F}(B))$.

Now we prove (b) for $t = q$.

Let $u, v \in \mathbf{F}(B)$, $u = \rho^i(\mathbf{x})$, $v = \rho^j(\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in R(\mathbf{F}(B))$, $\mathbf{x} = x_1^{m+sk}$, $\mathbf{y} = y_1^{m+rk}$, $|\mathbf{x}| = |u| \leq q$, $|\mathbf{y}| = |v| \leq q$ and let $\varphi(u) = \varphi(v)$. Since $\mathbf{x}, \mathbf{y} \in R(\mathbf{F}(B))$ and $|\mathbf{x}| \leq q, |\mathbf{y}| \leq q$, the part (a) implies that $\psi(u), \psi(v) \in R(\mathbf{H})$. Further, since $h_i(\psi(\mathbf{x})) = \varphi(u) = \varphi(v) = h_j(\psi(\mathbf{y}))$, *(Inj.1.)* and *(Inj.2.)* for $\mathbf{H} = (H, h)$, imply that $i = j$, $h(\psi(\mathbf{x})) = h(\psi(\mathbf{y}))$, and $\psi(\mathbf{x}) = \psi(\mathbf{y})$ as words. But this implies that $s = r$ and for each $1 \leq p \leq m + sk$, $\varphi(x_p) = \varphi(y_p)$. Now, by the induction hypothesis, since $|x_p| \leq q-1$ and $|y_p| \leq q-1$, it follows that $x_p = y_p$. Hence, $\mathbf{x} = \mathbf{y}$, and so $u = v$.

A direct application of (b) implies that φ is an injection, and so it is an isomorphism. Hence, $\mathbf{H} = (H, h)$ is a free (n, m)-semigroup with a basis $P = B$. □

4. Examples

The following examples will show that the class of injective VVS is larger than the class of free VVS.