Example 4.1. Let $\mathbf{F} = (F(a), f)$ be the free (n,m)-semigroup generated by one element a, and let $m \geq 2$. Let $Q = F(a)\backslash\{a\}$, and let g, as an (n,m)-operation on Q, be the restriction of the (n,m)-operation f on $F(a)$. Then $\mathbf{Q} = (Q,g)$ is an injective VVS, but is not free. In this example, the set of prime elements is not empty, but it is not a generating set.

For example, if $n = 3, m = 2$, then $x_i = \rho^i(aaa), y_i = \rho^i(aaaa), i = 1,2$, are four different elements in Q, and: $g_i(x_1x_2y_1y_2) = \rho^i(aaaaaaa) = g_i(y_1y_2x_1x_2)$, which implies that $\mathbf{Q} = (Q,g)$ is not a free VVS.

In the above example the non free injective VVS is a vector valued subsemigroup of a free VVS, which is injective. The following example shows that it is possible to have a vector valued subsemigroup of a free (and so of an injective) VVS, which is not injective.

Example 4.2. Let $\mathbf{F} = (F(a), f)$ be the free (n,m)-semigroup generated by one element a, and let $m \geq 2$. For the elements of $F(a)$ we have defined their norm. In this example, $|a| = 1$, and $|\rho^i(x_1^t)| = \sum_{i=1}^t |x_i|$. Let $Q \subseteq F(a)$ be the set of all the elements whose norm is ≥ 4. Thus, $a \notin Q, \rho^i(aaa) \notin Q$. Let g, as an (n,m)-operation on Q, be the restriction of the (n,m)-operation f on $F(a)$. It follows directly from the definition that $\mathbf{Q} = (Q,g)$ satisfies *(Inj.1.)* We will show that it does not satisfy the condition *(Inj.2.)* Let $\mathbf{x} = \rho^1(\rho^1(aaa)\rho^2(aaa)a)\rho^2(aaaa)a$, and let $\mathbf{y} = \rho^1(aaaa)\rho^2(\rho^1(aaa)\rho^2(aaa)a)a$. Then $\mathbf{x} \neq \mathbf{y}$, and they are irreducible in $\mathbf{Q} = (Q,g)$, although they are reducible in $\mathbf{F} = (F(a), f)$. The definition of g implies that $g_i(\mathbf{x}) = f_i(\mathbf{x}) = \rho^i(aaaaa) = f_i(\mathbf{y}) = g_i(\mathbf{y}), i = 1,2$, i.e. $g(\mathbf{x}) = g(\mathbf{y})$. Hence, g is not an injection on the irreducible elements.

The next two examples are similar. The VVS in both of these examples are injective VVS, but they are not vector valued subsemigroups of a free VVS. In the first one, the set of prime elements is empty, while in the second one the set of prime elements is not empty, but it is not a generating set.

Example 4.3. Let $\mathbf{Q} = (Q, f)$ be a (3,2)-semigroup with the presentation

$$< a, b | f(aaa) = ba >,$$

i.e. a (3,2)-semigroup generated by two elements a and b with the relation $f(aaa) = ba$. We will give a short description of $\mathbf{Q} = (Q, f)$. Let $A_0 = \{a, b\}$, and let A_p be defined. We say that an $\mathbf{x} \in A_p^n, n \geq 3$, is not "good" if it has one of the following forms;

$$\mathbf{x}'aaa\mathbf{x}''; \qquad \mathbf{x}'\rho^1(\mathbf{u})\rho^2(\mathbf{u})\mathbf{x}''; \qquad \mathbf{x}'ab^k a\mathbf{x}'', k \geq 1,$$

where in $b^k = bb\ldots b$, the element b appears k times. Otherwise, we say that $\mathbf{x}$ is "good". Next we define:

$$A_{p+1} = A_p \cup \{\rho^i(\mathbf{x}) | \mathbf{x} \in A_p^n, n \geq 3, i = 1,2, \text{ and } \mathbf{x} \text{ is "good"}\}.$$

Let $Q = \cup_{s=0}^{\infty} A_s$. We have the same notion of "good" elements in $Q^{2,1}$. We define the norm $|x_1^n|$ of an $x_1^n \in Q^n$, for $n \geq 1$ as follows:

$$|a| = |b| = 1; \quad |x_1^n| = \sum_{i=1}^{n} |x_i|; \quad |\rho^j(\mathbf{x})| = |\mathbf{x}|.$$

We also define a **size** $[x_1^n]$ of an $x_1^n \in Q^n$, for $n \geq 1$ by:

$$[a] = 1; \quad [b] = 2; \quad [\rho^j(\mathbf{x})] = [\mathbf{x}]$$

and

$$[x_1^n] = \sum_{i=1}^{n} (|x_1^{i-1}| + 1) \cdot [x_i],$$

where by definition $|x_1^0| = 0$. Then, it can be checked by induction on the size, that $\mathbf{Q} = (Q, f)$ is a (3,2)-semigroup, where $f : Q^{2,1} \to Q^2$ is defined by:

(d.1) $f(aaa) = ba$;

(d.2) $f(\mathbf{x}'\rho^1(\mathbf{u})\rho^2(\mathbf{u})\mathbf{x}'') = f(\mathbf{x}'\mathbf{u}\mathbf{x}'')$, if not (d.1);

(d.3.0) $f(\mathbf{x}'aaa\mathbf{x}'') = f(\mathbf{x}'ba\mathbf{x}'')$, if not (d.1) and (d.2);

....

(d.3.k) $f(\mathbf{x}'ab^k a\mathbf{x}'') = f(\mathbf{x}'b^k aa\mathbf{x}'')$, for $k \geq 1$, if not (d.1), (d.2) and (d.3.j) for $0 \leq j < k$;

(d.4) $f(\mathbf{x}) = \rho^1(\mathbf{x})\rho^2(\mathbf{x})$, otherwise, i.e. for $\mathbf{x}$ "good".

It follows directly from the definition that $\mathbf{Q} = (Q, f)$ satisfies *(Inj.1.)*, and that the set of prime elements in (Q, f) is empty. A sequence $\mathbf{x} \in Q^{2,1}$ is irreducible if it is not of the form $\mathbf{x}'f(\mathbf{u})\mathbf{x}''$, which implies that $\mathbf{x} \neq \mathbf{x}'ba\mathbf{x}''$, since $ba = f(aaa)$. So, the only difference between the irreducible and "good" elements in $Q^{2,1}$ is that irreducible elements can have the form $\mathbf{x}'aaa\mathbf{x}''$. If $\mathbf{x}, \mathbf{y}$ are irreducible and $f(\mathbf{x}) = f(\mathbf{y})$, then both of them are "good", or both of them are not "good". If $\mathbf{x}, \mathbf{y}$ are "good", then $f(\mathbf{x}) = f(\mathbf{y})$ implies $\mathbf{x}=\mathbf{y}$. If $\mathbf{x}, \mathbf{y}$ are not "good" then: $\mathbf{x} = \mathbf{u}_1 a^k \mathbf{v}_1$ and $\mathbf{y} = \mathbf{u}_2 a^t \mathbf{v}_2$, for some $k, t \geq 3$, where $\mathbf{u}_i$ does not end on a, and $\mathbf{v}_i$ does not begin by a, for i=1,2, and these a^k and a^t are the first appearances of a^s in $\mathbf{x}$ and $\mathbf{y}$. Then, $f(\mathbf{x}) = f(\mathbf{u}_1 ba^{k-2}\mathbf{v}_1), f(\mathbf{y}) = f(\mathbf{u}_2 ba^{t-2}\mathbf{v}_2)$, and these will be the first appearances of ba in $f(\mathbf{x})$ and $f(\mathbf{y})$. All this implies that $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{v}_1 = \mathbf{v}_2$, and moreover that k=t. After finitely many such conclusions it will follow that $\mathbf{x}=\mathbf{y}$. Hence, $\mathbf{Q} = (Q, f)$ is an injective (3,2)-semigroup. The fact that $f(aaa) = ba$, implies not only that $\mathbf{Q} = (Q, f)$ is not a free (3,2)-semigroup, but also that $\mathbf{Q} = (Q, f)$ is not a (3,2)-subsemigroup of a free (3,2)-semigroup.

Example 4.4. Let (Q, f) be a (3,2)-semigroup with the presentation

$$< a, b, c | f(aaa) = ba >,$$

i.e. a (3,2)-semigroup generated by three elements a, b and c with the relation $f(aaa)$=ba. A similar discussion as in Example 4.3, implies that (Q, f) is an injective (3,2)-semigroup, but it is not a free (3,2)-semigroup and is not a (3,2)-subsemigroup of a free (3,2)-semigroup. The set P of prime elements in (Q, f) is not empty, but (Q, f) is not generated by P.

5. Commutative and fully commutative VVS

For a set Q, let $Q^{(t)}$ be its symmetric cartesian product, for $t \geq 1$, and let $\pi : Q^t \to Q^{(t)}$ be the projection map. Thus, $\pi(\mathbf{x}) = \pi(\mathbf{y})$ if and only if $\mathbf{y}$ is a permutation of $\mathbf{x}$. When there is no confusion we will denote $\pi(\mathbf{x})$ by $\mathbf{x}$. Let $Q^{(+)}$ be the union of all the symmetric cartesian products $Q^{(t)}$, for $t \in \mathbf{N}$. We denote by the same letter $\pi : Q^+ \to Q^{(+)}$ the projection map. This definition, and the fact that Q^+ is the free semigroup with the basis Q, implies that $Q^{(+)}$ is the free commutative semigroup with the basis Q.

Let $n = m + k$, be as in part 1. An (n, m)-semigroup (Q, f) is called *commutative*, (denoted by CVVS) if for each $\mathbf{x}, \mathbf{y} \in Q^n, \pi(\mathbf{x}) = \pi(\mathbf{y})$ implies $f(\mathbf{x}) = f(\mathbf{y})$. If (Q,f) is a commutative (n,m)-semigroup, then it can be described as a pair (Q,f), where $f : Q^{(n)} \to Q^m$, such that $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}{=}\mathbf{uvw} \in Q^{(n+k)}, \mathbf{y}, \mathbf{u} \in Q^{(n)}$. Using the same notion of length of $\mathbf{x}$ as in part 1, let $Q^{(m,k)} = \{\pi(\mathbf{x}) | \mathbf{x} \in Q^+, |\mathbf{x}| = m + sk\}$. A commutative (n,m)-semigroup can be thought of as a pair (Q,f), where $f : Q^{(m,k)} \to Q^m$, such that $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}{=}\mathbf{uvw} \in Q^{(m,k)}$ and $\mathbf{y}, \mathbf{u} \in Q^{(m,k)}$. All the notions from VVS are defined in the same way for CVVS, replacing Q^n and $Q^{m,k}$ by $Q^{(n)}$ and $Q^{(m,k)}$. The description of free CVVS is the same as the description of free VVS, using free commutative semigroups and symmetric cartesian product, i.e. using the free commutative algebra $\mathbf{F} = (F; \rho^1, \rho^2, \ldots, \rho^m)$ with the basis B, of type $\Omega = \{\rho_r^j | j \in \mathbf{N}_m, r \geq 1\}, F^{(+)}, F^{(m,k)}, F^{(m+rk)}, F(B)^{(+)}, F(B)^{(m,k)}$ and $F(B)^{(m+rk)}$, instead of their noncommutative analogs. The definition of injective CVVS is the same as the definition of injective VVS, and the same theorem holds.

Theorem 5.1. *A CVVS is free if and only if it is injective and is generated by the set of its prime elements.*

A fully commutative (n,m)-semigroup (denoted by FCVVS) is a pair $\mathbf{Q} = (Q,f)$ where $f : Q^{(n)} \to Q^{(m)}$, such that $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}{=}\mathbf{uvw} \in Q^{(n+k)}, \mathbf{y}, \mathbf{u} \in Q^{(n)}$. A fully commutative (n,m)-semigroup can be thought of as a pair (Q,f), where $f : Q^{(m,k)} \to Q^{(m)}$, such that $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}{=}\mathbf{uvw} \in Q^{(m,k)}$ and $\mathbf{y}, \mathbf{u} \in Q^{(m,k)}$.

The main difference of VVS and CVVS from FCVVS is the fact that in the last case there are no component operations, i.e. the fully commutative (n,m)-operation $f : Q^{(n)} \to Q^{(m)}$ can not be considered as m n-ary operations. On the other hand, there is a natural functor H from CVVS to FCVVS, and there are infinitely many functors K from FCVVS to CVVS, such that $H \circ K$ is the identity. The functor H is defined by:

If $\mathbf{Q} = (Q,f)$ is a commutative (n,m)-semigroup, then $H(\mathbf{Q}) = (Q,g)$ is the fully commutative (n,m)-semigroup defined by $g = \pi \circ f$, where $\pi : Q^m \to Q^{(m)}$ is the projection map.

For any map $\xi : Q^{(m)} \to Q^m$ such that $\pi \circ \xi = id$, and a fully commutative (n,m)-semigroup $\mathbf{Q} = (Q,f)$, $K(\mathbf{Q}) = (Q,g)$ is a commutative (n,m)-semigroup, defined by $g = \xi \circ f$.

The definitions of H and K imply that $H \circ K = id$.

The notions of generating sets, subsemigroups, and homomorphisms are defined as usual in the category of FCVVS. The free FCVVS with the basis B is a FCVVS $\mathbf{F}$ generated by B such that any map from B to a FCVVS $\mathbf{Q}$ extends to a FCVVS homomorphism form $\mathbf{F}$ to $\mathbf{Q}$. We note here that the previously mentioned extension in almost all the cases is not unique. Description of free FCVVS, using the description of free CVVS and the above mentioned functors is as follows.

Proposition 5.2. *If $\mathbf{F} = (F(B), f)$ is the free commutative (n,m)-semigroup with the basis B, then the fully commutative (n,m)-semigroup $H(\mathbf{F}) = (F(B), \pi \circ f)$, is the free fully commutative (n,m)-semigroup with the basis B.*

Since for FCVVS we do not have the component operations, we will state the notion for injective FCVVS in its language.

A FCVVS $\mathbf{Q} = (Q, f)$ is called *injective*, if:

(Finj.1.) For each $\mathbf{x}, \mathbf{y} \in Q^{(m)}$ in the image of f, $cn(\mathbf{x})$ consists of exactly m elements, and if $cn(\mathbf{x}) \cap cn(\mathbf{y}) \neq \emptyset$, then $cn(\mathbf{x}) = cn(\mathbf{y})$; and

(Finj.2.) If $\mathbf{x}, \mathbf{y} \in R(\mathbf{Q})$ and $f(\mathbf{x}) = f(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$, i.e. the restriction of f on $R(\mathbf{Q})$ is an injection.

It is easy to check the following:

Proposition 5.3. *Let $\mathbf{Q} = (Q, f)$ be a CVVS and let $H(\mathbf{Q}) = (Q, g)$ be its image under the functor H. Then:*

(a) If (Q, f) is injective CVVS, then (Q, g) is injective FCVVS;

(b) If P is the set of primes in (Q, f), then P is the set of primes in (Q, g).

The next proposition shows the converse.

Proposition 5.4. *Let $\mathbf{Q} = (Q, f)$ be a FCVVS, and let $\xi : Q^{(m)} \to Q^m$ be a map, such that $\pi \circ \xi = id$. Let $K(\mathbf{Q}) = (Q, g)$ be its image under the functor K for the map ξ. Then:*

(a) If (Q, g) is generated by B, then (Q, g) is generated by B;

(b) If P is the set of prime elements in (Q, f), then P is the set of prime elements in (Q, g);

(c) If (Q,f) is injective FCVVS, then (Q,g) is injective CVVS.

Proof. *(a)* If $a \in Q$ is such that $a \in cn(f(\mathbf{x}))$ for some $\mathbf{x} \in B^{(m,k)}$, then it follows that $a \in cn(g(\mathbf{x}))$. Further, an easy induction implies that B is a generating set for (Q,g).

(b) If $a \in Q$ is not prime in (Q,f), then $a \in cn(f(\mathbf{x}))$ for some $\mathbf{x} \in Q^{(m,k)}$, which implies that $a \in cn(g(\mathbf{x}))$, i.e. a is not prime in (Q,g). Conversely, if a is not prime in (Q,g), then $a \in cn(g(\mathbf{x}))$ for some $\mathbf{x} \in Q^{(m,k)}$, which implies that $a \in cn(f(\mathbf{x}))$, i.e. a is not prime in (Q,f).

(c) Let im $\mathrm{g}_j \cap$im $\mathrm{g}_i \neq \emptyset$. Then, there are $\mathbf{x}, \mathbf{y} \in Q^{(m,k)}$, such that $g_j(\mathbf{x}) = g_i(\mathbf{y})$. Let $\mathbf{u} = g(\mathbf{x})$ and $\mathbf{v} = g(\mathbf{y})$. Then $g_j(\mathbf{x}) = g_i(\mathbf{y})$ together with $\pi(\mathbf{u}) = \pi \circ \xi \circ f(\mathbf{x}) = f(\mathbf{x}) = \mathbf{a}$ and $\pi(\mathbf{v}) = \pi \circ \xi \circ f(\mathbf{y}) = f(\mathbf{y}) = \mathbf{b}$ and *(Finj.1.)* implies that $cn(\mathbf{a}) = cn(\mathbf{b})$. Further, since $cn(\mathbf{a})$ and $cn(\mathbf{b})$ have exactly m elements, by *(Finj.1.)*, it follows that $\mathbf{a} = \mathbf{b} = f(\mathbf{x}) = f(\mathbf{y}) = \pi(\mathbf{u}) = \pi(\mathbf{v})$ has exactly m elements. Also, $\mathbf{u} = g(\mathbf{x}) = \xi \circ f(\mathbf{x}) = \xi(\mathbf{a}) = \xi(\mathbf{b}) = \xi \circ f(\mathbf{y}) = g(\mathbf{y}) = \mathbf{v}$. If $i \neq j$, then $cn(\pi(\mathbf{u})) = cn(\pi(\mathbf{v}))$ would have less than m elements. Hence, i=j. Next, for $g_j(\mathbf{x}) = g_j(\mathbf{y})$, the same discussion as above shows that $g(\mathbf{x}) = g(\mathbf{y})$, i.e. (Q,g) satisfies Inj. 1.

If $\mathbf{x}, \mathbf{y} \in Q^{(m,k)}$ are irreducible for (Q,g), then they are also irreducible for (Q,f). So, if $g(\mathbf{x}) = g(\mathbf{y})$, then $\xi \circ f(\mathbf{x}) = \xi \circ f(\mathbf{y})$, and after composing with π, it follows that $f(\mathbf{x}) = \pi \circ \xi \circ f(\mathbf{x}) = \pi \circ \xi \circ f(\mathbf{y}) = f(\mathbf{y})$. Since (Q,f) is injective, it follows that $\mathbf{x}=\mathbf{y}$, implying that (Q,g) is injective CVVS. □

Now, Propositions 5.2, 5.3 and 5.4 imply the following

Theorem 5.5. *A FCVVS is free if and only if it is injective and is generated by the set of prime elements.*

References

[1] Čupona, Ǵ., Vector valued semigroups, Semigroup Forum 26 (1983), 65–74.

[2] Čupona, Ǵ., Celakoski, N., Markovski, S. and Dimovski, D., Vector valued groupoids, semigroups and groups, in: "Vector Valued Semigroups and Groups", pp. 1–79, Maced. Acad. Sci. and Arts, 1988.

[3] Čupona, Ǵ., Markovski, S., Dimovski, D. and Janeva, B., A combinatorial theory of vector valued semigroups; in: "Vector Valued Semigroups and Groups", pp. 141–183, Maced. Acad. Sci. and Arts, 1988.

[4] Dimovski, D., Free vector valued semigroups, in: Proc. 5th Conf. "Algebra & Logic", (Cetinje, 1986), pp. 55–62, University of Novi Sad, 1988.

[5] Trpenovski, B. and Čupona, Ǵ., $[m, n]$-groupoids, Bulletin Maced. Math. Phys. Soc. 21 (1970), 19–29.

Received December 31, 1998.