Прилози, Одд. мат. тех. науки, МАНУ, 17, 1–2, с. 5–17 (1996)
Contributions, Sec. Math. Tech. Sci., MANU, 17, 1–2, p. 5–17 (1996)
ISSN 0351–3246
UDC: 512.548.4

FREE GROUPOIDS WITH $(xy)^2 = x^2y^2$

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Abstract: We investigate free objects in the variety $\mathcal{V}$ of groupoids which satisfy the law $(xy)^2 = x^2y^2$ [1]. The main results and necessary preliminary definitions are stated in Introduction. Corresponding generalizations, concerning the law $(xy)^n = x^ny^n$, are considered in the last part of the paper.

0. Introduction

First we state some necessary preliminaries.

Let $\mathbf{G} = (G, \cdot)$ be a **groupoid**, i.e. an algebra with a binary operation: $(x, y) \mapsto xy$ on G. If $a, b, c \in G$ are such that $a = bc$, then we say that b and c are **divisors** of a in $\mathbf{G}$. A sequence $a_1, a_2, \ldots$ of elements of G is a **divisor chain** in $\mathbf{G}$ if a_{i+1} is a divisor of a_i. We say that $a \in G$ is a **prime** in $\mathbf{G}$ if the set of divisors of a in $\mathbf{G}$ is empty. Thus, primes in $\mathbf{G}$ can be only the last members of divisor chains in $\mathbf{G}$.

Throughout the paper we will always write "a free groupoid" instead of "a free groupoid in the variety of all groupoids". It will be denoted by $\mathbf{F} = (F, \cdot)$, and its basis by B. (We write $\mathbf{F} = F(B)$ when it is necessary to emphasize the basis B.) It is well known (see, for example, [1], I.1) that the following properties characterize $\mathbf{F}$:

a) $ab = cd \Rightarrow a = c, b = d$, i.e. the mapping $(a, b) \mapsto ab$ is injective.

b) Every divisor chain in $\mathbf{F}$ is finite.

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[1] As usual: $x^2 = xx$.

Then the set B of primes in $\mathbf{F}$ is nonempty and it is the unique basis of $\mathbf{F}$.

If $\mathbf{G} = (G, \cdot)$ is a given groupoid, then for any nonnegative integer k we define a transformation (k): $x \mapsto x^{(k)}$ of G in the following way:

$$x^{(0)} = x, \qquad x^{(k+1)} = x^{(k)}x^{(k)}. \tag{0.1}$$

From the condition a) we obtain that, in a free groupoid $\mathbf{F}$, (k) is injective, for any $k \geq 0$. Thus, for each $k \geq 0$, there exists an injective partial transformation $(-k)$: $x \mapsto x^{(-k)}$ defined in $\mathbf{F}$ as follows:

$$y^{(-k)} = x \Leftrightarrow y = x^{(k)}. \tag{0.2}$$

For any $u \in F$, there exists the largest integer $[u] = m$, such that $u^{(-m)} \in F$. (The integer $[u]$ will be called the **exponent** of u in F.)

The following subset R of F will play an important role in the paper. Namely, if B is the basis of $\mathbf{F}$, then we define R as the least subset of F such that $B \subseteq R$, and if $u = vw \in F \backslash B$, then:

$$u \in R \Leftrightarrow [v,\, w \in R \quad \text{and} \quad (v = w \text{ or } \min\{[v],\, [w]\} = 0)]. \tag{0.3}$$

Recall that we denoted by $\mathcal{V}$ the variety of groupoids which satisfy the law

$$(xy)^2 = x^2y^2. \tag{0.4}$$

If $\mathbf{G} \in \mathcal{V}$, then we call $\mathbf{G}$ a **$\mathcal{V}$-groupoid**, and if it is free in $\mathcal{V}$, we say that it is **$\mathcal{V}$-free**.

Now we are ready to state the main results.

Theorem 1. If $u,\, v \in R$, $m = \min\{[u],\, [v]\}$ and $u * v$ is defined by:

$$u * v = \left(u^{(-m)}v^{(-m)}\right)^{(m)}, \tag{0.5}$$

then $\mathbf{R} = (R, *)$ is a $\mathcal{V}$-free groupoid and the set B (i.e. the basis of F) is the unique basis of $\mathbf{R}$.

Theorem 2. A $\mathcal{V}$-groupoid $\mathbf{H} = (H, \cdot)$ is $\mathcal{V}$-free iff the following conditions hold

(i) Every divisor chain in $\mathbf{H}$ is finite.

(ii) $x^2 = y^2 \Rightarrow x = y$.

(iii) $xy = uv,\ x \neq y,\ u \neq v \Rightarrow x = u,\ y = v.$ [2)]

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[2] "$p, q, \ldots$" means "$p \& q \& \ldots$".

(iv) $x^2 = yz,\ y \neq z \Rightarrow (\exists u,\ v)\ (x = uv,\ y = u^2,\ z = v^2)$.

Then the set P of primes in $\mathbf{H}$ is nonempty and the unique basis of $\mathbf{H}$.

Theorem 3. If $\mathbf{H}$ is a $\mathcal{V}$–free groupoid, then there exist subgroupoids $\mathbf{G}$, $\mathbf{Q}$ of $\mathbf{H}$ such that $\mathbf{G}$ is not $\mathcal{V}$–free, and $\mathbf{Q}$ is $\mathcal{V}$–free with an infinite rank.

The next results concern a sequence of functors in the variety $\mathcal{V}$. Namely, if $\mathbf{G}$ is a groupoid and k is a nonnegative integer, then we define the groupoid $\mathbf{G}^{(k)} = (G,\ (k))$ as follows:

$$x\ (k)\ y = (xy)^{(k)}. \tag{0.6}$$

(Note that the same symbol (k) is used in (0.6) with two different meanings: as an operation of G on the left, and as a transformation on the right side.)

Theorem 4. If $\mathbf{H}$ is a $\mathcal{V}$–free groupoid and $k \geq 1$, then:

1) $\mathbf{H}^{(k)} \in \mathcal{V}$

2) $\mathbf{H}^{(k)}$ is not $\mathcal{V}$–free, and

3) The subgroupoid $\mathbf{Q}$ of $\mathbf{H}^{(k)}$ generated by the basis B of $\mathbf{H}$ is a $\mathcal{V}$–free groupoid with the basis B.

In §i, $1 \leq i \leq 4$, we prove *Th. i*, and in §5 we consider the law $(xy)^n = x^n y^n$, where $n \geq 3$.

1. *A canonical description of $\mathcal{V}$–free groupoids*

First we will introduce a norm of the elements of F and state some lemmas in order to prove *Th.* 1. (As was mentioned in Introduction, we denote by B the basis of a given free groupoid $\mathbf{F} = (F,\ \cdot)$.)

The **norm** in F is defined as the homomorphism $u \mapsto |u|$ from $\mathbf{F}$ into the additive groupoid of positive integers, which is an extension of the mapping $B \to \{1\}$. Thus:

$$|vw| = |v| + |w|, \qquad |b| = 1, \tag{1.1}$$

for all $v,\ w \in F,\ b \in B$.

In the proof of *Th.* 1 we will use some of the following relations, where $u, v \in F$ and k, m are integers.

$$u^{(k)} \in F \Leftrightarrow k + [u] \geq 0 \tag{1.2}$$

$$k + [u] \geq 0 \Rightarrow \left|u^{(k)}\right| = 2^k|u| \tag{1.3}$$

$$k+[u]\geq 0,\ k+m+[u]\geq 0\Rightarrow \left(u^{(k)}\right)^{(m)}=u^{(k+m)} \tag{1.4}$$

$$k+[u]\geq 0,\ m-k+[v]\geq 0\Rightarrow (u^{(k)}=v^{(m)}\Leftrightarrow u=v^{(m-k)} \tag{1.5}$$

$$k+[u]\geq 0\Rightarrow \left(u^{(k)}\in R\Leftrightarrow u\in R\right) \tag{1.6}$$

We will also use the following two lemmas, which can be also easily shown.

Lemma 1.1. If $\varphi\colon \mathbf{Q}\to\mathbf{G}$ is a homomorphism, then

$$x\in Q,\quad m\geq 0\Rightarrow \varphi\left(x^{(m)}\right)=\varphi(x)^{(m)}.\ \square$$

Lemma 1.2. If $\mathbf{G}\in\mathcal{V}$ and $x,\ y\in G$, $m\geq 0$, then

$$(xy)^{(m)}=x^{(m)}\,y^{(m)}.\ \square$$

Now we can prove *Theorem* 1.

First, if $u\in R$ is such that $[u]=m$, then by (1.4) we have

$$u*u=\left(u^{(-m)}\,u^{(-m)}\right)^{(m)}=\left(\left(u^{(-m)}\right)^{(1)}\right)^{(m)}=u^2. \tag{1.7}$$

Let $u,\ v\in R$ be such that $u\neq v$, and $\min\{[u],\ [v]\}=m$. Then $\left[u^{(-m)}\right]=0$ or $\left[v^{(-m)}\right]=0$, wich implies $u^{(-m)}\,v^{(-m)}\in R$, and by (1.6) we obtain $u*v\in R$. Thus:

$$u,\ v\in R\Rightarrow u*v\in R, \tag{1.8}$$

i.e. $\mathbf{R}=(R,\ *)$ is a groupoid.

Moreover, if $\min\{[u],\ [v]\}=m$, then:

$$\begin{aligned}(u*v)*(u*v)&=(u*v)^2=(u*v)^{(1)}=\left(u^{(-m)}\,v^{(-m)}\right)^{(m+1)},\\ (u*u)*(v*v)&=u^2*v^2=\left((u^2)^{(-m-1)}\,(v^2)^{(-m-1)}\right)^{(m+1)}\\ &=\left(u^{(-m)}\,v^{(-m)}\right)^{(m+1)},\end{aligned}$$

and this implies that $\mathbf{R}\in\mathcal{V}$.

If $u,\ v\in R$ are such that $uv\in R$, then $u=v$ or $\min\{[u],\ [v]\}=0$, and thus we have:

$$u,\ v,\ uv\in R\Rightarrow u*v=uv, \tag{1.9}$$

and so B is a generating set of R. Clearly, B is the set of primes in $\mathbf{R}$.