According to *L.* 2.2, we have:

Proposition 3.3. If $a \in Q$, then there exists a positive integer $L_Q(a)$ such that $L_Q(a)$ is the largest length of divisor chains in $\mathbf{Q}$ with the first member a. Moreover: $L_Q(a) \leq L(a)$. □

We will show:

Proposition 3.4. If $b \in B$ and $\mathbf{Q}$ is generated by b^2, b^2b, then $\mathbf{Q}$ is not $\mathcal{V}$–free.

Proof. Clearly: b^2, $b^2b \in Q$, $(b^2)^2b^2 = (b^2b)^2 \in Q$ and $b^2 \neq b^2b$, but there is no $v \in Q$ such that $b^2 = v^2$. Thus $\mathbf{Q}$ does not satisfy (iv), i.e. $\mathbf{Q}$ is not $\mathcal{V}$–free. ($\{b^2,\ b^2b\}$ is the set of primes in $\mathbf{Q}$.) □

By *Th.* 2 and *Pr.* 3.1, we have:

Proposition 3.5. $\mathbf{Q}$ is $\mathcal{V}$–free if:

$$u \neq v,\ uv \in Q\ u^2, v^2 \in Q \Rightarrow u, v \in Q\,, \tag{3.1}$$

for any u, $v \in H$. □

As a concequence of *Pr.* 3.5 we have:

Propositon 3.6. Each of the following conditions is sufficient for $\mathbf{Q}$ to be $\mathcal{V}$–free:

$$x^2 \in Q \Rightarrow x \in Q\,, \tag{3.2}$$

$$u \neq v,\ uv \in Q \Rightarrow u,\ v \in Q\,. \ \square \tag{3.3}$$

The following property will help to complete the proof of *Th.* 3.

Propositon 3.7. If $[u] = 0$ for every prime in $\mathbf{Q}$, then $\mathbf{Q}$ is $\mathcal{V}$–free.

Proof. It is enough to show that $\mathbf{Q}$ satisfies the condition (3.2) and this can be shown by induction on $L_Q(x^2)$. □

Propositon 3.8. Let $b \in B$, $a_1 = b^2b$, $a_{k+1} = a_kb$, $A = \{a_k \mid k \geq 1\}$. If $\mathbf{Q}$ is generated by A, then $\mathbf{Q}$ is $\mathcal{V}$–free with an infinite rank.

Proof. All the elements of A are primes in $\mathbf{Q}$, and then apply *Pr.* 3.7. □

Propositon 3.9. Let $C = \{b^2\} \cup A$, where A is as in *Pr.* 3.8. If $\mathbf{S}$ is the groupoid generated by C, then $\mathbf{S}$ is not $\mathcal{V}$–free, and all the elements of C are primes in S. □

4. *Some properties of the functors* (k) *in* $\mathcal{V}$

The proofs of the following three statements are obvious.

Propositon 4.1. $\mathbf{G} \in \mathcal{V} \Rightarrow G^{(k)} \in \mathcal{V}$. □

Propositon 4.2. If $\mathbf{G} = (G, \cdot)$, $\mathbf{S} = (S, \cdot) \in \mathcal{V}$ and $\varphi: G \to S$ is a homomorphism from $\mathbf{G}$ into $\mathbf{S}$, then $\varphi: \mathbf{G}^{(k)} \to \mathbf{S}^{(k)}$ is a homomorphism from $\mathbf{G}^{(k)}$ into $\mathbf{S}^{(k)}$ as well. □

Thus for every $k \geq 0$, (k) is a functor in $\mathcal{V}$.

Propositon 4.3. If $k, n \geq 0$ and $\mathbf{G} \in \mathcal{V}$, then $\left(\mathbf{G}^{(k)}\right)^{(n)} = \mathbf{G}^{(kn+n)}$. □

Below we assume that $\mathbf{H}$ is a $\mathcal{V}$–free groupoid, with the basis B, and that k is a positive integer. The subgroupoid of $\mathbf{H}^{(k)}$ generated by B will be denoted by $\mathbf{Q}$. Also (i), (ii), (iii) and (iv) are the conditions stated in *Th.* 2.

The following statements 4.4–4.6 are obvious or they can be easily shown.

Propositon 4.4. If $x, y, u, v \in H$, then:

$$x(k)y = u(k)v \Leftrightarrow xy = uv\,. \quad \square$$

Propositon 4.5. If $k \geq 1$, then B is a proper subset of the set P of primes in $\mathbf{H}^{(k)}$.

(Each element $b \in B$ is prime in $\mathbf{H}^{(k)}$, and for every $u \in H$, $b \in B$, we have $ub \in P$, $ub \notin B$.) □

Propositon 4.6. $\mathbf{H}^{(k)}$ satisfies (i), (ii) and (iii) of *Th.* 2, but for $k \geq 1$, $\mathbf{H}^{(k)}$ is not $\mathcal{V}$–free.

(Namely, if $b \in B$, then: $b^2b\,(k)\,b^2b = (b^2)^2\,(k)\,b^2$, $(b^2)^2 \neq b^2$, but b^2 is a prime in $\mathbf{H}^{(k)}$, for $k \geq 1$. Thus $\mathbf{H}^{(k)}$ does not satisfy (iv).) □

In order to complete the proof of *Th.* 4, first we will show the following

Lemma 4.7. If $x, y, z \in Q$, $y \neq z$ and $x^2 = yz$, then there exist $\gamma, \delta \in Q$ such that $\gamma \neq \delta$ and $x = (\gamma\delta)^{(k)}$.

Proof. The equality $x^2 = yz$ implies that $[yz] \geq 1$, and (by (2.5)) we have $[y], [z] \geq 1$ and $x = (yz)^{(-1)} = y^{(-1)}z^{(-1)}$. Thus: $x \in Q \backslash B$, and so there exist $\gamma, \delta \in Q$ such that $x = \gamma(k)\delta = (\gamma\delta)^{(k)}$. It remains to show that there exist different γ, δ with the above property.

Suppose that different γ, δ with the mentioned property do not exist and put $x = \alpha_0$. Then there exists a (unique) $\alpha_1 \in Q$ such that $\alpha_0 = \alpha_1^{(k+1)}$. Let the sequence $\alpha_0, \alpha_1, \ldots, \alpha_i, \ldots$ be such that $\alpha_i = \alpha_{i+1}^{(k+1)}$, for every i. Since $|\alpha_i| > |\alpha_{i+1}|$, the sequence is finite. Let α_n be its last member. By the definition of the sequence we have:

$$\alpha_0 = y^{(-1)}\,z^{(-1)}, \qquad \alpha_1 = y^{(-k-2)}\,z^{(-k-2)},$$
$$\alpha_2 = y^{(-2k-3)}\,z^{(-2k-3)}, \ldots, \quad \alpha_n = y^{(-nk-n-1)}\,z^{(-nk-n-1)}.$$

By the last equlity, there exist u, $v \in Q$ such that $\alpha_n = u(k)v = (uv)^{(k)}$. Clearly, $u \neq v$ (since α_n is the last member of the sequence). From the equality $(uv)^{(k)} = y^{(-nk-n-1)}\,z^{(-nk-n-1)}$ we have:

$$u = y^{(-nk-n-k-1)}, \qquad v = z^{(-nk-n-k-1)}, \qquad \text{i.e.}$$
$$y = u^{(nk+n+k+1)}, \qquad z = v^{(nk+n+k+1)}.$$

Therefore:

$$x = \alpha_0 = y^{(-1)}\,z^{(-1)} = u^{(nk+n+k)}\,v^{(nk+n+k)} =$$
$$= \left(u^{(nk+n)}\,v^{(nk+n)}\right)^{(k)} = (\gamma\delta)^{(k)},$$

where $\gamma = u^{(nk+n)} \neq v^{(nk+n)} = \delta$. □

Propositon 4.8. **Q** is $\mathcal{V}$–free.

Proof. Let x, y, $z \in Q$ be such that $x^2 = uz$, $y \neq z$. By *L.* 4.7, there exist γ, $\delta \in Q$ such that $\gamma \neq \delta$, $x = (\gamma\delta)^{(k)}$. By $x^2 = yz$ it follows that $\gamma^{(k+1)}\,\delta^{(k+1)} = yz$, i.e $y = \gamma^{(k+1)}$, $z = \delta^{(k+1)}$. Therefore: $x = \gamma(k)\delta$, $y = \gamma(k)\gamma$, $z = \delta(k)\delta$. Thus **Q** satisfies the condition (iv), and from *Pr.* 4.6 it follows that **Q** satisfies (i), (ii), (iii) as well. □

5. *On the equation* $(xy)^n = x^n y^n$

Denote by $\mathcal{V}_n$ the variety of groupoids which satisfy the identity

$$(xy)^n = x^n y^n, \tag{5.1}$$

where n is a given positive integer. Here the powers are defined in the usual way, i.e. by:

$$x^1 = x, \qquad x^{k+1} = x^k x. \tag{5.2}$$

By the above considerations, $\mathcal{V}_1$ is the variety of all groupoids, and $\mathcal{V}_2$ the variety $\mathcal{V}$. Further on we assume that n is a fixed integer and $n \geq 2$.

Define $x^{(k)}$, for $k \geq 0$, by:

$$x^{(0)} = x\,, \qquad x^{(k+1)} = \left(x^{(k)}\right)^n . \tag{5.3}$$

Note that (5.3), for $n = 2$, coincides with (0.1).

As before, $\mathbf{F} = (F, \cdot)$ denotes a free groupoid with the basis B. Since the implication

$$x^k = y^m \;\Rightarrow\; x = y\,, \quad k = m \tag{5.4}$$

is true in $\mathbf{F}$, the mapping $x \mapsto x^{(m)}$ is an injective transformation of F. Thus we can define $x^{(-k)}$ and $[x]$, as in the special case $n = 2$.

It is easy to show that (1.2)–(1.6), *L*. 1.1 and *L*. 1.2 are true for any $n \geq 2$.

Now we will define F_n as the least subset of F such that $B \subseteq F_n$ and: $vw \in F_n \Leftrightarrow$

$$\left[\left(w \in F_n,\ v = w^{n-1}\right) \text{ or } (v,\, w \in F_n \text{ and } \min\{[v], [w]\} = 0)\right] . \tag{5.5}$$

Therefore, $F_2 = R$ where R is defined by (0.3). Note that the implication $vw \in F_n \Rightarrow v,\, w \in F_n$, for $n \geq 3$, is not true. (For example, if $b \in B$ and $n = 3$, then $b^{(2)} = \left(b^3\right)^2 \cdot b^3 \in F_3$, but $\left(b^3\right)^2 \notin F_3$.)

The following statement is a generalization of *Th*. 1.

Theorem 1′. $\mathbf{F}_n = (F_n, *)$ is a $\mathcal{V}_n$–free groupoid with the unique basis B. Here:

$$u * v = \left(u^{(-m)}\, v^{(-m)}\right)^{(m)} ,$$

where $u,\, v \in F_n$ and $m = \min\{[u], [v]\}$. □.

This generalization is obtained by substituing R by F_n. The situation with the other theorems is similar, except with *Th*. 4. Namely, the definition of the operation (k), given by (0.6), does make sense for $n \geq 3$ also, but it is easy to show that $\mathbf{F}_n^{(k)} \notin \mathcal{V}_n$, for $n \geq 3$.

The statements (ii), (iii) and (iv) of *Th*. 2, in the formulation of *Th*. 2′ (besides the substitution of $\mathcal{V}$ by $\mathcal{V}_n$), obtain the following forms:

(ii′) $x^n = y^n \Rightarrow x = y$.

(iii′) $xy = uv,\quad x \neq y^{n-1},\quad u \neq v^{n-1} \Rightarrow x = u,\ \ y = v$.

(iv′) $x^n = yz,\ \ y \neq z^{n-1} \Rightarrow (\exists u,\, v)x = uv,\ \ y = u^n,\ \ z = v^n$.

According to *Th*. 3′, note that if $\mathbf{H}$ is a $\mathcal{V}_n$–free groupoid and if $\mathbf{Q}$ is the subgroupoid of $\mathbf{H}$ generated by $A = \{a_p | p \geq 1\}$, where $a_p = b^{n+p}$ (b is an element of the basis B), then A is the basis of $\mathbf{Q}$. Therefore, $\mathbf{Q}$ has an infinite rank. If $\mathbf{S}_p$

is generated by $\{b^n, a_p\}$, then $\mathbf{S}_p$ is not $\mathcal{V}_n$–free, and $\{b_n, a_p\}$ is the set of primes in $\mathbf{S}_p$. The groupoid $\mathbf{S}$ generated by $C = \{b^n\} \cup A$ is not $\mathcal{V}_n$–free, and C is the set of primes in $\mathbf{S}$.

For a given positive integer n, there are $(2n-2)!/n!(n-1)! = A_n$ different possibilities of defining n–th powers, i.e. transformations $x \mapsto x^n$ in groupoids (see, for example [2], III.2, or [7], I.4). Therefore, there exist A_n varieties of groupoids each of which is defined by an equality of the form (5.1).

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Резиме

СЛОБОДНИ ГРУПОИДИ СО РАВЕНСТВО $(xy)^2 = x^2y^2$

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Се разгледуваат слободните објекти во многуобразието групоиди што го задоволуваат равенството $(xy)^2 = x^2y^2$. Главните резултати се формулираат во воведот. Во наредните четири раздели се испитуваат својствата на слободните објекти и се даваат докази на резултатите формулирани во воведот. Во последниот раздел се разгледува поопштиот случај на групоиди што го задоволуваат равенството $(xy)^n = x^ny^x$.

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