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# COMPLEX COMMUTATIVE VECTOR VALUED GROUPS

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## INTRODUCTION

The notion of vector valued groupoids, semigroups and groups, were introduced in [24] and [1]. Several mathematicians from Skopje, Novi Sad, and Niš, have developed this theory and obtained interesting results. One of the main results was the description of the free vector valued semigroups and groups. From the known results for the vector valued structures, it follows that these structures on the one hand are similar to the usual algebraic binary structures, but on the other they do have new ideas and specific properties. As a part of this theory is the theory of the fully commutative vector valued algebraic structures and this monograph is mainly concerned with these structures.

We will give a description of this monograph, by giving a brief description of each chapter and each section. In this description we will state almost all the definitions and results from the main text.

In the first chapter at the begining we give the basic definitions and known facts about the vector valued groupoids, semigroups and groups, and then several general definitions and properties of commutative vector valued groupoids semigroups and groups are developed.

For a positive integer $p$, $Q^p$ denotes the $p$-th Cartesian power of $Q$. We use the notation $x=a_1^p$ instead of $x=(a_1, a_2, \dots, a_p)$ for elements $x \in Q^p$. Let $n,m,k$ be positive integers and let $n = m + k$.

**Definition 0.1.** A map $f: Q^n \to Q^m$ is called an **(n,m)-operation**, and the pair $(Q,f)$ is called an **(n,m)-groupoid.** An $(n,m)$-groupoid $(Q,f)$ is called an **(n,m)-semigroup** if the operation $f$ is associative, i.e. for every $1 \le i \le k$ and every $x_1^{n+k} \in Q^{n+k}$,

$$f(x_1^i f(x_{i+1}^{i+n}) x_{i+n+1}^{n+k}) = f(f(x_1^n) x_{n+1}^{n+k}). \qquad (0.1)$$

An (n,m)-semigroup is called an **(n,m)-group** if for each $a\in Q^k$, $b\in Q^m$, the equations

$$f(ax)=b=f(ya) \qquad (0.2)$$

have solutions $x,y\in Q^m$.

Thus, the notions of (n,1)-groupoids, (n,1)-semigroups and (n,1)-groups, are the same as the notions of n-groupoids, n-semigroups and n-groups, i.e. are the same as the notions of groupoids, semigroups and groups, in the case n=2.

There are several definitions for the commutative analogs of the above notions. We need the following definitions and conventions.

Let $Q^{(m)}$ be the m-th permutation product of Q, i.e. let $Q^{(m)} = Q^m/\approx$ where $\approx$ is the equivalence on $Q^m$ defined by

$$x_1^m \approx y_1^m \Leftrightarrow x_1, x_2, \dots, x_m \text{ is a permutation of } y_1, y_2, \dots, y_m. \qquad (0.3)$$

If (Q,f) is an (n,m)-group such that $f(x_1^n) = f(y_1^n)$ for each $x_1^n \approx y_1^n$ from $Q^n$, i.e. the map f factors through $Q^{(n)}$, then we say that (Q,f) is totally commutative (n,m)-groupoid. It is shown in [6] that if (Q,f) is totally commutative (n,m)-group, then $|Q|=1$, i.e. Q is one element set.

Another definition of commutative (2m,m)-groupoids and (2m,m)-grous is examined in [2].

**Definition 0.2.** A map $f:Q^{(n)}\to Q^{(m)}$ is called a **commutative (n,m)-operation** on Q, and the pair (Q,f) is called a **commutative (n,m)-groupoid**. A commutative (n,m)-groupoid is called a **commutative (n,m)-semigroup**, if the operation f is associative, i.e. for each $1\le i\le k$ and each $x_1^{n+k}\in Q^{(n+k)}$,

$$f(x_1^i f(x_{i+1}^{i+n}) x_{i+n+1}^{n+k}) = f(f(x_1^n) x_{n+1}^{n+k}) . \qquad (0.4)$$

A commutative (n,m)-semigroup (Q,f) is called a **commutative (n,m)-group** if for each $a\in Q^{(k)}$, $b\in Q^{(m)}$, the equation

$$f(ax)=b \qquad (0.5)$$

has solution $x\in Q^{(m)}$.

For shorter notations, we will write "com(n,m)-..." instead of "commutative (n,m)-....".

The following Theorem, shown in [5] is a generalization of the associative law for semigroups.

**Theorem 0.1.** Let $(Q,f)$ be a com(n,m)-semigroup. For $s\geq 1$, let $f^{(s)}:Q^{(m+s+k)}\to Q^{(m)}$ be defined inductively by:

$$f^{(1)}=f;\quad f^{(s+1)}(xy)=f(f^{(s)}(x),y);\ x\in Q^{(m+sk)},\ y\in Q^{(k)}. \tag{0.6}$$

Then:

(i) For each $s\geq 1$, $f^{(s)}$ is well defined;

(ii) For each $s\geq 1$, $(Q,f^{(s)})$ is a com(m+sk,m)-semigroup; and

(iii) For each $s,t\geq 1$, $x\in Q^{(m+sk)}$, $y\in Q^{(tk)}$,

$$f^{(t)}(f^{(s)}(x)\ y) = f^{(s+t)}(xy).\ \blacksquare$$

In the second section of chapter one, we recall some results from [5], about the relations between com(n,m)-groups and usual commutative groups.

Let $(Q,f)$ be a given com(n,m)-semigroup, $n-m=k\geq 1$, and let p be the least non-negative integer, such that $m+p\equiv 0 \pmod k$, and let s be the least non-negative integer such that $k(s-1) < m \leq ks$ and $m+p=ks$.

**Definition 0.3.** Define a binary operation $*$ on $Q^{(m)}$ by:

$$a*b=f(acb) \tag{0.7}$$

where $c\in Q^{(p)}$ for $p\geq 1$; and c is the empty symbol for p=0, i.e.

$$a*b=f(ab). \tag{0.8}$$

for p=0.

It is shown that $*$ is well defined and that $(Q^{(m)},*)$ is a semigroup. In the case $p\geq 1$, the operation $*$ depends on c.

We say that $(Q^{(m)},*)$ is a **derived semigroup for** $(Q,f)$.

It is obvious that $(Q^{(m)},*)$ is unique for $p=0$, and by an example it is shown , that for $p\geq 1$ it is possible to have two derived semigroups for one com(n,m)-semigroup which are **not even** isomorphic. Proposition I. 2.1 shows that for com(n,m)-groups, $(Q^{(m)},*)$ is unique up to isomorphism, and is called the **derived group** for $(Q,f)$. One of the characterizations of com(n,m)-groups is:

**Proposition 0.2.** A com(n, m)-semigroup (Q, f) is a com(n, m)-group iff its derived semigroup $(Q^{(m)}, *)$ is a commutative group. ■

In section I.3, we give a description of com(n, m)-semigroups and com(n, m)-groups via transformations on $Q^{(m)}$. Let (Q, f) be a com(n, m)-groupoid, $n-m=k\geq 1$, $m+p=ks$, and $k(s-1)<m\leq ks$, $p\geq 0$.

**Definition 0.4.** Let $Hom(Q^{(m)},Q^{(m)})=\{\varphi\,|\,\varphi:Q^{(m)}\to Q^{(m)}\}$ be the semigroup of all the maps from $Q^{(m)}$ to $Q^{(m)}$, i.e. transformations on $Q^{(m)}$, with the operation composition of maps. For each $x\in Q^{(k)}$ we define $\varphi_x:Q^{(m)}\to Q^{(m)}$ by:

$$\varphi_x(a)=f(ax). \tag{0.9}$$

Let $H'(Q,f)=\{\varphi_x\,|\, x\in Q^{(k)}\}$, and let $H(Q,f)=\langle H'(Q,f)\rangle$ be the subsemigroup of $Hom(Q^{(m)},Q^{(m)})$, generated by $H'(Q,f)$.

**Proposition 0.3.** A com(n, m)-groupoid (Q, f) is a com(n, m)-semigroup iff H(Q, f) is commutative. ■

**Theorem 0.4.** A com(n, m)-groupoid (Q, f) is a com(n, m)-group iff H(Q, f) is a commutative group. ■

**Proposition 0.5.** A com(n, m)-semigroup (Q, f) is a com(n, m)-group, iff for each $x\in Q^{(k)}$, $\varphi_x$ is a bijection. ■

**Proposition 0.6.** Let (Q, f) be a com(m+k, m)-group. Then, the derived group $(Q^{(m)}, *)$ is isomorphic to the group H(Q, f). ■

Next we discuss the question when a com(m+sk, m)-group is induced by a com(m+k, m)-group, and later give a way of obtaining com(n, m)-groups from com(n, m)-semigroups.

**Definition 0.5.** We say that an element $a\in Q$ is **singular** if the map $\varphi_b$, denoted by $\varphi(a)$ is not a bijection, where $b=a^k$. Let $\mathcal{R}$ be the set of all the singular elements in Q.

**Theorem 0.7.** Let (Q, f) be a com(n, m)-semigroup and $Q\setminus\mathcal{R}\neq\emptyset$. Then $(Q\setminus\mathcal{R}, g)$ is a com(n, m)-group, where $g(x)=f(x)$. ■

In section I.4 we generalize the construction of the examples given in [23]. Let $(Q,+,\cdot)$ be a commutative ring with a multiplicative unit 1.

**Definition 0.6.** Let $\{\wp^t\,|\, t\geq 0\}$ be a sequence of maps $\wp^t:Q^{(+)}\to Q$ which satisfies the following conditions:

(i) $\varphi^0(x) = 1$;

(ii) $\varphi^r(x) = 0$ for $x \in Q^{(t)}$ and $t < r$ ;

(iii) $\varphi^r(xy) = \sum_{0\le s\le r} \varphi^{r-s}(x)\varphi^s(y)$ ;

where the sum and the product are from the ring $(Q,+,\cdot)$.

The maps $\varphi^t$ are called **symmetric functions.**

For given symmetric functions $\varphi^t$ let $\varphi_r : Q^{(r)} \to Q^r$ be defined by

$$\varphi_r(a^r) = (\varphi^1(a^r),\ \varphi^2(a^r),\ \dots,\ \varphi^{r-1}(a^r),\ \varphi^r(a^r)).$$

**Proposition 0.8.** Let $\varphi^t$ be symmetric functions on $(Q,+,\cdot)$ such that $\varphi^1(x)=x$ for all $x \in Q$. Then, for all $s$ and $1 \le t \le s$,

$$\varphi^t(a_1^s) = \sum_{\{i_1,\dots,i_t\}\in\mathcal{P}_t(M_s)} a_{i_1}\cdot a_{i_2}\cdot \ldots \cdot a_{i_t} . \qquad (0.10)$$

where $\mathcal{P}_t(A) = \{X \mid X \subseteq A,\ |A| = t\}$.

**Definition 0.7.** If there is a map $f : Q^{(n)} \to Q^{(m)}$ such that the diagram

$$\begin{array}{ccc} Q^n & \xrightarrow{\ \ g\ \ } & Q^m \\ \uparrow{\scriptstyle \varphi_n} & & \uparrow{\scriptstyle \varphi_m} \\ Q^{(n)} & \xrightarrow{\ \ f\ \ } & Q^{(m)} \end{array}$$

is commutative, we say that the com(n,m)-groupoid $(Q,f)$ is **obtained** from the (n,m)-groupoid $(Q,g)$ via the symmetric functions $\varphi^t$.

In the case when $\varphi_m$ is a bijection, such an f always exists, i.e. we can take $f = \varphi_m^{-1} \circ g \circ \varphi_n$. In such a case, for each $p \in Q^k$, we will denote the transformation $\varphi_m \circ \varphi_p \circ \varphi_m^{-1} : Q^m \to Q^m$ by $\bar{\varphi}_p$.

**Proposition 0.9.** Let $(Q,+,\cdot)$ be an algebraically closed field, and let $\varphi^t$ be symmetric functions on $Q$ satisfying the condition (0.10). Then, for every $q \ge 1$, $\varphi_q : Q^{(q)} \to Q^q$ is a bijection. Moreover,

$$\varphi_q(a_1^q) = b_1^q \ \text{ iff } \ (x+a_1)\dots(x+a_q) = b_1 + b_2x + \dots + b_qx^{q-1} + x^q, \qquad (0.11)$$

for each $x \in Q$. ∎

In the case when $(Q,f)$ is a com(n,m)-groupoid obtained from an (n,m)-groupoid $(Q,g)$ via symmetric functions, it is possible from some known facts about $(Q,g)$ to get some information about $(Q,f)$. The most complicated condition to be checked almost always is the associativity. In general, almost nothing can be said about the associativity of $(Q,f)$

even if we have plenty of information about $(Q,g)$, but there are cases when certain conditions on $(Q,g)$ imply the associativity of $(Q,f)$. Next, we will set up this situation.

Let $(Q,+,\cdot)$ be an algebraically closed field, and let $g:Q^n \to Q^m$ be defined by:

$$g(x) = A\cdot x + b, \tag{0.12}$$

where $A = [\alpha_{ij}]_{m\times n}$ is an $n\times m$-matrix over $Q$, $b=[\alpha_{i0}]$ is an $m\times 1$-matrix over $Q$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, and where we identify the elements (i.e. the vectors) from $Q^t$ with vector columns, i.e. with $t\times 1$-matrices. Using the component representation $(g_1,\ldots g_m)$ of $g$, where for each $i$, $g_i:Q^n\to Q$ is the i-th component of $g$, the definition (0.12) can be restated as:

$$g_i(a_1^n) = \alpha_{i0} + \sum_{1\le r\le n} \alpha_{ir}\cdot a_r \tag{0.13}$$

where $\alpha_{ij}, a_r \in Q$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, $r=1,2,\ldots,n$, and the multiplication and the addition are in $(Q,+,\cdot)$.

**Theorem 0.10.** Let $(Q,g)$ be as above, and let $(Q,f)$ be a com(n,m)-groupoid obtained from $(Q,g)$ via the symmetric functions of type (0.10). Then, $(Q,f)$ is a com(n,m)-semigroup iff for each $i\in\{1,2,\ldots,m\}$, $s\in\{1,2,\ldots,k\}$ and $r\in\{1,2,\ldots,m+k\}$,

$$\sum_{0\le j\le m} \alpha_{i(s+j)}(\alpha_{j(r-1)} - \alpha_{(j+1)r}) = 0 \tag{0.14}$$

where we use the following conventions:

$$\alpha_{00}=1, \quad \alpha_{0i} = \alpha_{si} = 0 \text{ for each } 1\le i\le m+k \text{ and } s>m. \tag{0.15}$$

With the convention as in (0.15), for each $1\le s\le k$, let $A_s=[\alpha_{i(s+j)}]_{m\times(m+1)}$, let $A_0=[\alpha_{j(q-1)}]_{(m+1)\times n}$ and let $A_{m+1}=[\alpha_{(j+1)q}]_{(m+1)\times n}$, where $1\le i\le m$, $0\le j\le m$, and $1\le q\le n$. Then the condition (0.14) can be stated as

$$A_s(A_0-A_{m+1})=0. \quad ■$$

In section I.5 we give several examples of com(n,m)-semigroups and com(n,m)-groups, taken from [23], where the above mentioned construction is used for the algebraically closed field $\mathbb{C}$.

The previous known nontrivial examples of commutative vector valued structures were defined on two element sets and on the sets of complex numbers $\mathbb{C}$, where the property of $\mathbb{C}$ being algebraically closed was crucial. These examples led us to the notion of affine commutative semigroups and groups which are examined in detail in chapter two, section 1.

Let $(\mathbb{C},f)$ be a com(m+k,m)-groupoid over $\mathbb{C}$ obtained from an (m+k,m)-groupoid $(\mathbb{C},g)$ via the symmetric functions I (4.1), as in I.(4.5).

**Definition 0.8.** A com(m+k,m)-groupoid $(\mathbb{C},f)$ as above is said to be an **affine** com(m+k,m)-groupoid over $\mathbb{C}$.

Next, we introduce the notions of characteristic vectors and characteristic polynomials, which are used for detailed examination of affine com(n,m)-semigroups.

In the same chapter, section 2, using the results of the previous section, we prove Theorems 2.1 and 2.2. As an important consequence of Theorem 2.2 we will prove Theorem 2.4 about the affine semigroups and groups. Namely, for a given affine com(m+k,m)-semigroup (or group) with non-zero characteristic polynomial, we will be able to find (the number of) those affine com(m+1,m)-semigroups (groups) which induce the given semigroup (group). It is interesting that practically it is impossible to prove Theorems 2.1 and 2.2 without using the results of II.1, but on the other hand these two theorems do not say anything about the affine com(n,m)-semigroups and groups, alone.

Let us denote by $A(m;\mathbb{C})$ the set of all the affine complex transformations on $\mathbb{C}^m$, i.e. of all the non-singular complex matrices of the form

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix} \quad (0.16)$$

**Definition 0.9.** Let a non-zero vector $h=h_o^m\in\mathbb{C}^{m+1}$ be given. We denote by $G_h$ (or simply by G) the subset of all matrices of $A(m;\mathbb{C})$ such that the following m vectors

$$\begin{array}{l} (1-\alpha_{11},\ \alpha_{10}-\alpha_{21},\ \ldots,\alpha_{(m-1)0}-\alpha_{m1},\ \alpha_{m0}) \\ (-\alpha_{12},\ \alpha_{11}-\alpha_{22},\ \ldots,\alpha_{(m-1)1}-\alpha_{m2},\ \alpha_{m1}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (-\alpha_{1m},\ \alpha_{1(m-1)}-\alpha_{2m},\ldots,\alpha_{(m-1)(m-1)}-\alpha_{mm},\alpha_{m(m-1)}) \end{array} \quad (0.17)$$

are collinear with the vector h.

**Theorem 0.11.** Let a non-zero vector $h=h_0^m \in \mathbb{C}^{m+1}$ be given. The set of matrices $G_h$ with the matrix multiplication is a closed commutative Lie subgroup of $A(m;\mathbb{C})$ with complex dimension m. ■

**Theorem 0.12.** Let $h=h_0^m \in \mathbb{C}^{m+1}$ be a given non-zero vector and assume that the polynomial

$$P(t) = h_m - h_{m-1}t + h_{m-2}t^2 - \dots +(-1)^m h_0 t^m \tag{0.18}$$

has exactly s different roots. Then the corresponding group $G_h$ satisfies

$$G_h \cong \underbrace{C_1 \times C_1 \times \dots \times C_1}_{s} \times \underbrace{C_0 \times C_0 \times \dots C_0}_{m-s} . \tag{0.19}$$

where $C_1 = \mathbb{C}\setminus\{0\}$ with the operation multiplication of complex numbers, and $\mathbb{C}_0 = \mathbb{C}$ with the operation addition of complex numbers. ■

**Theorem 0.13.** Let an affine com(m+k,m)-semigroup (group) be given such that its characteristic polynomial is non-zero, and considered as a polynomial of one variable has s different roots. Then there exist exactly $k^s$ distinct affine com(m+1,m)-semigroups (groups) which induce the given affine com(m+k,m)-semigroup (group). Specially, each affine com(m+k,m)-group on $\mathbb{C}$ is induced by one affine com(m+1,m)-group on $\mathbb{C}$. ■

As a generalization of the affine com(n,m)-semigroups and com(n,m)-groups we obtain the notions of projective com (n,m)-semigroups and com-(n,m)-groups, in section II.3.

Complex com(m+k,m)-groupoids $(\mathbb{C},f)$ can be examined via polynomial mappings

$$\phi : P_{m+k} \to P_m$$

where

$$f(z_1^{m+k})=w_1^m \iff \phi((t-z_1)\dots(t-z_{m+k})) = (t-w_1)\dots(t-w_m).$$

If the coefficients of the polynomial $\phi(\rho)$ (except the coefficient 1 in front of $t^m$) are linear functions of the coefficients of the polynomial $\rho$ with degree m+k, then the complex com(m+k,m)-groupoid determined in this way is called affine in II.2. The affine com(n,m)-group structures can be generalized in the following way. We can consider the polynomial mappings

$$\phi^* : P^*_{m+k} \to P^*_m$$

instead of the polynomial mappings

$$\phi : P_{m+k} \to P_m$$

where for a positive integer n,

$$P_n^* = \{a_0t^n - a_1t^{n-1} + \ldots + (-1)^n a_n \mid (a_0, a_1, \ldots, a_n) \in \mathbb{C}P^n\}$$

and where $\mathbb{C}P^n$ is the n-dimensional complex projective space.

Further, if $\rho \in P_n^*$ and $\deg(\rho) = r < n$, then we say that $\infty$ is a root of the polynomial $\rho$ with multiplicity n-r. The converse also holds. Indeed, if the numbers $z_1, \ldots, z_n \in \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ which may not be different, are given, then there exists unique polynomial $\rho \in P_n^*$ such that $z_1, \ldots, z_n$ are roots of $\rho$. Thus, a polynomial mapping $\phi^* : P^*_{m+k} \to P^*_m$ corresponds to each com(m+k, m)-groupoid on $\mathbb{C}^*$, and conversely, a com(m+k, m)-groupoid corresponds to each such mapping.

In general we can write

$$\phi^*(a_0t^{m+k} - a_1t^{m+k-1} + \ldots + (-1)^{m+k}a_{m+k})$$
$$= F_0t^m - F_1t^{m-1} + F_2t^{m-2} - \ldots + (-1)^m F_m \tag{0.20}$$

where $F_i$, $0 \le i \le m$, are functions of $a_0, a_1, \ldots, a_{m+k}$.

**Definition 0.10.** If $F_i$ $(i=0,1,..,m)$ are linear functions of $a_0, a_1, \ldots, a_{m+k}$, i.e.

$$F_i(a_0, a_1, \ldots, a_{m+k}) = \sum_{0 \le j \le m+k} \alpha_{ij} a_j \qquad (0 \le i \le m) \tag{0.21}$$

then the corresponding com(m+k, m)-groupoid on $\mathbb{C}^*$ which is induced by the polynomial mapping (0.20) is called a **projective** com(m+k, m)-groupoid. If this groupoid is associative, then it is called a **projective** com(m+k, m)-semigroup. A com(m+k, m)-group on a non-empty subset of $\mathbb{C}^*$ which is obtained by removing the singular elements from a projective com(n+k, m)-group is called a **projective** com(m+k, m)-group.

In the rest of this section we examine in detail the projective com-(n, m)-semigroups and com(n, m)-groups on $\mathbb{C}^*$.

In chapter three we examine com(n, m)-semigroups and com(n, m)-groups, defined on topological spaces and manifolds, which preserve the topological structures, i.e. we examine topological com(n, m)-semigroups and com(n, m)-groups.

**Definition 0.11.** Let X be a topological space and n a positive integer. The **n-fold permutation product of** X, is the topological factor space $X^{(n)} = X^n/\approx$, where $X^n$ is the n-fold topological product of X and $\approx$ is defined by (0.3).

**Definition 0.12.** Let Q be a topological space and let (Q,f) be a com(n,m)-groupoid. We say that (Q,f) is a **topological (continuous)** com(n,m)-groupoid, if the map $f:Q^{(n)}\to Q^{(m)}$ is continuous. If (Q,f) is a topological com (n,m)-groupoid, and (Q,f) is a com(n,m)-semigroup, then we say that (Q,f) is a **topological (continuous)** com(n,m)-semigroup.

Next in this section we give several characterizations of topological com(n,m)-semigroups.

In section III.3 we examine topological com(n,m)-semigroups (Q,f) which are com(n,m)-groups and whose solutions of the equations f(a,x) = b depend continuously on a and b.

**Definition 0.13.** Let (Q,f) be a com(n,m)-group and let $f':Q^{(k)}\times Q^{(m)}\to Q^{(m)}$ be the map defined by

$$f'(a,b) = (\varphi_a)^{-1}(b),\quad a\in Q^{(k)},\ b\in Q^{(m)} \tag{0.22}$$

where $\varphi_a$ is defined by (0.9).

We say that (Q,f) is a **topological com(n,m)-group** if (Q,f) is a topological com(n,m)-semigroup and f' is a continuous map.

Topological com(n,m)-groups have special properties.

**Proposition 0.14.** Let (Q,f) be a topological com(n,m)-group. Then, for each $a\in Q^{(k)}$, $\varphi_a:Q^{(m)}\to Q^{(m)}$ is a homeomorphism. ■

**Proposition 0.15.** Let (Q,f) be a topological com(m+k,m)-semigroup. Then, (Q,f) is a topological **com(m+k,m)-group** iff for each $c\in Q^{(p)}$, the derived group $(Q^{(m)},*)$ is a commutative topological group. ■

**Corollary 0.16.** If (Q,f) is a topological com(n,m)-group, then $Q^{(m)}$ is homogeneous topological space, i.e. for each $x,y\in Q^{(m)}$, there exists a homeomorphism $h:Q^{(m)}\to Q^{(m)}$ such that h(x) = y. ■

**Corollary 0.17.** For $m\geq 2$, there do not exist topological com(n,m)-groups on the space of real numbers $\mathbb{R}$. ■

**Proposition 0.18.** If (Q,f) is an affine or projective com(n,m)-group, then it is a topological com(n,m)-group. ■

In section III.3 we consider topological com(m+k,m)-groups (M,f) where the topological space M is n-dimensional manifold, i.e. M is Haus-