such that the vectors

$$(-\alpha_{01}, \alpha_{00}-\alpha_{11}, \alpha_{10}-\alpha_{21},\ldots, \alpha_{(m-1)0}-\alpha_{m1}, \alpha_{m0})$$
$$(-\alpha_{02}, \alpha_{01}-\alpha_{12}, \alpha_{11}-\alpha_{22},\ldots, \alpha_{(m-1)1}-\alpha_{m2}, \alpha_{m1})$$
$$- - - - - - - - - - - - - - - - - - -$$
$$(-\alpha_{0m},\alpha_{0(m-1)}-\alpha_{1m},\alpha_{1(m-1)}-\alpha_{2m},\ldots,\alpha_{(m-1)(m-1)}-\alpha_{mm},\alpha_{m(m-1)})$$

are equal to h in $\mathbb{C}P^{m+1}$.

In the special case when $h_0=0$ all of the α_{0i} have to be 0 for $1\le i\le m$, i.e. $\alpha_{01}=\alpha_{02}=\ldots=\alpha_{0m}=0$. Since the matrix (3.19) has to be non-singular it follows that $\alpha_{00}\neq 0$. But the matrix (3.19) belongs to $PGL(m;\mathbb{C})$, and thus it is determined up to a scalar multiple. If we put $\alpha_{00}=1$ then we obtain the group G_h defined in §1.2 for $h=(h_1,h_2,\ldots,h_{m+1})$. This gives a connection between the "affine" and "projective definition" of G_h.

Analogous to Theorem 2.1 one can prove the following theorem for the projective case:

Theorem 3.6. Let a vector $h\in\mathbb{C}P^{m+1}$ be given. The set of matrices G_h with the matrix multiplication is a closed Lie subgroup of $PGL(m;\mathbb{C})$ with complex dimension m. ■

In analogy with the proof of Theorem 2.1 one can verify that G_h coincides with the derived group $(G^{(m)},\bullet)$ for arbitrary projective com(m+1,m)-group (G,f) whose characteristic polynomial is

$$h_0t^{m+1} - h_1t^m + h_2t^{m-1} - \ldots +(-1)^{m+1}h_{m+1}.$$

The following theorems can be proved in the same way as Theorems 2.2, 1.11 and 2.3 for the affine case.

Theorem 3.7. Let a vector $h=h_0^m\in\mathbb{C}P^{m+1}$ be given, and let the polynomial

$$P(t) = h_0t^{m+1} - h_1t^m + h_2t^{m-1} - \ldots +(-1)^{m+1}h_{m+1}$$

has exactly s different roots in $\mathbb{C}^*$ $(s\in\{1,\ldots,m+1\})$. Then

$$G_h \cong \underbrace{C_1\times\ldots\times C_1}_{s-1}\times\underbrace{C_0\times\ldots\times C_0}_{m+1-s}. \quad ■ \tag{3.20}$$

Theorem 3.8. Let a non-singular projective com(m+k,m)- semigroup be given, and let the corresponding characteristic polynomial be $\Delta^*(z_1^k)$.

Then there exists a polynomial Δ'^{*} of one variable such that

$$\Delta^{*}(z_1^k) = \Delta'^{*}(z_1)\cdot\Delta'^{*}(z_2)\cdot \ \ldots \ \cdot\Delta'^{*}(z_k) \tag{3.21}$$

and $\deg\Delta'^{*}(z)\le m+1$. ■

Theorem 3.9. Let a non-singular projective com(m+k,m)- semigroup on $\mathbb{C}^{*}$ (group on $\mathbb{C}^{*}\setminus\mathcal{R}$) be given, and suppose that its characteristic polynomial has exactly s different roots in $\mathbb{C}^{*}$. Then the given projective com(m+k,m)-semigroup (group) is induced by k^{s-1} different projective com(m+1,m)-semigroups on $\mathbb{C}^{*}$ (groups on $\mathbb{C}^{*}\setminus\mathcal{R}$). ■

In this chapter we have not considered the question about isomorphism between two projective (affine) com(m+k,m)- groups (semigroups). Having in mind Theorem 3, the following question appears naturally. If two com(m+k,m)-groups (semigroups) are isomorphic, is the isomorphism induced by a bilinear transformation? If we know that the answer of this question is affirmative, then using compositions of the matrix transformations (3.4), (3.5) and (3.6) we would be able to deduce whether two com(m+k,m)- groups (semigroups) are isomorphic or not.

III. TOPOLOGICAL COMMUTATIVE (n, m)-SEMIGROUPS AND GROUPS

III.1. Topological com(n, m)-semigroups

In the previous chapter we have examined in detail the affine and projective semigroups and groups on subsets of $\mathbb{C}$ or $\mathbb{C}^*$. We were mainly interested in the maximal com(n, m)-groups on subsets of $\mathbb{C}$ or $\mathbb{C}^*$. We have shown that if the characteristic polynomial is the zero polynomial, then these groups have only a few elements, while if the characteristic polynomial is not the zero polynomial, then these groups are defined on subsets obtained by removing at least one and at most $m+1$ points from $\mathbb{C}$ or $\mathbb{C}^*$. In this sense all of them are maximal, but they may have com(n, m)-subgroups.

On the other hand, if we consider $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{C}P^n$ as topological spaces, then all the affine and projective groups examined in the previous chapter are obtained by the symmetric functions and linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^m$, i.e. from $\mathbb{C}P^n$ to $\mathbb{C}P^m$, and such functions are continuous. In this chapter we will examine topological com(n, m)-semigroups and groups.

In this section we will examine com(n, m)-groupoids defined on topological spaces such that the (n, m)-operation is continuous. For this we have to recall some known facts given in ([25], [15]), about the n-fold permutation products of a topological space X.

Definition 1.1. Let X be a topological space and n a positive integer. The **n-fold permutation product of** X, is the topological factor space $X^{(n)} = X^n/\approx$, where X^n is the n-fold topological product of X and $\approx$ is the ker of the natural projection $\pi_n : X^n \to X^{(n)}$

as defined in 1.1, i.e. $x_1^n \approx y_1^n$ iff $x_1, x_2, \ldots, x_n$ is a permutation of $y_1, y_2, \ldots, y_n$.

Since π_n is a quotient map, it is a continuous surjection. Moreover, it is shown in [25] that π_n is an open and closed map. A simple argument shows that $\pi_n \times \pi_m : X^{n+m} \to X^{(n)} \times X^{(m)}$ is an open continuous surjection. Since $\lambda^{(n)}$ is a factor space, it follows that a map $f: X^{(n)} \to Y$ is continuous iff $f \circ \pi_n$ is continuous from X to Y. The above remarks imply that a map $g: X^{(n)} \times X^{(m)} \to Y$ is continuous iff the map $g \circ (\pi_n \times \pi_m): X^{n+m} \to Y$ is continuous.

The initial studies of the permutation topological products have been done in [20] and [19], where the name "symmetric" product has been used. Here we use the name "permutation" product as it is used in [25]. Later, [21], [14], [15] have described the homotopical and homological properties of permutation products.

Several results about permutation products are stated in [25]. Some of them are the following:

(1) [17] a) For n odd, $(S^1)^{(n)} \cong S^1 \times I^{n-1}$, i.e. the n-fold permutation product of the circle S^1, is homeomorphic to the trivial (n-1)-bundle over S^1;

b) For n even, $(S^1)^{(n)} \cong S^1 \tilde{\times} I^{n-1}$, i.e. the n-fold permutation product of the circle S^1, is homeomorphic to the nontrivial (nonorientable) (n-1)-bundle over S^1. For example, for n=2, $(S^1)^{(2)}$ is homeomorphic to the Mobious band.

2) [16] $(S^2)^{(n)} \cong (\mathbb{C}^*)^{(n)} \cong \mathbb{C}P^n$, i.e. the n-fold permutation product of the 2-dimensional sphere S^2, (one point compactification $\mathbb{C}^*$ of the complex plane $\mathbb{C}$) is homeomorphic to the n-dimensional complex projective space $\mathbb{C}P^n$.

3) [25] For k≥0, if P_k is a set of k distinct points in $\mathbb{C}$, then $(\mathbb{C} \setminus P_k)^{(n)} \cong (\mathbb{C} \setminus \{0\})^k \times \mathbb{C}^{n-k}$, where $(\mathbb{C} \setminus \{0\})^0$ is a one point space.

Definition 1.2. Let Q be a topological space and let (Q,f) be a com(n,m)-groupoid. We say that (Q,f) is a **topological (continuous)** com(n,m)-groupoid, if the map $f: Q^{(n)} \to Q^{(m)}$ is continuous. If (Q,f) is a topological com(n,m)- groupoid, and (Q,f) is a com(n,m)-semigroup, then we say that (Q,f) is a **topological (continuous)** com(n,m)-semigroup.

For a com(m+k,m)-semigroup (Q,f) where Q is a topological space, let $[Q^{(m)},Q^{(m)}]$ be the space of all the continuous maps from $Q^{(m)}$ to $Q^{(m)}$ with the compact-open topology, and let $\Psi:Q^{(k)}\to[Q^{(m)},Q^{(m)}]$ be the map defined by $\Psi(a) = \varphi_a$, where φ_a is defined in I, (3.1).

Proposition 1.1. Let Q be a topological space, and let (Q,f) be a com(m+k,m)-semigroup. Then: (Q,f) is a topological com(m+k,m)-semigroup iff Ψ is a continuous map.

Proof. The map Ψ is continuous iff the map $\Psi':Q^{(k)}\times Q^{(m)}\to Q^{(m)}$ defined by $\Psi'(a,b) = \Psi(a)(b) = \varphi_a(b)$ is continuous. The conclusion now follows from the following facts: the last map Ψ' is continuous iff the map $\Psi'\circ(\pi_k\times\pi_m)$ is continuous; the map $\Psi'\circ(\pi_k\times\pi_m)$ is equal to the map $f\circ\pi_{m+k}$; and $f\circ\pi_k$ is continuous iff f is continuous. ∎

The next proposition shows that topological com(m+k,m)-semigroups induce topological com(m+ks,m)-semigroups.

Proposition 1.2. Let (Q,f) be a topological com(m+k,m)-semigroup, and let (Q,g) be the induced com(m+sk,m)-semigroup by (Q,f). Then (Q,g) is a topological com(m+sk,m)-semigroup.

Proof. The proof is by induction on s. Let $b\in Q^{(k)}$, $a\in Q^{(m+(s-1)k)}$, and $g(a,b) = f(h(a),b)$, where (Q,h) is the com(m+(s-1)k,m)-semigroup induced by (Q,f). Now, if we consider the map $g\circ\pi_{m+sk}$, it can be written as the composition $h'\circ(h,id)\circ(\pi_{m+(s-1)k},\pi_k)$, where $h':Q^{(m)}\times Q^{(k)}\to Q^{(m)}$ is defined by $h'(c,d)=f(c,d)$. The facts that: $h'\circ(\pi_m,\pi_k) = f\circ\pi_{m+k}$, $f\circ\pi_{m+k}$ is continuous and (π_m,π_k) is an open surjection, imply that h' is continuous. Hence, it follows that $g\circ\pi_{m+sk}$ is continuous which is equivalent to g being continuous. ∎

We will show that the affine and projective com(n,m)-semigroups are topological. For this we need the following facts:

Proposition 1.3. (a) ([25], [16]) For each $m\geq1$, $\mathcal{S}_m:\mathbb{C}^{(m)}\to\mathbb{C}^m$ is a homeomorphism.

(b) ([25], [16]) For each $m\geq1$, $(\mathbb{C}^*)^{(m)}$ is homeomorphic to $\mathbb{C}P^m$, via a homeomorphism $\mathcal{P}_m$ defined by $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$ and the identifications:

$$\mathbb{C}P^m = 1\times\mathbb{C}^m \cup 1^2\times\mathbb{C}^{m-1} \cup \ldots \cup 1^{m-2}\times\mathbb{C}^2 \cup 1^{m-1}\times\mathbb{C}^1 \cup 1^m, \quad \text{and}$$

$$(\mathbb{C}^*)^{(m)} = (\mathbb{C} \cup \infty)^{(m)} = \mathbb{C}^{(m)} \cup \infty\times\mathbb{C}^{(m-1)} \cup\ldots\cup \infty^{m-1}\times\mathbb{C}^{(1)} \cup \infty^m. \blacksquare$$

Proposition 1.4. If (Q,f) is an affine or projective com(n,m)-semigroup, then it is a topological com(n,m)-semigroup.

Proof. Let $(\mathbb{C},f)$ be an affine com(n,m)-semigroup, i.e. $f=\varphi_m^{-1}\circ g\circ\varphi_n$, where $g:\mathbb{C}^n\to\mathbb{C}^m$ is an affine transformation. Because g is continuous and φ_m, φ_n are homeomorphisms, it follows that f is continuous, i.e. $(\mathbb{C},f)$ is a topological com(n,m)-semigroup.

Let $(\mathbb{C}^*,f)$ be a projective com(n,m)-semigroup. Then $f=\mathcal{P}_m^{-1}\circ g\circ\mathcal{P}_n$ where $g:\mathbb{C}P^n\to\mathbb{C}P^m$ is a linear transformation. Since g is continuous and $\mathcal{P}_m$, $\mathcal{P}_n$ are homeomorphisms, it follows that f is continuous, i.e. $(\mathbb{C}^*,f)$ is a topological com(n,m)-semigroup. ■

III.2. Topological commutative (n,m)-groups

In this section we will examine topological com(n,m)-semigroups (Q,f) which are com(n,m)-groups and whose solutions of the equations $f(a,x) = b$ depend continuously on a and b. First we give the definition.

Definition 2.1. Let (Q,f) be a com(n,m)-group and let $f':Q^{(k)}\times Q^{(m)}\to Q^{(m)}$ be the map defined by

$$f'(a,b) = (\varphi_a)^{-1}(b), \quad a\in Q^{(k)}, \; b\in Q^{(m)} \tag{2.1}$$

where φ_a is defined in I, (3.1).

We say that (Q,f) is a **topological com(n,m)-group** if (Q,f) is a topological com(n,m)-semigroup and f' is a continuous map.

Topological com(n,m)-groups have special properties.

Proposition 2.1. Let (Q,f) be a topological com(n,m)-group. Then, for each $a\in Q^{(k)}$, $\varphi_a:Q^{(m)}\to Q^{(m)}$ is a homeomorphism.

Proof. Since (Q,f) is a com(n,m)-group it follows that for each $a\in Q^{(k)}$, φ_a is a bijection.

Next, $\varphi_a\circ\pi_m = f\circ\pi_n\circ(a,\mathrm{id})$, where $(a,\mathrm{id}):Q^m\to Q^n$ is the map defined by $(a,\mathrm{id})(b) = (a,b)$, which shows that $\varphi_a\circ\pi_m$ is a continuous map. Hence, φ_a is continuous.

If we denote again by (a,id) the map from $Q^{(m)}$ to $Q^{(k)}\times Q^{(m)}$ defined by $(a,\mathrm{id})(b) = (a,b)$, it follows that $\varphi_a^{-1} = f'\circ(a,\mathrm{id})$. Since f' and (a,id) are continuous, it follows that φ_a^{-1} is continuous. ∎

In a similar way as in Proposition 1.1, we can characterize topological com(n,m)-groups. Let $\mathrm{Homeo}(Q^{(m)})$ be the subspace of all the self homeomorphisms on $Q^{(m)}$, of the space $[Q^{(m)},Q^{(m)}]$. The above Proposition says that if (Q,f) is a topological com(n,m)-group, then $\Psi(Q^{(k)}) \subseteq \mathrm{Homeo}(Q^{(m)})$.

Proposition 2.2. Let (Q,f) be a topological $com(m+k,m)$-semigroup. Then, (Q,f) is a topological $com(m+k,m)$-group iff $\Psi(Q^{(k)}) \subseteq Homeo(Q^{(m)})$ and the map $\lambda: Q^{(k)} \to [Q^{(m)}, Q^{(m)}]$ defined by $\lambda(a,b) = \varphi_a^{-1}(b)$ is continuous.

Proof. If (Q,f) is a topological $com(n,m)$-group, then the map λ is continuous because the map f' defined by (2.1) is continuous, and $\Psi(Q^{(k)}) \subseteq Homeo(Q^{(k)})$ by Proposition 2.1.

Conversely, let $\Psi(Q^{(k)}) \subseteq Homeo(Q^{(k)})$ and λ be continuous. Then for each $a \in Q^{(k)}$, φ_a is bijection, and so (Q,f) is a $com(m+k,m)$-group. Since λ is continuous, it follows that f' is continuous, and f is continuous by the assumption that (Q,f) is a topological $com(m+k,m)$-semigroup. ■

Proposition 2.3. Let (Q,f) be a topological $com(m+k,m)$-group, and let (Q,g) be the induced $com(m+sk,m)$-group by (Q,f). Then (Q,g) is a topological $com(m+sk,m)$-group.

Proof. Proposition 1.2 implies that (Q,g) is a topological $com(m+k,m)$-semigroup. For each $d = a_1 a_2 \dots a_s \in Q^{(sk)}$, $a_i \in Q^{(k)}$, $\varphi_d = \varphi_{a_1} \circ \varphi_{a_2} \circ \dots \circ \varphi_{a_s}$. So, $\Psi(Q^{(sk)}) \subseteq Homeo(Q^{(m)})$. Next, $\lambda: Q^{(sk)} \to [Q^{(m)}, Q^{(m)}]$ is continuous because the map $g': Q^{(sk)} \times Q^{(m)} \to Q^{(m)}$ defined by $g'(d,b) = \varphi_d^{-1}(b)$ can be written as the composition of two continuous maps, $f' \circ (id, h')$ where (Q,h) is the induced topological $com(m+(s-1)k,m)$-group, and $h': Q^{((s-1)k)} \times Q^{(m)} \to Q^{(m)}$ is defined by (2.1). So, Proposition 2.2 implies that (Q,g) is a topological $com(m+sk,m)$-group. ■

Next, let (Q,f) be a topological $com(m+k,m)$-semigroup,. and let (Q,f) be a $com(m+k,m)$-group. Let m,k,p, and s be as in I.2, i.e. $m+p \equiv 0 \pmod k$, $m+p=ks$, $k(s-1) < m \leq ks$. For $c \in Q^{(p)}$ let $(Q^{(m)}, \ast)$ be the derived group for (Q,f) defined by I.(2.1) or I.(2.2).

Proposition 2.4. Let (Q,f) be a topological $com(m+k,m)$-semigroup. Then, (Q,f) is a topological $com(m+k,m)$-group iff for each $c \in Q^{(p)}$, $(Q^{(m)}, \ast)$ is a commutative topological group.

Proof. Proposition I.2.1 implies that (Q,f) is a $com(m+k,m)$-group iff $(Q^{(m)}, \ast)$ is a commutative group. The map $\ast: Q^{(m)} \times Q^{(m)} \to Q^{(m)}$ can be written as the composition $g \circ (id, c)$, where

$(id,c)(x,y)=(x,y,c)$ for $x,y\in Q^{(m)}$, and (Q,g) is the induced com(m+sk,m)-group, where $m+p=ks$. If $a\in Q^{(m)}$ and $c\in Q^{(p)}$, then $ac=d_1d_2\ldots d_s$ for $d_i\in Q^{(k)}$, and so, $\varphi_{ac}=\varphi_{d_1}\circ\varphi_{d_2}\circ\ldots\circ\varphi_{d_s}$.

Let (Q,f) be a topological com(m+k,m)-group. Then (Q,g) is a topological group, and so $\bullet$ is continuous as a composition of two continuous maps. From the definition of $\bullet$, it follows that $a^{-1}=\varphi_{ac}^{-1}\circ\varphi_{ac}^{-1}(a) = g'\circ((id,c),g'\circ(id,c))(a)$, i.e. $^{-1}:Q^{(m)}\to Q^{(m)}$ is a continuous map. Hence $(Q^{(m)},\bullet)$ is a topological group.

Conversely, let $(Q^{(m)},\bullet)$ be a topological group. It is enough to show that f' is a continuous map. Since $f'(x,a) = \varphi_x^{-1}(a)$, for $x\in Q^{(k)}$ and $a\in Q^{(m)}$, i.e. $a = f(x,f'(x,a))$ it follows that $a\bullet e = f(xecf'(x,a))$, where e is the neutral element in $(Q^{(m)},\bullet)$, which implies that $a\bullet e = f(xe)\bullet f'(x,a)$. So, $f'(x,a) = (f(xe))^{-1}\bullet a\bullet e$, which implies that f' is continuous, because f, $\bullet$ and $^{-1}$ are continuous maps. ∎

Corollary 2.5. If (Q,f) is a topological com(n,m)-group, then $Q^{(m)}$ is a homogeneous topological space, i.e. for each $x,y\in Q^{(m)}$, there exists a homeomorphism $h:Q^{(m)}\to Q^{(m)}$ such that $h(x) = y$.

Proof. The proof follows from the previous Proposition, and the fact that topological groups are homogeneous spaces. ∎

Corollary 2.6. For $m\geq 2$, there do not exist topological com(n,m)-groups on the space of real numbers $\mathbb{R}$.

Proof. The statement follows from Corollary 2.4 and the fact that for $m\geq 2$, $\mathbb{R}^{(m)}$ is not homogeneous space. ∎

Proposition 2.7. If (Q,f) is an affine or projective com(n,m)-group, then it is a topological com(n,m)-group.

Proof. If (Q,f) is an affine or projective com(n,m)-group, then Proposition 1.4 implies that it is a topological com(n,m)-semigroup, and Theorems II.2.3 and II.3.9 imply that (Q,f) is induced by an affine or projective com(m+1,m)-group respectively. Now, Proposition 2.3 implies that it is enough to consider only affine or projective com(m+1,m)-groups (Q,f). If (Q,f) is an affine com(m+1,m)-group, then it is shown in II.2 that $(Q^{(m)},\bullet)$ is a Lie group, and if (Q,f) is a projective com(m+1,m)-group, then it is shown in II.3 that $(Q^{(m)},\bullet)$ is a Lie group. So, in both cases $(Q^{(m)},\bullet)$ is a topological group, and by

Proposition 2.4. (Q,f) is a topological com(n,m)-group. ∎

We have a similar Proposition as Proposition I.1.4 for topological com(n,m)-groupoids, -semigroups and -groups.

Proposition 2.8. Let (Q,f) be a topological com(n,m)-groupoid, -semigroup or -group, and let $h:Q\to P$ be a homeomorphism. Then (P,g) for $g = h^{(m)}\circ f\circ (h^{(n)})^{-1}$, is also a topological com(n,m)-groupoid, -semigroup or -group, respectively, where $h^{(t)}$ is defined by I.(1.10). ∎

The above Proposition does not preserve the property of being an affine or projective com(n,m)-group, which is shown by the following example:

Example. Let $(\mathbb{C}\setminus\{0\},f)$ be the affine com(3,2)-group of Example 5, from I.5, for $\lambda=0$. Let $h:\mathbb{C}\setminus\{0\}\to\mathbb{C}\setminus\{0\}$ be the homeomorphism defined by $h(z) = z\cdot|z|$, where $|z|$ is the length of $z\in\mathbb{C}$. Note that the inverse homeomorphism for h is $h^{-1}(z) = z/\sqrt{|z|}$. Via this homeomorphism, we obtain a topological com(3,2)-group $(\mathbb{C}\setminus\{0\},g)$, where g is defined by $g(xyz) = (uv)$ for

$$u/\sqrt{|u|} + v/\sqrt{|v|} = x/\sqrt{|x|} + y/\sqrt{|y|} + z/\sqrt{|z|} \quad \text{and}$$

$$uv/\sqrt{|uv|} = xyz/\sqrt{|xyz|}.$$

Then, $(\varphi_2\circ g\circ \varphi_3^{-1})(a,b,c) = (F(a,b,c),c)$, where F is not a linear function of a, b and c. Hence $(\mathbb{C}\setminus\{0\},g)$ is not an affine semigroup. ∎

The above example is an example of a topological com(n,m)-group which is not affine, but it is isomorphic to an affine com(n,m)-group. The next example shows that there are topological com(n,m)-groups which are not even homeomorphic to an affine com(n,m)-group, but for them, m=1.

Example. From the Lie group theory, every complex abelian Lie group, of dimension 1 is isomorphic to a factor group of the group of complex numbers with the usual addition of complex numbers, $(\mathbb{C},+)$, over a discrete subgroup D. A discrete subgroup of $(\mathbb{C},+)$ is isomorphic to one of the following three groups: $D = \{0\}$, $D \cong \mathbb{Z}$ and $D \cong \mathbb{Z} + i\mathbb{Z}$. If $D \cong \mathbb{Z}$, w.l.o.g. we can assume that $D = \{2k\pi i \mid k\in\mathbb{Z}\}$, and so, the map $\exp:(\mathbb{C},+)/D \to (\mathbb{C}\setminus\{0\},\cdot)$ defined by $\exp(z)=e^z$, gives an isomorphism of Lie groups. It can be checked that these three groups as algebraic groups are not isomorphic. Since every topological group over a manifold is isomorphic to a Lie group, it follows that every topological complex com(2,1)-group is isomorphic to one of the three groups: $(\mathbb{C},+)$, $(\mathbb{C}\setminus\{0\},\cdot)$ and $(\mathbb{C},+)/\mathbb{Z}+i\mathbb{Z}$. The first two groups are affine and projective, and by Theorems II.2.2 and II.3.7 it follows that up to isomorphism there are only two affine, i.e. projective complex (2,1)-groups. Hence the group $(\mathbb{C},+)/\mathbb{Z}+i\mathbb{Z}$ is not affine. Moreover, this group is defined on the torus, and the torus is not homeomorphic to a subset of $\mathbb{C}$ or $\mathbb{C}^{*}$.

III.3. Locally euclidean topological com(m+k,m)-groups

In this section we will consider topological com(m+k,m)-groups (M,f) where the topological space M is an n-dimensional manifold, i.e. M is a Hausdorff topological space and each point $p \in M$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$. We will call these groups locally euclidean topological com(m+k,m)-groups. Specially if m=k=1 then it is known [18] that each locally euclidean group is a Lie group, and hence M is a differentiable manifold.

Proposition 3.1. If (M,f) is a locally euclidean topological com(m+k,m)-group, then $M^{(m)}$ is a manifold of dimension $m \cdot \dim M$.

Proof. Let us suppose that (M,f) is a locally euclidean topological com(m+k,m)-group. We choose m mutually different elements $a_1,\dots,a_m \in M$. Then the point $(a_1,\dots,a_m) \in M^{(m)}$ has a neighborhood which is homeomorphic to $\mathbb{R}^{m\dim M}$. Since $M^{(m)}$ must be homogeneous topological space, we obtain that each point of $M^{(m)}$ has a neighborhood which is homeomorphic to $\mathbb{R}^{m\dim M}$. ∎

Let us suppose that (M,f) is a topological com(m+k,m)-group. We know from Proposition I. 3.4, that the derived group $(M^{(m)},*)$ is a topological commutative group. Specially if k=1 the group $(M^{(m)},*)$ is just the induced com(2m,m)-group considered as a usual (2,1)-group, and if k=m the group $(M^{(m)},*)$ is the same com(2m,m)-group considered as a usual (2,1)-group.

Proposition 3.2 If (M,f) is a connected locally euclidean topological com(m+k,m)-group, then the derived group $(M^{(m)},*)$ is a Lie group, and moreover,

$$M^{(m)} \cong \mathbb{R}^t \times (S^1)^q. \tag{3.1}$$

Proof. Since (M,f) is a topological com(m+k,m)-group, Proposition 2.4 implies that $(M^{(m)},*)$ is a topological group. Using the fact that (M,f) is a locally euclidean topological com(m+k,m)-group, it follows from Proposition 3.1 that $M^{(m)}$ is manifold. Thus $(M^{(m)},\bullet)$ is a connected locally euclidean topological group and hence it is a connected Lie group. Moreover, it is a commutative Lie group and from the discussion in section II.2 it follows that $M^{(m)}$ has the form (3.1). ■

Theorem 3.3. Let (M,f) be a connected locally euclidean topological com(m+k,m)-group. Then this group is induced by k^s connected locally euclidean topological com(m+1,m)-groups where s is the rank of the homology group $H_1(M^{(m)},\mathbb{Z})$. Specially if $M^{(m)}$ is a simply-connected manifold, then the corresponding com(m+k,m)-group is induced by exactly one com(m+1,m)-group.

Proof. It was proved in Theorem I.3.5, that a given com(m+k,m)-group (M,f) is induced by a com(m+1,m)-group (M,f') iff there exists a mapping $\varphi'_r:M^{(m)}\to M^{(m)}$ for each $r\in M$ such that

$$\varphi_{r_1,\dots,r_k} = \varphi'_{r_1}\circ\varphi'_{r_2}\circ\dots\circ\varphi'_{r_k}$$

for each $r_1^k\in M^{(m)}$. Moreover, each of these com(m+1,m)-groups is given by the following mapping

$$\varphi'_r = \varphi_{ra^{k-1}}\circ u_i^{-1} \tag{3.2}$$

where u_i is a solution of the following equation

$$(\varphi_{a^k})^{k-1} = (u_i)^k \tag{3.3}$$

in the group $H(M,f)$ where a is an arbitrary element in M.

Since (M,f) is a connected locally euclidean topological com(m+k,m)-group it follows from Proposition 3.2 that

$$M^{(m)}\cong\mathbb{R}^t\times(S^1)^q$$

for some non-negative integers t and q. Since $s=\operatorname{rank}H_1(M^{(m)},\mathbb{Z})$ we obtain that q=s. It follows from the Propositions 3.2 and I.3.4, that the group $H(M,f)$ is isomorphic to the commutative Lie group $(\mathbb{R}^t\times(S^1)^q,\bullet)$ and thus the equation (3.3) has exactly k^s roots in this Lie group. Hence the given com(m+k,m)-group is induced by exactly k^s com(m+1,m)-groups. We see that each of these **com**(m+1,m)-groups $(\mathbb{C},h)$ is topological because

$$h(x_1^{m+1}) = \varphi'_{x_1}(x_2^{m+1}) = \varphi_{x_1a^{k-1}}\circ(u_i)^{-1}(x_2^{m+1})$$

continuously depends on $x_1^m\in M^{(m+1)}$, and the solution x_1^m of the equation

$h(zx_1^m)=y_1^m$ i.e. $\varphi_{za^{k-1}}\circ(u_i)^{-1}(x_1^m)=y_1^m$ continuously depends on $z, y_1, \dots, y_m$.

Since M is a connected manifold, each of these groups is a connected locally euclidean topological com(m+1, m)-group.

Finally if $M^{(m)}$ is a simply-connected manifold, then $s=\text{rank}H_1(M^{(m)}, \mathbb{Z})=0$ and $k^s=1$. ■

Proposition 3.4. If $(\mathbb{R},f)$ is a topological com(n, 1)-group, then $(\mathbb{R},f)$ is isomorphic to $(\mathbb{R},g)$ where $g(r_1^n)=r_1+r_2+\dots+r_n$.

Proof. Since $\mathbb{R}$ is locally euclidean, it follows from Theorem 3.3, that $(\mathbb{R},f)$ is induced from a topological com(2, 1)-group $(\mathbb{R},g)$. But, up to isomorphism, there is only one commutative topological group on $\mathbb{R}$, and that is the group with the addition of real numbers, i.e. $g(r,t) = r+t$. Hence, $g(r_1^n) = r_1 + r_2 + \dots + r_n$. ■

We shall give an example showing that Theorem 3.3 does not hold if we omit the assumption that M is a connected manifold.

Example. Let $M=\{2t+1 \mid t\in\mathbb{Z}\}$ and let us define $f(x,y,z)= x+y+z$ for $x,y,z\in M$. Then (M,f) is a locally euclidean topological com(3, 1)-group with dimM=0, but M is not a connected manifold. It can be checked that $H(M,f)=\{2t \mid t\in\mathbb{Z}\}$ with $(2s)\circ(2q)=2s+2q$. Since the equation $x\circ x=2$ does not have a solution in $H(M,f)$, the given com(3, 1)-group (M,f) is not induced by any com(m, 1)-group.

Propositions 3.1 and 3.2 play an important role in the theory of locally euclidean topological com(m+k, m)-groups. Indeed, if (M,f) is a connected locally euclidean topological com(m+k, m)-group, then $M^{(m)}$ must be a manifold and moreover

$$M^{(m)} \cong \mathbb{R}^t \times (S^1)^q.$$

But these two requirements are rarely satisfied. For example if dimM=1, then $M \cong \mathbb{R}$ or $M \cong S^1$. But $\mathbb{R}^{(m)}$ and $(S^1)^{(m)}$ are not manifolds of type $\mathbb{R}^t \times (S^1)^q$ because they are manifolds with a boundary. It is proved in [25] that if $M^{(m)}$ is a manifold ($m\geq 2$), then dimM=2. Indeed it is proved that if $m\geq 3$, then the points $x_1^m\in M^{(m)}$ such that $x_i=x_j$ for some $i\neq j$, do not have neighborhoods homeomorphic to $\mathbb{R}^{m\dim M}$. It is also proved in [25] that $(S^2)^{(m)} \cong \mathbb{C}P^m$. But $\mathbb{C}P^m$ does not have the form $\mathbb{R}^t \times (S^1)^q$. Moreover, using the fact that $M^{(m)}$ is always an orientable manifold, one can prove that M should also be an orientable manifold. Now using all these facts we obtain the following theorem:

Theorem 3.5. If (M,f) is a locally euclidean topological com(m+k,m)-group for $m\geq 2$, then $\dim M=2$, M is an orientable manifold and M is not homeomorphic to the sphere S^2. ∎

There are some other restrictions on the manifold M for the possibility of defining a topological com(n,m)-group on M.

(1) If M is a compact and orientable 2-dimensional manifold, then $M^{(m)}$ is also a compact manifold. If M admits a topological com(n,m)-group structure, and M is compact, then by the isomorphism (3.1), we have that $M^{(m)} \cong (S^1)^{2m}$.

(2) If M is an arbitrary manifold and $m\geq 2$, then it is proved in [20] that

$$\pi_1(M^{(m)}) \cong H_1(M,\mathbb{Z}). \tag{3.4}$$

So, if M admits a connected topological com(n,m)-group structure, then $M^{(m)} \cong \mathbb{R}^{2m-q}\times(S^1)^q$, for $0\leq q\leq 2m$. Then, (3.4) implies that $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^q$.

Above we have given only restrictions when M does not admit a topological com(n,m)-group structure. The results in chapters II and III, show that the manifolds homeomorphic to one of the manifolds: $\mathbb{C}$, $\mathbb{C}\backslash\{1\}$, $\mathbb{C}\backslash\{1,2\}$, ..., $\mathbb{C}\backslash\{1,2,\dots,m\}$, admit a topological com(n,m)-group structure. Moreover, this result implies that for each $0\leq t\leq m$,

$$(\mathbb{C}\backslash\mathcal{A}_t)^{(m)} \cong \mathbb{C}^{m-t}\times(\mathbb{C}\backslash\{0\})^t, \tag{3.5}$$

where $\mathcal{A}_t$ is a set of t distinct points in $\mathbb{C}$.

These homeomorphisms are proved also in [25].

We note that the isomorphism (3.5) can be written as

$$(\mathbb{C}\backslash\mathcal{A}_t)^{(m)} \cong \mathbb{R}^{2m-t}\times(S^1)^t, \tag{3.6}$$

for $0\leq t\leq m$. The fact that $(\mathbb{C}\backslash\{1,2,\dots,2m\})^{(m)}$ is not a compact topological space and $(S^1)^{2m}$ is a compact space, implies that it is not possible to obtain homeomorphisms of type (3.6) for each $0\leq t\leq 2m$.

All of the above discussion suggests a consideration not only of locally euclidean topological com(n,m)-groups, but complex topological com(n,m)-groups, i.e. a consideration of com(n,m)-groups on complex manifolds.

III.4. Classification of locally euclidean topological com(n,m)-groups

In this section we will give a classification of locally euclidean topological com(n,m)-groups under certain conjectures, made under the influence of the discussion from the last section III.3.

Conjecture 4.1. Let M be a given topological manifold and let $m\geq 2$.

(i) If $0\leq t\leq m$, and $M^{(m)} \cong \mathbb{R}^{2m-t}\times(S^1)^t$ which is equivalent to $M^{(m)} \cong \mathbb{C}^{m-t}\times(\mathbb{C}\setminus\{0\})^t$, then $M \cong \mathbb{C}\setminus\mathcal{A}_t$ where $\mathcal{A}_t$ is a set of t distinct complex numbers.

(ii) If $t\geq m$, then there does not exist a topological manifold such that $M^{(m)} \cong \mathbb{R}^{2m-t}\times(S^1)^t$.

With this Conjecture we have the following Corollary:

Corollary 4.1. If M is a connected topological manifold and (M,f) is a topological com(n,m)-group for $m\geq 2$, and if Conjecture 4.1 holds, then M is homeomorphic to one of the manifolds $\mathbb{C}\setminus\mathcal{A}_t$, for $0\leq t\leq m$.

Proof. Theorem 3.5 implies that $\dim M=2$, Proposition 3.1 implies that $M^{(m)}$ is a manifold with dimension $m\cdot\dim M = 2m$, Proposition 3.2 implies that $M^{(m)} \cong \mathbb{R}^q\times(S^1)^t \cong \mathbb{R}^{2m-t}\times(S^1)^t$, and Conjecture 4.1, (i) implies that $M \cong \mathbb{C}\setminus\mathcal{A}_t$ for some $0\leq t\leq m$. ∎

Next, let (M,f) be a given connected locally euclidean topological com(n,m)-group, and let $m\geq 2$. Since, by Theorem 3.3, it is induced by a locally euclidean topological com(m+1,m)-group (M,h), the examination of (M,f) can be reduced to the examination of (M,h). So, we will consider only connected, locally euclidean topological com(m+1,m)-groups, i.e. we assume that (M,f) is a connected locally euclidean topological com(m+1,m)-group. Assuming that Conjecture 4.1 holds, we may suppose that $M \cong \mathbb{C}\setminus\mathcal{A}_t$ for some $1\leq t\leq m$. Then,

$$M^{(m)} \cong (\mathbb{C}\setminus\{0\})^t\times\mathbb{C}^{m-t}$$

We denote the induced com(2m,m)-group, considered as an ordinary group, i.e. the derived group, by $(M^{(m)},*)$. We know that in this case $(M^{(m)},*)$ is a commutative Lie group, and after the identification of $M^{(m)}$ with $(\mathbb{C}\setminus\{0\})^t\times\mathbb{C}^{m-t}$, the operation $*$ on $M^{(m)}$ is given by:

$$(z_1^t, z_{t+1}^m) * (u_1^t, u_{t+1}^m) = (v_1^t, w_{t+1}^m) \tag{4.1}$$

where $v_i = z_i \cdot u_i$ and $w_i = z_{t+i} + u_{t+i}$. In fact, the group $(M^{(m)},*)$ coincides with the direct product $(C_1)^t\times(C_0)^{m-t}$ of the groups $C_0 = (\mathbb{C},+)$ and $C_1 = (\mathbb{C}\setminus\{0\},\cdot)$ where $+$ is the addition and $\cdot$ is the multiplication of complex numbers.

Under the above assumption, and after the identification of $M^{(m)}$ with $(C_1)^t\times(C_0)^{m-t}$, the set

$$S = \{f(ze) \mid z\in M\}$$

where e is the identity in the group $(M^{(m)},*)$ is a smooth 1-dimensional complex submanifold of $(C_1)^t\times(C_0)^{m-t}$, homeomorphic to M as topological manifolds. Let $g:\mathbb{C}\to S$ be the map defined by $g(z) = f(ze)$, i.e.

$$g(z) = f(ze) = (g_1(z),\ldots,g_t(z),\gamma_1(z),\ldots,\gamma_q(z)),$$

where $q=m-t$, and $(g_1,\ldots,g_t,\gamma_1,\ldots,\gamma_q)$ are the component maps of the map g. Since the first t coordinates of the points in S are in $C_1 = \mathbb{C}\setminus\{0\}$, it follows that $g_i(z) \neq 0$ i.e. $g_i:\mathbb{C}\to\mathbb{C}\setminus\{0\}$ for $1\le i\le t$, while $\gamma_i:\mathbb{C}\to\mathbb{C}$. From the definition of g, it follows that g is a homeomorphism, i.e. g identifies M with S. Moreover, we have the following equivalences:

$$\begin{aligned} f(z_1^{m+1}) = w_1^m &\Leftrightarrow f(z_1ez_2e\ldots z_{m+1}e) = f(w_1ew_2e\ldots w_me)\\ &\Leftrightarrow f(z_1e)*\ldots*f(z_{m+1}e) = f(w_1e)*\ldots*f(w_me)\\ &\Leftrightarrow g(z_1)*\ldots*g(z_{m+1}) = g(w_1)*\ldots*g(w_m), \end{aligned}$$

which are equivalent to:

$$\begin{cases} g_1(z_1)\cdot\ldots\cdot g_1(z_{m+1}) = g_1(w_1)\cdot\ldots\cdot g_1(w_m)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ g_t(z_1)\cdot\ldots\cdot g_t(z_{m+1}) = g_t(w_1)\cdot\ldots\cdot g_t(w_m)\\ \gamma_1(z_1)+\ldots+\gamma_1(z_{m+1}) = \gamma_1(w_1)+\ldots+\gamma_1(w_m)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \gamma_q(z_1)+\ldots+\gamma_q(z_{m+1}) = \gamma_q(w_1)+\ldots+\gamma_q(w_m) \end{cases} \tag{4.2}$$

In fact, the identification g of S with M, defines a locally euclidean topological com(m+1,m)-group (S,f') defined by $f'(g(z_1),\dots,g(z_{m+1})) = (g(w_1),\dots,g(w_m))$ iff (4.2) holds. It is easy to check that for $k\geq 1$, the induced com(m+k,m)-group (S,f') satisfies the following equivalence:

$$f'(g(z_1),\dots,g(z_{m+k})) = (g(w_1),\dots,g(w_m)) \Longleftrightarrow$$

$$\Longleftrightarrow \quad \begin{cases} g_1(z_1)\cdot\ldots\cdot g_1(z_{m+k}) = g_1(w_1)\cdot\ldots\cdot g_1(w_m) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ g_t(z_1)\cdot\ldots\cdot g_t(z_{m+k}) = g_t(w_1)\cdot\ldots\cdot g_t(w_m) \\ \gamma_1(z_1)+\ldots+\gamma_1(z_{m+k}) = \gamma_1(w_1)+\ldots+\gamma_1(w_m) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \gamma_q(z_1)+\ldots+\gamma_q(z_{m+k}) = \gamma_q(w_1)+\ldots+\gamma_q(w_m) \end{cases} \qquad (4.3)$$

which implies the following Propositions:

Proposition 4.2. Every connected locally euclidean topological com(m+k,m)-group (M,f), where $M \cong \mathbb{C}\setminus\mathcal{A}_t$, for some $0\leq t\leq m$, is isomorphic to a topological com(m+k,m)-group (S,f') defined by (4.3). ∎

Proposition 4.3. If Conjecture 4.1 holds, then every connected locally euclidean topological com(m+k,m)-group is isomorphic to a topological com(m+k,m)-group (S,f') defined by (4.3). ∎

Definition 4.1. We will say that a topological com(m+k,m)-group has a **semi canonical form**, if it is defined by (4.3), for some S.

The semi canonical forms have the following important characteristics:

(1) The formulas (4.3) are the same for each locally euclidean topological com(m+k,m)-group (M,f), where $M \cong \mathbb{C}\setminus\mathcal{A}_t$ for some $0\leq t\leq m$, and the com(m+k,m)-groups (M,f) depend only on the choice of the set S. But the set S can not be chosen arbitrarily.

(2) The sets S satisfy the following "property P":

Property P : For each $u\in(C_1)^t\times(C_0)^{m-t}$, there exists a unique element $v_1^m\in S^{(m)}$, such that $u = v_1 * v_2 * \ldots * v_m$.

Of special interest is the following Proposition:

Proposition 4.4. Each subset S of $(C_1)^t\times(C_0)^{m-t}$ with the property P defines a com(m+k,m)-group (S,f), where $f(z_1^{m+k}) = w_1^m$ iff

$z_1 * z_2 * \ldots * z_{m+k} = w_1 * w_2 * \ldots * w_m$.

Proof. The Property P, the associativity and commutativity of $*$ imply that (S,f) is a com(m+k,m)-semigroup. The Property P and the fact that $((C_1)^t \times (C_0)^q, *)$ is a group, imply that (S,f) is a com(m+k,m)-group. ∎

Having all this in mind, we can reduce the examination of all the locally euclidean topological com(n,m)-groups to the examination of all the submanifolds S in $(C_1)^t \times (C_0)^{m-t}$ of complex dimension 1, which satisfy the Property P. As a direct consequence of Property P, it follows that $S^{(m)} \cong (C_1)^t \times (C_0)^{m-t}$. But this is only a necessary condition for S to define a topological com(n,m)-group, and does not say almost anything about the way in which this 1-dimensional complex surface is embedded in $(C_1)^t \times (C_0)^{m-t}$.

The determination of the surface S, is equivalent to the determination of the smooth functions $g_i : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ and $\gamma_i : \mathbb{C} \to \mathbb{C}$, which define the subset S of $(C_1)^t \times (C_0)^{m-t}$ by

$$S = \{ (g_1(z), \ldots, g_t(z), \gamma_1(z), \ldots, \gamma_q(z) \mid z \in \mathbb{C} \}$$

The condition that S satisfies Property P is equivalent to the condition that the system of equations

$$\begin{cases} g_1(z_1) \cdot \ldots \cdot g_1(z_{m+k}) = u_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ g_t(z_1) \cdot \ldots \cdot g_t(z_{m+k}) = u_t \\ \gamma_1(z_1) + \ldots + \gamma_1(z_{m+k}) = u_{t+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \gamma_q(z_1) + \ldots + \gamma_q(z_{m+k}) = u_m \end{cases} \tag{4.4}$$

has a unique solution $z_1^m \in (\mathbb{C} \setminus \mathcal{A}_t)^{(m)}$, for each $u_1^m \in (C_1)^t \times (C_0)^q$. In such a case, the surface S', defined for example by the functions

$$\begin{aligned} g_i' &= g_i \quad \text{for } i=1,3,4,5,\ldots,t, \\ \gamma_j' &= \gamma_j \quad \text{for } j=1,3,4,5,\ldots,q, \\ g_2'(z) &= g_1(z) \cdot g_2(z), \quad \text{and} \\ \gamma_2'(z) &= \gamma_1(z) + \gamma_2(z), \end{aligned}$$

satisfies also Property P. In fact, the meaning of this is that we have chosen a new coordinate system in $(C_1)^t \times (C_0)^{m-t}$. As a special case of all such transformations from one surface to another, we have the following "inner transformation". Namely, for a surface defined by functions g_i and γ_j and a diffeomorphism $\phi: \mathbb{C}\to\mathbb{C}$, the surface S' defined by the functions $g_i' = g_i'\circ\phi$ and $\gamma_j' = \gamma_j\circ\phi$ satisfies also Property P. We said that these kinds of transformations are "inner", because they depend on the choice of coordinates. The inner transformations (i.e. the change of coordinates) induce on the newly parameterized surface S a new com(n,m)-group, but this new group is isomorphic to the old one. All of the above discussion can be stated in the following Theorem:

Theorem 4.5. If Conjecture 4.1. holds, then any two connected, locally euclidean com(n,m)-groups determining the same surface S, are isomorphic. ∎

Because of the above theorem we will be interested in the surfaces S (with Property P) only as sets, forgetting about their parameterizations, since this gives us characterizations of com(n,m)-groups up to isomorphism.

The following Proposition gives a way of constructing new surfaces with Property P from a given one.

Proposition 4.6. (i) Let ϕ be an automorphism of the group $(C_1)^t\times(C_0)^q$, where q=m-t, and let S be a subset of $(C_1)^t\times(C_0)^q$ with Property P. Then the set $S' = \phi(S)$ satisfies also Property P. Moreover, the com(m+k,m)-groups defined via S and S' are isomorphic.

(ii) Let S be a subset of $(C_1)^t\times(C_0)^q$ with Property P. Then, for given $\lambda\in(C_1)^t\times(C_0)^q$, the set S' defined by

$$S' = \lambda * S = \{\lambda * z \mid z\in S\}$$

also satisfies Property P.

Proof. (i) The fact that ϕ is an automorphism implies directly the conclusions.

(ii) For a given $u\in(C_1)^t\times(C_0)^q$ there is a unique $z_1^m\in S^{(m)}$, such that $(\lambda)^{-m}*u = z_1*z_2*\dots*z_m$, where $(\lambda)^{-m}$ is the inverse to $\lambda*\lambda*\dots*\lambda$ in the group $(C_1)^t\times(C_0)^q$. Hence, $u = (\lambda*z_1)*(\lambda*z_2)*\dots*(\lambda*z_m)$, i.e. S' satisfies Property P. ∎

Next, we will examine the system (4.4). The fact that this system has a unique solution $x=z_1^m\in(\mathbb{C}\setminus\mathcal{A}_t)^{(m)}$ for arbitrary $y=u_1^m\in(C_1)^t\times(C_0)^q$

implies that (4.4) is equivalent to the system:

$$\begin{cases} \varphi^1(x) = \theta_1(y) \\ \varphi^2(x) = \theta_2(y) \\ \ldots\ldots\ldots\ldots \\ \varphi^m(x) = \theta_m(y) \\ (z_1-a_1)(z_2-a_1)\ldots(z_m-a_1) \neq 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (z_1-a_t)(z_2-a_t)\ldots(z_m-a_t) \neq 0, \end{cases}$$

where $\theta:(C_1)^t\times(C_0)^q\to\mathbb{C}^m$ is a vector valued function whose component functions are $\theta_i:(C_1)^t\times(C_0)^q\to\mathbb{C}$, and $\mathcal{A}_t=\{a_1,\ldots,a_t\}$.

The inequalities in the above system are equivalent to:

$$\begin{cases} (a_1)^m-(a_1)^{m-1}\cdot\varphi^1(x)+(a_1)^{m-2}\cdot\varphi^2(x)+\ldots+(-1)^m\cdot\varphi^m(x)=r_1(y)\neq 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (a_t)^m-(a_t)^{m-1}\cdot\varphi^1(x)+(a_t)^{m-2}\cdot\varphi^2(x)+\ldots+(-1)^m\cdot\varphi^m(x)=r_t(y)\neq 0, \end{cases}$$

where $r_i(y)$ are functions of y.

The system

$$\begin{cases} (a_1)^m-(a_1)^{m-1}\cdot\varphi^1(x)+(a_1)^{m-2}\cdot\varphi^2(x)+\ldots+(-1)^m\cdot\varphi^m(x)=r_1(y) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (a_t)^m-(a_t)^{m-1}\cdot\varphi^1(x)+(a_t)^{m-2}\cdot\varphi^2(x)+\ldots+(-1)^m\cdot\varphi^m(x)=r_t(y) \end{cases}$$

thought of as a system with unknowns $\varphi^{q+1}(x),\ldots,\varphi^m(x)$ has a unique solution because $\det K = \pm\prod_{1\le i<j\le t}(a_i-a_j)\neq 0$, where $K = \left[(-1)^{q+j}\cdot(a_s)^{t-j}\right]$, $1\le s,j\le t$ is the $t\times t$ matrix for the system.

So, the system (4.4) has the following equivalent form:

$$\begin{cases} \varphi^1(x) = \theta_1(y) \\ \varphi^2(x) = \theta_2(y) \\ \ldots\ldots\ldots\ldots \\ \varphi^q(x) = \theta_q(y) \\ (z_1-a_1)(z_2-a_1)\ldots(z_m-a_1) = r_1(y) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (z_1-a_t)(z_2-a_t)\ldots(z_m-a_t) = r_t(y). \end{cases}$$

Next, the system

$$\begin{cases} \varphi^1(x) = \theta_1(y) \\ \varphi^2(x) = \theta_2(y) \\ \dots\dots\dots\dots \\ \varphi^q(x) = \theta_q(y) \end{cases}$$

is equivalent to the system

$$\begin{cases} z_1 + z_2 + \dots + z_m = r_{t+1}(y) \\ z_1^2 + z_2^2 + \dots + z_m^2 = r_{t+2}(y) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ z_1^q + z_2^q + \dots + z_m^q = r_{t+q}(y) = r_m(y). \end{cases}$$

The above discussion shows that the system (4.4) is equivalent to the system

$$\begin{cases} (z_1-a_1)(z_2-a_1)\dots(z_m-a_1) = r_1(y) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (z_1-a_t)(z_2-a_t)\dots(z_m-a_t) = r_t(y). \\ z_1 + z_2 + \dots\dots + z_m = r_{t+1}(y) \\ z_1^2 + z_2^2 + \dots\dots + z_m^2 = r_{t+2}(y) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ z_1^q + z_2^q + \dots\dots + z_m^q = r_m(y). \end{cases} \tag{4.5}$$

where $\rho: (C_1)^t\times(C_0)^q \to (C_1)^t\times(C_0)^q$, defined by $\rho=(r_1,\dots,r_m)$, is a diffeomorphism.

Now, we are able to give examples of surfaces with Property P.

Let S be the surface defined by the functions

$$\begin{cases} g_i(z) = z - a_i & \text{for } 1\le i\le t \\ \gamma_j(z) = z^j & \text{for } 1\le j\le q=m-t , \end{cases} \tag{4.6}$$

i.e.

$$S = \{(z-a_1,\ z-a_2,\dots,z-a_t,\ z,\ z^2,\dots,z^q) \mid z\in\mathbb{C}\} \subseteq (C_1)^t\times(C_0)^q.$$

The above discussion implies that S has Property P. Next, Proposition 4.6.(ii) implies that for this S, the surface $\lambda*S$, where $\lambda=(\alpha_1,\dots,\alpha_t,\beta_1,\dots,\beta_q)\in(C_1)^t\times(C_0)^q$ has Property P and is defined by the functions

$$\begin{cases} g_i(z) = \alpha_i(z - a_i) & \text{for } 1\le i\le t \\ \gamma_j(z) = z^j + \beta_j & \text{for } 1\le j\le q=m-t . \end{cases} \tag{4.7}$$

The surface $\lambda*S$ will be denoted by S_λ. Thus, S_e is the surface defined by the functions (4.6) where e is the identity element in the group $(C_1)^t\times(C_0)^q$. Next, if we take an automorphism $\phi = (\phi_1,\dots,\phi_m)$ of $(C_1)^t\times(C_0)^q$, for example:

$$\begin{aligned} \phi_i(u_1^t,v_1^q) &= (u_1)^{\alpha_{i1}}\cdot\ldots\cdot(u_t)^{\alpha_{it}} \qquad 1\le i\le t \\ \phi_{t+j}(u_1^t,v_1^q) &= \sum_{1\le s\le q} \beta_{js}\cdot u_s \qquad 1\le j\le q\ , \end{aligned} \tag{4.8}$$

where $\alpha_{is}\in\mathbb{Z}$, $1\le i,s\le t$, with

$$\det\left[\alpha_{is}\right] = \pm 1\ , \tag{4.9}$$

and $\beta_{js}\in\mathbb{C}$, $1\le j,s\le q$, with

$$\det\left[\beta_{js}\right] \ne 0\ . \tag{4.10}$$

Proposition 4.6.(i) implies that the surface defined by the functions

$$\begin{aligned} g_i(z) &= \left[\alpha_1(z-a_1)\right]^{\alpha_{i1}}\cdot\ldots\cdot\left[\alpha_t(z-a_t)\right]^{\alpha_{it}} \qquad 1\le i\le t \\ \gamma_j(z) &= \sum_{1\le s\le q} \beta_{js}\cdot(z^s+\beta_s) \qquad 1\le j\le q\ , \end{aligned} \tag{4.11}$$

also satisfies Property P.

We have the following remarks:

Remark 4.1. For arbitrary $\lambda\in(C_1)^t\times(C_0)^q$ and a surface S given by the functions (4.11), the surface $\lambda\bullet S$ is defined by functions of the same form (4.11) but with different parameters.

Remark 4.2. The automorphisms of form (4.8) do not cover all the automorphisms of $(C_1)^t\times(C_0)^q$. For example, the automorphism $\phi(u_1^t,v_1^q) = (\bar{u}_1^t,v_1^q)$, where $\bar{z}$ denotes the conjugate of z, is not of the form (4.8).

Remark 4.3. If the functions g_i, γ_j from (4.11) are taken as the function g_i, γ_j in (4.3), then, it is possible to transform (4.3) by the following transformations (I), (II) and (III), to a system of the same form (4.3), but whose functions g_i, γ_j are of the form (4.7), of course with different parameters. The transformations (I), (II) and (III) are the following:

(I) The j-th equation is multiplied by the i-th equation, for $1\le i,j\le t$;

(II) The i-th and the j-th equation exchange their places, for $1\le i,j\le t$ or $t+1\le i,j\le m$;

(III) The i-th equation is multiplied by a constant $c\in\mathbb{C}$ and added to the j-th equation, for $t+1 \le i, j \le m$.

Remark 4.4. The surface defined by the functions (4.11) with the same coordinatization of $\mathbb{C}^{(m)}$, defines an affine com(n,m)-group whose characteristic polynomial has t distinct roots of multiplicity 1. Conversely, each such affine com(n,m)-group can be determined by functions of the form (4.11). It will be proved in another paper.

The above discussion led us to the following Conjecture:

Conjecture 4.2. Each connected, locally euclidean topological com(n,m)-group is isomorphic to an affine com(n,m)-group.

Example. Let us consider the special case $m = 2$, $t = 0$. Then, Conjecture 4.2 and Remark 4.4 imply that in order to classify all connected locally euclidean topological com(n,2)-groups on $\mathbb{C}$, it is enough to clarify the ones which are determined by functions (4.11) for $m=2$ and $t=0$. So, let $\lambda = (\alpha, \beta)$. Then S_λ is defined by the functions $\gamma_1(z) = z+\alpha$ and $\gamma_2(z) = z^2+\beta$, and the induced com(3,2)-group $(\mathbb{C}, f)$ is defined by: $f(u_1^3) = w_1^2 \iff \sum_{1\le i\le 3} ((u_i)^s + \alpha) = \sum_{1\le j\le 2} ((w_j)^s + \beta)$ for $s=1,2$. The function $z \to z-\alpha$, determines a com(3,2)-group, $(\mathbb{C}, f)$ given by

$$f(u_1^3) = w_1^2 \iff$$
$$\iff \begin{cases} u_1+u_2+u_3 = w_1+w_2 \\ (u_1)^2+(u_2)^2+(u_3)^2-2\alpha(u_1+u_2+u_3)+\beta-\alpha^2=(w_1)^2+(w_2)^2-2\alpha(w_1+w_2) \end{cases}$$
$$\iff \begin{cases} u_1+u_2+u_3 = w_1+w_2 \\ u_1u_2+u_1u_3+u_2u_3+(\alpha^2-\beta)/2=w_1+w_2 \end{cases},$$

and this is an affine com(3,2)-group on $\mathbb{C}$. If $\beta=\alpha^2$, then we obtain the additive com(3,2)-group from Example I.5.1. If $\beta\neq\alpha^2$ then by the linear transformation $z \to z\cdot\sqrt{\beta-\alpha^2}/\sqrt{2}$ we obtain the com(3,2)-group defined by

$$f(u_1^3) = w_1^2 \Leftrightarrow \begin{cases} \varphi^1(u_1^3)=\varphi^1(w_1^2) \\ \varphi^2(u_1^3)=\varphi^2(w_1^2)+1 \end{cases},$$

and this is the com(3,2)-group from Example I.5.2. We know that these two groups are not isomorphic, and so, by Conjecture 4.2, it follows that up to isomorphism there are exactly two topological com(3,2)-groups, i.e. com(n,2)-groups on $\mathbb{C}$.

REFERENCES

[1] G.Čupona; Vector valued semigroups, Semigroup Forum, Vol. 26 (1983), 65-74.

[2] G.Čupona, D.Dimovski; On a class of vector valued groups; Proc. of Conf. Alg. and Logic, Zagreb, (1984), 29-37.

[3] G.Čupona, N.Celakoski, S.Markovski, D.Dimovski; Vector valued groupoids semigroups and groups; Vector valued semigroups and groups, Skopje, (1987), 1-79.

[4] G.Čupona, A.Samardžisk, N.Celakoski; Fully commmutative vector valued groupoids, Proc. of Conf. "Algebra and Logic", Sarajevo 1987, Novi Sad (1989), 29-42.

[5] G.Čupona, D.Dimovski, A.Samardžiski; Fully commutative vector valued groups, Prilozi, MANU, VIII 2, 1987, (1990) 5-17.

[6] D.Dimovski; Some existence conditions for vector valued groups, God. Zbor. Matem. Fak., 33-34 (99-100), Skopje, (1982-1983), 99-103.

[7] D.Dimovski; Examples of vector valued groups; Prilozi, MANU, VI.2., Skopje, 1985, (1987) 105-114.

[8] D.Dimovski, Free vector valued semigroups; Proc.of Conf. Alg. and Logic, Cetinje, 1986, Novi Sad (1987) 55-62.

[9] D.Dimovski, K.Trenčevski; One-dimensional (4,2)-Lie groups; Vector valued semigroups and groups, MANU, Skopje, (1987), 91-102.

[10] D.Dimovski; Free (n+1,n)-groups; Vector valued semi- groups and groups, MANU, Skopje, (1987), 103-122

[11] D.Dimovski, K.Trenčevski; Nonexistence of continuous (4,3)-groups on ℝ, Proc. of Conf. Alg. and Logic, Sarajevo, 1987, Novi Sad (1989), 49-59.

[12] D.Dimovski, B.Janeva, S.Ilić; Free (n,m)-groups, Communications in Algebra, vol, 19, no. 3, (1991) 965-980.

[13] D.Dimovski; Groups with unique product structures, Journal of Algebra, (1992).

[14] A.Dold; Homology of symmetric products and other functors of complexes, Ann. of Math. 75 (1962), 331-341.

[15] В.Б.Федорчук, О некаториx, геометрических своиствах ковариантних функторов, Усп. Мат. Наук, Т. 39, в. 5(239) (1984), 167-208.

[16] S.D.Liao; On the topology of cyclic products on spheres, Trans. Amer. Math. Soc. 77 (1954) 520-551.551.

[17] H.R.Morton; Symmetric product of the circle, Proc. Camb. Phil. Soc. 63, (1967), 349-352.

[18] М.М.Постников, Групи и алгебри Ли, Наука,Москва (1982).

[19] M.Richardson; On the homology characters of symmetric products, Duke. Math. J. 1, (1935), 50-59.

[20] P.A.Smith; The topology of involutions, Proc. Nat. Acad. Sci. 19, (1933), 612-618.

[21] R.Swan; The homology of cyclic products, Trans. Amer. Math. Soc. 95 (1960), 27-68.

[22] K.Trenčevski; A note on non-existence for some classes of continuous (3,2)-groups, Proc.of Conf. Alg. and Logic, Cetinje, 1986, Novi Sad (1987) 205-208.

[23] K.Trenčevski; Some examples of fully commutative vector valued groups, Prilozi, IX 1-2, MANU, (1988), 27-37.

[24] Б.Трпеновски, Ѓ.Чупона; [m,n]-групоиди, Билтен ДМФ СРМ Скопје, 21, (1970) 19-29.

[25] C.H.Wagner; Symmetric, cyclic and permutation products of manifolds, Dissert. Math. (Rozrawy math.) 182, (1980), 3-48.

I N D E X

NOTATIONS

Р Е З И М Е

Поимите за векторско вредносни групоиди, полугрупи и групи се воведени од проф. Ѓ. Чупона и проф. Б. Трпеновски. Оваа теорија беше развивана и беа добиени интересни резултати од повеќе математичари од Скопје, Нови Сад и Ниш. Еден од главните резултати беше описот на слободните векторско вредносни полугрупи и групи. Од познатите резултати за векторско вредносни структури прозлегува дека овие структури од една страна се слични на обичните алгебарски бинарни структури, но од друга страна тие имаат нови идеи и специфични особини. Еден дел од оваа теорија е теоријата на потполно комутативни векторско вредносни алгебарски структури, и оваа монографија е пред сѐ посветена на овие структури.

Монографијата содржи три глави.

Во првата глава на почетокот се дадени основните дефиниции и познати факти за векторско вредносните групоиди, полугрупи и групи, а потоа се добиени повеќе нови дефиниции, особини и теореми за комутативните векторско вредносни полугрупи и групи.

Познатите нетривијални примери на комутативни векторско вредносни структури се добиени на двоелементни множества и на множества од комплексни броеви $\mathbb{C}$, каде особината дека $\mathbb{C}$ е алгебарски затворено поле е од суштинско значење. Сите овие примери нѐ наведуваат на поимот за афини комутативни полугрупи и групи, и тие се детално проучувани во втората глава. Во истата глава се докажани и повеќе резултати во врска со Лиевите групи, користејќи ја притоа теоријата за комутативни векторско вредносни структури. Овие резултати би било многу комплицирано да се докажуваат без

користење на комутативните векторско вредносни полугрупи и групи, а од друга страна пак тие се користат во проучувањето на таквите полугрупи и групи. Како генерализација на афините комутативни векторско вредносни полугрупи и групи во истата глава се добиени и проучувани проективните полугрупи и групи.

Во третата глава се проучувани векторско вредносните полугрупи и групи дефинирани на тополоки простори и многуобразија, кои ја запазуваат тополошката структура, т.е. се проучувани тополошките комутативни векторско вредносни полугрупи и групи. Специјално, покажано е дека ако е дадена таква векторско вредосна група на многуобразие, тогаш димензијата на многуобразието мора да биде 2. На крајот од третата глава е даден опис на локално евклидовите тополошки комутативни групи.

CONTENTS