

Let us suppose that the given affine $\text{com}(m+k, m)$ -semigroup is given by the matrix $A = [\alpha_{ij}]$ ($i \in \{0, 1, \dots, m\}$, $j \in \{0, 1, \dots, m+k\}$) and let

$$A_i = \begin{bmatrix} \alpha_{0i} & \alpha_{0(i+1)} & \cdots & \alpha_{0(i+m)} \\ \alpha_{1i} & \alpha_{1(i+1)} & \cdots & \alpha_{1(i+m)} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{mi} & \alpha_{m(i+1)} & \cdots & \alpha_{m(i+m)} \end{bmatrix}$$

for $i \in \{0, \dots, k\}$. Then $\bar{\varphi}_{r, a_1, \dots, a_{k-1}}$ can be written in the following form:

$$\bar{\varphi}_{r, a_1, \dots, a_{k-1}} = \sum_{0 \leq j \leq k-1} (A_j \varphi^j + r A_{j+1} \varphi^j)$$

where $\varphi^j = \varphi^j(a_1^{k-1})$. Thus

$$\begin{aligned} (\bar{\varphi}'_r)_{is} &= \left[\sum_{j=0}^{k-1} (A_j + r A_{j+1}) \varphi^j X^{-1} \right]_{is} = \\ &= \sum_{t=0}^m \sum_{j=0}^{k-1} (\alpha_{i(j+t)} + r \alpha_{i(j+t+1)}) \varphi^j X_{ts}^{-1} = \\ &= \sum_{t=0}^m \sum_{j=0}^{k-1} \alpha_{i(j+t)} \varphi^j X_{ts}^{-1} + r \sum_{t=0}^m \sum_{j=0}^{k-1} \alpha_{i(j+t+1)} \varphi^j (X_{(t+1)(s+1)}^{-1} + \lambda h_t) \end{aligned}$$

for $i, s \in \{0, 1, \dots, m\}$, because for arbitrary s the vector $X_{ts}^{-1} - X_{(t+1)(s+1)}^{-1}$ is collinear with h , which follows from the fact that $X^{-1} \in G_h$. We can use the fact that $\sum_{t=0}^m \alpha_{i(j+t+1)} h_t = 0$ because the matrix $A = [\alpha_{ij}]$ satisfies the associativity law, i.e. A determines a $\text{com}(m+k, m)$ -semigroup. Let us denote by B the following $(m+1) \times (m+2)$ matrix $B = [\beta_{is}]$, where $\beta_{is} = \sum_{t=0}^m \sum_{j=0}^{k-1} \alpha_{i(j+t)} \varphi^j X_{ts}^{-1}$, where $i \in \{0, 1, \dots, m\}$ and $j \in \{0, 1, \dots, m+1\}$. Using these facts we obtain finally

$$\begin{aligned} (\bar{\varphi}'_r)_{is} &= \beta_{is} + r \sum_{t=0}^m \sum_{j=0}^{k-1} \alpha_{i(j+t+1)} \varphi^j X_{(t+1)(s+1)}^{-1} = \\ &= \beta_{is} + r \sum_{t=0}^m \sum_{j=0}^{k-1} \alpha_{i(j+t)} \varphi^j X_{t(s+1)}^{-1} = \beta_{is} + r \beta_{i(s+1)}. \end{aligned}$$

Thus $\bar{\varphi}'_r$ has the form (2.9), and the given $\text{com}(m+k, m)$ -semigroup is induced by the $\text{com}(m+1, m)$ -semigroup which is determined by the matrix B .

Finally if (C, f) is an affine $\text{com}(m+k, m)$ -group, then the characteristic polynomial does not have roots and $s=0$. Hence $k^s=1$ and (C, f) is induced by only one affine $\text{com}(m+k, m)$ -group. ■

We notice that the proof of Theorem 2.3 enables us to find all k^S affine $\text{com}(m+k, m)$ -semigroups (groups) which induce the given non-singular affine $\text{com}(m+k, m)$ -semigroup (group). Now we will give another way of finding all those k^S affine $\text{com}(m+1, m)$ -semigroups (groups) if $h_m \neq 0$.

Let us suppose that the given affine $\text{com}(m+k, m)$ -group is given by the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1(m+k)} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{m(m+k)} \end{bmatrix}$$

and let the required affine $\text{com}(m+1, m)$ -group be given by the following matrix:

$$B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \beta_{10} & \beta_{11} & \beta_{12} & \dots & \beta_{1(m+1)} \\ \dots & \dots & \dots & \dots & \dots \\ \beta_{m0} & \beta_{m1} & \beta_{m2} & \dots & \beta_{m(m+1)} \end{bmatrix}$$

Let us put $r_1 = \dots = r_k = 0$ in the equality

$$\bar{\varphi}_{r_1, \dots, r_k} = \bar{\varphi}'_{r_1} \cdot \dots \cdot \bar{\varphi}'_{r_k}$$

Then we obtain

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \beta_{10} & \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ \beta_{m0} & \beta_{m1} & \beta_{m2} & \dots & \beta_{mm} \end{bmatrix}^k = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mm} \end{bmatrix} \quad (2.30)$$

Since $h_m \neq 0$, the matrix on the right hand side of (2.30) is non-singular.

Thus (2.30) is an equality in the group G_h and it has k^S solutions.

Using the fact that $h_m \neq 0$, each matrix solution

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \beta_{10} & \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ \beta_{m0} & \beta_{m1} & \beta_{m2} & \dots & \beta_{mm} \end{bmatrix}$$

of (2.30) can uniquely be extended with one vector-column

$$(0, \beta_{1(m+1)}, \beta_{2(m+1)}, \dots, \beta_{m(m+1)})^T$$

such that the following vector

$$(-\beta_{1(m+1)}, \beta_{1m} - \beta_{2(m+1)}, \dots, \beta_{mm})$$

is collinear with h . Hence we obtain the required matrix B .

11.3. Projective $\text{com}(m+k, m)$ -semigroups and groups

The complex $\text{com}(m+k, m)$ -groupoids (\mathbb{C}, f) can be examined via polynomial mappings

$$\phi : P_{m+k} \rightarrow P_m$$

such that

$$f(z_1^{m+k}) = w_1^m \Leftrightarrow \phi((t-z_1)\dots(t-z_{m+k})) = (t-w_1)\dots(t-w_m).$$

If the coefficients of the polynomial $\phi(\rho)$ (except the coefficient 1 in front of t^m) are linear functions of the coefficients of the polynomial ρ with degree $m+k$, then the complex $\text{com}(m+k, m)$ -groupoid determined in this way was called affine in 11.2. The affine $\text{com}(n, m)$ -group structures can be generalized in the following way. We can consider the polynomial mappings

$$\phi^* : P_{m+k}^* \rightarrow P_m^*$$

instead of the polynomial mappings $\phi : P_{m+k} \rightarrow P_m$ where for a positive integer n ,

$$P_n^* = \{a_0 t^n - a_1 t^{n-1} + \dots + (-1)^n a_n \mid (a_0, a_1, \dots, a_n) \in \mathbb{C}P^n\}$$

where $\mathbb{C}P^n$ is the n -dimensional complex projective space.

Further, if $\rho \in P_n^*$ and $\deg(\rho) = r < n$, then we say that ∞ is a root of the polynomial ρ with multiplicity $n-r$. The converse also holds. Indeed, if the numbers $z_1, \dots, z_n \in \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ which may not be different, be given, then there exists a unique polynomial $\rho \in P_n^*$ such that z_1, \dots, z_n are roots of ρ . Thus a polynomial mapping $\phi^* : P_{m+k}^* \rightarrow P_m^*$ corresponds to each $\text{com}(m+k, m)$ -groupoid on \mathbb{C}^* , and conversely, a $\text{com}(m+k, m)$ -groupoid corresponds to each such mapping.

In general we can write

$$\begin{aligned} \phi^*(a_0 t^{m+k} - a_1 t^{m+k-1} + \dots + (-1)^{m+k} a_{m+k}) \\ = F_0 t^m - F_1 t^{m-1} + F_2 t^{m-2} - \dots + (-1)^m F_m \end{aligned} \quad (3.1)$$

where $F_i, 0 \leq i \leq m$, are functions of a_0, a_1, \dots, a_{m+k} .

Definition 3.1. If F_i ($i=0,1,\dots,m$) are linear functions of a_0, a_1, \dots, a_{m+k} i.e.

$$F_i(a_0, a_1, \dots, a_{m+k}) = \sum_{0 \leq j \leq m+k} \alpha_{ij} a_j \quad (0 \leq i \leq m) \quad (3.2)$$

then the corresponding $\text{com}(m+k, m)$ -groupoid on \mathbb{C}^* which is induced by the polynomial mapping (3.1) is called a **projective** $\text{com}(m+k, m)$ -groupoid. If this groupoid is associative, then it is called a **projective** $\text{com}(m+k, m)$ -semigroup. A $\text{com}(m+k, m)$ -group on a non-empty subset of \mathbb{C}^* which is obtained by removing the singular elements from a projective $\text{com}(n+k, m)$ -group is called a **projective** $\text{com}(m+k, m)$ -group.

Let a $\text{com}(m+k, m)$ -groupoid (semigroup, group) (G, f) be given, and let an arbitrary bijection $\varphi : G \rightarrow G$ be given. Then φ induces a $\text{com}(m+k, m)$ -groupoid (semigroup, group) (G, f') as follows:

$$f'(z_1^{m+k}) = w_1^m \iff f(\varphi^{-1}(z_1), \dots, \varphi^{-1}(z_{m+k})) = (\varphi^{-1}(w_1), \dots, \varphi^{-1}(w_m)).$$

These two groupoids (semigroups, groups) (G, f) and (G, f') are isomorphic. If $G = \mathbb{C}^*$ and (\mathbb{C}^*, f) is a projective $\text{com}(m+k, m)$ -groupoid, then (\mathbb{C}^*, f') may not be a projective $\text{com}(m+k, m)$ -groupoid. In order to study the projective semigroups and groups, the following theorem has an important role.

Theorem 3.1. Let $\varphi : \mathbb{C}^* \rightarrow \mathbb{C}^*$ be an arbitrary bilinear transformation $\varphi(z) = (az+b)/(cz+d)$ ($ad-bc \neq 0$), and let (\mathbb{C}^*, f) be a projective $\text{com}(m+k, m)$ -groupoid. Then the induced $\text{com}(m+k, m)$ -groupoid by φ is projective.

Proof. Suppose that the projective $\text{com}(m+k, m)$ -groupoid is given by the linear functions

$$F_i(a_0, a_1, \dots, a_{m+k}) = \sum_{0 \leq j \leq m+k} \alpha_{ij} a_j \quad (0 \leq i \leq m)$$

and that the induced $\text{com}(m+k, m)$ -groupoid is determined by the functions $F_i(a_0, \dots, a_{m+k})$. We will prove that they are linear functions. Since each bilinear transformation can be represented as a composition of the following transformations 1. $\varphi(z) = 1/z$, 2. $\varphi(z) = z/\lambda$ ($\lambda \neq 0, \lambda \neq \infty$) and 3. $\varphi(z) = z - \lambda$ ($\lambda \neq \infty$), it is sufficient to prove the theorem in each of these special cases. Assume that $z_1, \dots, z_{m+k}, w_1, \dots, w_m \in \mathbb{C}^*$ are such that $f(z_1^{m+k}) = w_1^m$.

1. Let $\varphi(z)=1/z$. We will verify that its inverse transformation $\varphi^{-1}(z) = 1/z$ induces the following mapping:

$$\mathbb{C}P^{m+k} \ni (a_0, a_1, \dots, a_{m+k}) \rightarrow (a_{m+k}, a_{m+k-1}, \dots, a_0) \in \mathbb{C}P^{m+k}. \quad (3.3)$$

First let us suppose that $z_i \neq \infty$ and $z_i \neq 0$ for each $i \in \{0, 1, \dots, m+k\}$. Then $a_0 \neq 0$, $a_{m+k} \neq 0$ and $a_i/a_0 = \varphi^i(z_1^{m+k})$, for $(i \in \{1, \dots, m+k-1\})$. Hence we obtain

$$\begin{aligned} (\varphi^i)' &= \varphi^i(\varphi^{-1}(z_1), \dots, \varphi^{-1}(z_{m+k})) = \varphi^i(z_1^{-1}, \dots, z_{m+k}^{-1}) \\ &= \varphi^{m+k-i}(z_1^{m+k}) / \varphi^{m+k}(z_1^{m+k}) \\ &= (a_{m+k-i}/a_0) / (a_{m+k}/a_0) = (a_{m+k-i}/a_{m+k}), \end{aligned}$$

and thus, in $\mathbb{C}P^{m+k}$:

$$(1, \varphi^1', \varphi^2', \dots, \varphi^{m+k}') = (a_{m+k}, a_{m+k-1}, \dots, a_0).$$

Herewith we have proved that

$$(a_0, a_1, \dots, a_{m+k}) \rightarrow (a_{m+k}, a_{m+k-1}, \dots, a_0)$$

if $z_i \neq \infty, 0$ for each $i \in \{0, \dots, m+k\}$. Now let us suppose that among the numbers z_1, \dots, z_{m+k} there exist exactly r numbers which are equal to ∞ , and the others are not equal to zero. Without loss of generality we can assume that $z_{m+k} = z_{m+k-1} = \dots = z_{m+k-r+1} = \infty$. Then $a_0 = a_1 = \dots = a_{r-1} = 0$, $a_r \neq 0$ and

$$a_{r+i}/a_r = \varphi^i(z_1^{m+k-r}) \quad (i \in \{1, 2, \dots, m+k-r\}).$$

Hence we obtain

$$\begin{aligned} \varphi^i' &= \varphi^i(z_1^{-1}, \dots, z_{m+k-r}^{-1}, 0, \dots, 0) = \varphi^i(z_1^{-1}, \dots, z_{m+k-r}^{-1}) = \\ &= \varphi^{m+k-r-i}(z_1^{m+k-r}) / \varphi^{m+k-r}(z_1^{m+k-r}) = (a_{m+k-i}/a_r) / (a_{m+k}/a_r) \\ &= a_{m+k-i}/a_{m+k} \quad \text{for } i \in \{1, \dots, m+k-r\} \quad \text{and} \\ \varphi^i' &= 0 \quad \text{for } i \in \{m+k-r+1, \dots, m+k\}. \end{aligned}$$

Thus, in $\mathbb{C}P^{m+k}$

$$(1, \varphi^1', \dots, \varphi^{m+k}') = (a_{m+k}, a_{m+k-1}, \dots, a_{m+k-r}, 0, 0, \dots, 0),$$

and in this case we also have

$$(a_0, \dots, a_{m+k}) \rightarrow (a_{m+k}, \dots, a_0).$$

Analogously one can verify this statement in the special case when some of the numbers z_1, \dots, z_{m+k} are zeros. For the sake of simplicity we

skip those special cases like $z_i=0$, $z_i=\infty$, $w_i=0$ or $w_i=\infty$ for some i .

Further we obtain

$$\frac{F'_i(a_0, \dots, a_{m+k})}{F'_0(a_0, \dots, a_{m+k})} = \varphi^i(w_1^m) = \frac{\varphi^{m-i}(\varphi^{-1}(w_1^m)) F_{m-i}(a_{m+k}, \dots, a_0)}{\varphi^m(\varphi^{-1}(w_1^m)) F_m(a_{m+k}, \dots, a_0)}$$

for $i \in \{1, \dots, m\}$. The equality implies that

$$\begin{aligned} F'_i(a_0, \dots, a_{m+k}) &= F_{m-i}(a_{m+k}, \dots, a_0) = \\ &= \sum_{0 \leq j \leq m+k} \alpha_{(m-i)j} a_{m+k-j} = \sum_{0 \leq j \leq m+k} \alpha_{(m-i)(m+k-j)} a_j \end{aligned}$$

for $i \in \{0, 1, \dots, m\}$. Thus F'_i ($i \in \{0, 1, \dots, m\}$) are linear functions of a_0, a_1, \dots, a_{m+k} . Moreover we notice that

$$\alpha'_{ij} = \alpha_{(m-i)(m+k-j)} \quad (3.4)$$

for $i \in \{0, 1, \dots, m\}$ and $j \in \{0, 1, \dots, m+k\}$.

2. Let $\varphi(z) = z/\lambda$ ($\lambda \neq 0$, $\lambda \neq \infty$). It is easy to verify that in an analogous way as the one in the previous case, the inverse transformation $\varphi^{-1}(z) = \lambda z$ induces the following mapping:

$$\mathbb{C}P^{m+k} \ni (a_0, a_1, \dots, a_{m+k}) \rightarrow (a_0, \lambda a_1, \lambda^2 a_2, \dots, \lambda^{m+k} a_{m+k}) \in \mathbb{C}P^{m+k}.$$

Thus we obtain

$$\frac{F'_i(a_0^{m+k})}{F'_0(a_0^{m+k})} = \varphi^i(w_1^m) = \frac{1}{\lambda^i} \varphi^i(\varphi^{-1}(w_1^m)) = \frac{1}{\lambda^i} \frac{F_i(a_0, \lambda a_1, \dots, \lambda^{m+k} a_{m+k})}{F_0(a_0, \lambda a_1, \dots, \lambda^{m+k} a_{m+k})}$$

for $i \in \{1, \dots, m\}$. Hence we obtain

$$F'_i(a_0, a_1, \dots, a_{m+k}) = \lambda^{-i} F_i(a_0, \lambda a_1, \dots, \lambda^{m+k} a_{m+k}) = \lambda^{-i} \sum_{j=0}^{m+k} \alpha_{ij} \lambda^j a_j$$

for $i \in \{0, 1, \dots, m\}$. Thus F'_i are linear functions of a_0, \dots, a_{m+k} and moreover

$$\alpha'_{ij} = \lambda^{j-i} \alpha_{ij} \quad (3.5)$$

for $i \in \{0, 1, \dots, m\}$ and $j \in \{0, 1, \dots, m+k\}$.

3. Let $\varphi(z) = z - \lambda$ ($\lambda \neq \infty$). Analogously to (3.3), the inverse transformation $\varphi^{-1}(z) = z + \lambda$ induces the following mapping:

$$\mathbb{C}P^{m+k} \ni (a_0, a_1, \dots, a_{m+k}) \rightarrow (b_0, b_1, \dots, b_{m+k}) \in \mathbb{C}P^{m+k}$$

where $b_i = \sum_{0 \leq r \leq i} \lambda^r \binom{r+m+k-i}{r} a_{i-r}$ for $i \in \{0, 1, \dots, m+k\}$.

Using this fact, further we obtain

$$\begin{aligned} \frac{F'_i(a_0, \dots, a_{m+k})}{F'_0(a_0, \dots, a_{m+k})} &= \varphi^i(w_1^m) = \sum_{0 \leq r \leq i} (-\lambda)^r \binom{r+m-1}{r} \varphi^{i-r}(\varphi^{-1}(w_1^m)) = \\ &= \sum_{0 \leq r \leq i} (-\lambda)^r \binom{r+m-1}{r} \frac{F_{i-r}(b_0, b_1, \dots, b_{m+k})}{F_0(b_0, b_1, \dots, b_{m+k})} = \\ &= \sum_{0 \leq r \leq i} (-\lambda)^r \binom{r+m-1}{r} \sum_{0 \leq p \leq m+k} \alpha_{(i-r)p} b_p^p / \sum_{0 \leq p \leq m+k} \alpha_{0p} b_p^p \end{aligned}$$

for $i \in \{1, \dots, m\}$. Hence we get

$$\begin{aligned} F'_i(a_0, \dots, a_{m+k}) &= \sum_{r=0}^i (-\lambda)^r \binom{r+m-1}{r} \sum_{p=0}^{m+k} \alpha_{(i-r)p} b_p^p = \\ &= \sum_{r=0}^i (-\lambda)^r \binom{r+m-1}{r} \sum_{p=0}^{m+k} \alpha_{(i-r)p} \sum_{s=0}^p \lambda^s \binom{s+m+k-p}{s} a_{p-s}^{p-s} = \sum_{j=0}^{m+k} \alpha'_{ij} a_j^j \end{aligned}$$

for $i \in \{0, 1, \dots, m\}$, where

$$\alpha'_{ij} = \sum_{0 \leq r \leq i} (-\lambda)^r \binom{r+m-1}{r} \sum_{j \leq p \leq m+k} \alpha_{(i-r)p} \lambda^{p-j} \binom{m+k-j}{p-j}.$$

Putting $p=s+j$, we obtain

$$\alpha'_{ij} = \sum_{0 \leq r \leq i} (-\lambda)^r \binom{r+m-1}{r} \sum_{0 \leq s \leq m+k-j} \alpha_{(i-r)(s+j)} \lambda^s \binom{m+k-j}{s} \quad (3.6)$$

where $i \in \{0, 1, \dots, m\}$ and $j \in \{0, 1, \dots, m+k\}$. This finishes the proof of the theorem. ■

Now we will introduce a characteristic polynomial for a given projective $\text{com}(m+k, m)$ -groupoid on \mathbb{C}^* . Let (\mathbb{C}^*, f) be an arbitrary projective $\text{com}(m+k, m)$ -groupoid, and let $f(z_1^k x_1^m) = y_1^m$. The axiom for solvability reduces to the following. Suppose that $y_1, \dots, y_m \in \mathbb{C}^*$ are the roots of the following polynomial

$$F_0 t^m - F_1 t^{m-1} + F_2 t^{m-2} - \dots + (-1)^m F_m = 0$$

where $(F_0, \dots, F_m) \in \mathbb{C}P^m$, and that $z_1, \dots, z_k \in \mathbb{C}^*$ are the roots of the following polynomial

$$G_0 t^k - G_1 t^{k-1} + G_2 t^{k-2} - \dots + (-1)^k G_k = 0$$

where $(G_0, \dots, G_k) \in \mathbb{C}P^k$. The aim is to find numbers $x_1, \dots, x_m \in \mathbb{C}^*$ as the roots of a polynomial

$$H_0 t^m - H_1 t^{m-1} + H_2 t^{m-2} - \dots + (-1)^m H_m = 0$$

where $(H_0, \dots, H_m) \in \mathbb{C}P^m$. Then the equation $f(z_1^k x_1^m) = y_1^m$ can be written in

the following form:

$$\sum_{0 \leq t \leq n} \sum_{0 \leq r \leq \min(t, m)} \alpha_{it} H_r G_{t-r} = F_i \quad (3.7)$$

for each $i \in \{0, 1, \dots, m\}$ and where $G_{t-r} = 0$ for $t-r > k$. We notice that (3.7) is analogous to (2.26) for the affine $\text{com}(m+k, m)$ -groupoids. The equality (3.7) should be conceived as an equality in $\mathbb{C}P^m$ but not in \mathbb{C}^{m+1} . Thus the axiom of solvability reduces to the requirement that the system (3.7) has the unique solution $(H_0, H_1, \dots, H_m) \in \mathbb{C}P^m$. The main determinant of the linear system (3.7) is

$$\Delta^*(z_1^k) = \begin{vmatrix} \sum_{i=0}^k \alpha_{0i} G_i & \sum_{i=0}^k \alpha_{0(i+1)} G_i & \dots & \sum_{i=0}^k \alpha_{0(i+m)} G_i \\ \sum_{i=0}^k \alpha_{1i} G_i & \sum_{i=0}^k \alpha_{1(i+1)} G_i & \dots & \sum_{i=0}^k \alpha_{1(i+m)} G_i \\ \dots & \dots & \dots & \dots \\ \sum_{i=0}^k \alpha_{mi} G_i & \sum_{i=0}^k \alpha_{m(i+1)} G_i & \dots & \sum_{i=0}^k \alpha_{m(i+m)} G_i \end{vmatrix} \quad (3.8)$$

or shortly,

$$\Delta^*(z_1, \dots, z_k) = \det \left(\sum_{0 \leq i \leq k} \alpha_{j(i+s)} G_i \right) \quad (3.8')$$

where $j, s \in \{0, 1, \dots, m\}$. The polynomial (3.8) is called the **characteristic polynomial** for the considered projective $\text{com}(m+k, m)$ -groupoid (semigroup, group). Thus the axiom of solvability is satisfied iff the polynomial $\Delta^*(z_1, \dots, z_k)$ never becomes zero. The degree of this polynomial with respect to each argument is $\leq m+1$, while the degree of the characteristic polynomial in the affine case was $\leq m$. Specially if $k=1$, the characteristic polynomial can be written in the following form:

$$\Delta^*(z) = \begin{vmatrix} (-z)^{m+1} & (-z)^m & (-z)^{m-1} & \dots & 1 \\ \alpha_{00} & \alpha_{01} & \alpha_{02} & \dots & \alpha_{0(m+1)} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1(m+1)} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{m(m+1)} \end{vmatrix} \quad (3.9)$$

Now we will reveal the connection between the affine and projective group structures. It is given by the following Theorem.

Theorem 3.2. Each affine $\text{com}(m+k, m)$ -groupoid can be treated as a $\text{com}(m+k, m)$ -subgroupoid of a projective $\text{com}(m+k, m)$ -groupoid (\mathbb{C}^*, f^*) where $\infty \in \mathbb{C}^*$ is a singular element. Conversely, let (\mathbb{C}^*, f^*) be an arbitrary

projective $\text{com}(m+k, m)$ -groupoid such that there exists $\xi \in \mathbb{C}^*$ and $(\mathbb{C}^* \setminus \{\xi\}, f^*)$ is a projective $\text{com}(m+k, m)$ -subgroupoid from (\mathbb{C}^*, f^*) . Then the subgroupoid $(\mathbb{C}^* \setminus \{\xi\}, f^*)$ is isomorphic to an affine $\text{com}(m+k, m)$ -groupoid. Besides that, each projective $\text{com}(m+k, m)$ -group is isomorphic to an affine $\text{com}(m+k, m)$ -group.

Proof. Let us suppose that the affine $\text{com}(m+k, m)$ -groupoid (\mathbb{C}, f) is given by the following functions:

$$F_i(a_1, \dots, a_{m+k}) = \alpha_{i0} + \sum_{1 \leq r \leq m+k} \alpha_{ir} a_r \quad (i \in \{1, \dots, m\}).$$

If we put $\alpha_{00} = 1$ and $\alpha_{0i} = 0$ for $i \in \{1, \dots, m+k\}$ we obtain a projective $\text{com}(m+k, m)$ -groupoid (\mathbb{C}^*, f^*) which is given by the following functions:

$$F_i^*(a_0, a_1, \dots, a_{m+k}) = \sum_{0 \leq j \leq m+k} \alpha_{ij} a_j \quad (i \in \{0, \dots, m\}).$$

Now (\mathbb{C}, f) is a $\text{com}(m+k, m)$ -subgroupoid of (\mathbb{C}^*, f^*) because $F_i^*(1, a_1, \dots, a_{m+k}) = F_i(a_1, \dots, a_{m+k})$ and $F_0^*(1, a_1, \dots, a_{m+k}) = 1$. Further we shall prove that $\infty \in \mathbb{C}^*$ is a singular element for (\mathbb{C}^*, f^*) i.e. $\varphi_{\infty, \dots, \infty}$ is not a bijection. Let $f^*(\infty, \dots, \infty, z_1^m) = w_1^m$. In that case $a_0 = a_1 = \dots = a_{k-1} = 0$ and so $F_0^* = a_0 = 0$. Hence at least one of the elements w_1, \dots, w_m is ∞ and $\varphi_{\infty, \dots, \infty}$ is not a bijection. Thus $\infty \in \mathbb{C}^*$ is a singular element for (\mathbb{C}^*, f^*) , and the first part of the theorem is proven. Before we prove the remaining part of the theorem, it is convenient to consider the following:

Remark. We notice that according to the definitions of a characteristic polynomial for affine $\text{com}(m+k, m)$ -groupoids and projective $\text{com}(m+k, m)$ -groupoids, the characteristic polynomials for (\mathbb{C}^*, f^*) and (\mathbb{C}, f) are the same. The characteristic polynomial $\Delta(z_1^k)$ for (\mathbb{C}, f) has degree $\leq m$ with respect to each argument, but the characteristic polynomial $\Delta^*(z_1^k)$ for (\mathbb{C}^*, f^*) has degree $\leq m+1$ with respect to each argument. The above statement gives us the right to consider $\Delta^*(z_1^k)$ as a polynomial in P_{m+1}^* , for each argument. Indeed, if a polynomial of degree $\leq m$ is considered as an element of P_{m+1}^* , then ∞ is a root of it. On the other hand we saw that ∞ is a singular element of (\mathbb{C}^*, f^*) .

Now let us return to the proof of the theorem. Let (\mathbb{C}^*, f^*) be a projective $\text{com}(m+k, m)$ -groupoid and let $\xi \in \mathbb{C}^*$ be such that $(\mathbb{C}^* \setminus \{\xi\}, f^*)$ is subgroupoid of (\mathbb{C}^*, f^*) . We choose an arbitrary bilinear transformation Ψ , such that $\Psi(\xi) = \infty$. According to Theorem 3.1 the $\text{com}(m+k, m)$ -groupoid (\mathbb{C}, f) which is induced by the transformation Ψ is projective, and it is

isomorphic to $(\mathbb{C}^* \setminus \{\xi\}, f^*)$. We will prove that (\mathbb{C}, f) is an affine $\text{com}(m+k, m)$ -groupoid. Indeed, as it is a $\text{com}(m+k, m)$ -groupoid on \mathbb{C} , we can suppose that $a_0 \equiv 1$ and $F_0 \equiv 1$. Since

$$1 \equiv F_0(1, a_1, \dots, a_{m+k}) = \alpha_{00} + \sum_{1 \leq i \leq m+k} \alpha_{0i} a_i,$$

it follows that $\alpha_{00} = 1$ and $\alpha_{0i} = 0$ for $i \in \{1, \dots, m+k\}$, which means that (\mathbb{C}, f) is an affine $\text{com}(m+k, m)$ -groupoid.

Now let a projective $\text{com}(m+k, m)$ -group be given. In fact it is obtained from a non-singular projective $\text{com}(m+k, m)$ -semigroup by removing the singular elements. We will prove that the set of singular elements \mathcal{R} is not empty. We know that

$$\mathcal{R} = \{z \in \mathbb{C}^* \mid \Delta^*(z, \dots, z) = 0\}$$

and $\Delta^*(z, \dots, z)$ is a polynomial of degree $k(m+1)$. Thus $\Delta^*(z, \dots, z) = 0$ has at least one root. Specially, if $\Delta^*(z, \dots, z) = \text{const.}$ then $z = \infty$ is the unique root of it. Hence $\mathcal{R} \neq \emptyset$. Let $\theta \in \mathcal{R}$. We choose a bilinear transformation Ψ such that $\Psi(\theta) = \infty$. This bilinear transformation induces a projective $\text{com}(m+k, m)$ -group on $\mathbb{C}^* \setminus \Psi(\mathcal{R}) \subseteq \mathbb{C}$ which is isomorphic to the given projective $\text{com}(m+k, m)$ -group. So for the induced projective $\text{com}(m+k, m)$ -group we can put $a_0 \equiv 1$ and $F_0 \equiv 1$. Hence we obtain that $\alpha_{00} = 1$ and $\alpha_{0i} = 0$ for $i \in \{1, \dots, m+k\}$, and the induced projective $\text{com}(m+k, m)$ -group is affine. ■

The following theorem gives us the necessary and sufficient condition for a projective $\text{com}(m+k, m)$ -groupoid on \mathbb{C}^* to be a $\text{com}(m+k, m)$ -semigroup.

Theorem 3.3. A projective $\text{com}(m+k, m)$ -groupoid on \mathbb{C}^* which is given by the elements α_{ij} ($i \in \{0, 1, \dots, m\}$, $j \in \{0, 1, \dots, m+k\}$) is associative iff

$$d_s \cdot d_0 = 0, \tag{3.10}$$

for each $s = 1, 2, \dots, k$, where

$$d_s = \begin{bmatrix} \alpha_{0(s-1)} & \alpha_{0s} & \alpha_{0(s+1)} & \cdots & \alpha_{0(s+m)} \\ \alpha_{1(s-1)} & \alpha_{1s} & \alpha_{1(s+1)} & \cdots & \alpha_{1(s+m)} \\ \alpha_{2(s-1)} & \alpha_{2s} & \alpha_{2(s+1)} & \cdots & \alpha_{2(s+m)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{m(s-1)} & \alpha_{ms} & \alpha_{m(s+1)} & \cdots & \alpha_{m(s+m)} \end{bmatrix}$$

and

$$A_0 = \begin{bmatrix} -\alpha_{01} & -\alpha_{02} & \dots & -\alpha_{0(m+k)} \\ \alpha_{00}^{-\alpha_{11}} & \alpha_{01}^{-\alpha_{12}} & \dots & \alpha_{0(m+k-1)}^{-\alpha_{1(m+k)}} \\ \alpha_{10}^{-\alpha_{21}} & \alpha_{11}^{-\alpha_{22}} & \dots & \alpha_{1(m+k-1)}^{-\alpha_{2(m+k)}} \\ \dots & \dots & \dots & \dots \\ \alpha_{m0}^{-\alpha_{(m+1)1}} & \alpha_{m1}^{-\alpha_{(m+1)2}} & \dots & \alpha_{m(m+k-1)}^{-\alpha_{(m+1)(m+k)}} \end{bmatrix}$$

or shortly,

$$\sum_{-1 \leq j \leq m} \alpha_{i(s+j)} (\alpha_{j(r-1)}^{-\alpha_{(j+1)r}}) = 0 \quad (3.10')$$

where $i \in \{0, 1, \dots, m\}$, $r \in \{1, \dots, m+k\}$, $s \in \{1, \dots, k\}$ and $\alpha_{(-1)j} = 0$ for $j \in \{0, 1, \dots, m+k-1\}$.

Proof. According to Proposition 1.3.1 we will prove that the transformations φ_p and φ_q commute for each $p, q \in (\mathbb{C}^*)^{(m)}$ iff (3.10) holds. We know that the transformation φ_p is given by (3.7). Moreover, it is easy to see that it can be written in the following form:

$$A_0 G_0 + A_1 G_1 + \dots + A_k G_k \quad (3.11)$$

where $(G_0, G_1, \dots, G_k) \in \mathbb{C}P^k$ is such a k -tuple that $p = (p_1, \dots, p_k)$ and p_1, \dots, p_k are the roots of the polynomial equation

$$G_0 t^k - G_1 t^{k-1} + \dots + (-1)^k G_k = 0,$$

and

$$A_i = \begin{bmatrix} \alpha_{0i} & \alpha_{0(i+1)} & \dots & \alpha_{0(i+m)} \\ \alpha_{1i} & \alpha_{1(i+1)} & \dots & \alpha_{1(i+m)} \\ \dots & \dots & \dots & \dots \\ \alpha_{mi} & \alpha_{m(i+1)} & \dots & \alpha_{m(i+m)} \end{bmatrix} \quad (3.12)$$

for $i \in \{0, 1, \dots, k\}$.

Two transformations φ_p and φ_q commute iff the matrices $\bar{\varphi}_p = \sum_{0 \leq i \leq k} A_i G_i$ and $\bar{\varphi}_q = \sum_{0 \leq i \leq k} A_i G'_i$ commute. This is satisfied for arbitrary numbers G_i and G'_i ($i \in \{0, 1, \dots, k\}$) iff

$$A_i \cdot A_j = A_j \cdot A_i \quad (3.13)$$

for arbitrary $i, j \in \{0, 1, \dots, k\}$. On the other hand the condition (3.9) can easily be written in the following form:

$$A_{s+1} \cdot A' = A_s \cdot A'' \quad (3.14)$$

for $s \in \{0, 1, \dots, k-1\}$, where

$$A' = \begin{bmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0(m+k-1)} \\ \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1(m+k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m0} & \alpha_{m1} & \cdots & \alpha_{m(m+k-1)} \end{bmatrix}$$

and

$$A'' = \begin{bmatrix} \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0(m+k)} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1(m+k)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{m(m+k)} \end{bmatrix}$$

Further, the equality (3.14) can be written in the following form:

$$A_{s+1} \cdot A_i = A_s \cdot A_{i+1} \quad (3.15)$$

for $1, s \in \{0, 1, \dots, k-1\}$. Thus we only have to prove that the conditions (3.13) and (3.15) are equivalent.

It is easy to see that (3.13) is a consequence of (3.15), because applying the formula (3.15) $|j-i|$ times we obtain (3.13). In order to prove the converse we must use the fact that the matrices A_i are given by (3.12) and hence they are not mutually independent. We put $A_i = [x_i, x_{i+1}, \dots, x_{i+m}]$ ($i \in \{0, 1, \dots, k\}$) where x_j ($j \in \{0, 1, \dots, m+k\}$) is the following vector column

$$x_j = (\alpha_{0j}, \alpha_{1j}, \dots, \alpha_{mj})^T.$$

The formulas (3.13) can now be written in the following form:

$$A_i \cdot x_{j+r} = A_j \cdot x_{i+r}$$

$i, j \in \{0, 1, \dots, k\}$, $r \in \{0, 1, \dots, m\}$, or

$$A_i \cdot x_p = A_j \cdot x_q$$

where $i, j \in \{0, 1, \dots, k\}$, $p, q \in \{0, 1, \dots, m+k\}$ and $i+p=j+q$. Hence we obtain

$$A_{s+1} \cdot x_i = A_s \cdot x_{i+1}$$

for $s \in \{0, 1, \dots, k-1\}$, $i \in \{0, 1, \dots, m+k-1\}$. This implies the equality (3.14) i.e. (3.15) and the proof of the theorem is finished. ■

Similarly to Propositions 1.1 and 1.5, about the affine $\text{com}(n, m)$ -semigroups, one can prove the following two propositions:

Proposition 3.4. If (\mathbb{C}^n, f) is a projective $\text{com}(m+1, m)$ -semigroup, then the induced $\text{com}(m+k, m)$ -semigroup is also projective. ■

Proposition 3.5. Let (\mathbb{C}^*, f) be a $\text{com}(m+1, m)$ -semigroup, such that for each $p \in \mathbb{C}^*$, $\bar{\varphi}_p$ is a linear transformation on $\mathbb{C}P^m$. Then (\mathbb{C}^*, f) is a projective $\text{com}(m+1, m)$ -semigroup. ■

We notice that if we put $\alpha_{01} = \alpha_{02} = \dots = \alpha_{0(m+k)} = 0$ in the theorem 3.3, we obtain as a consequence the associativity law of the affine $\text{com}(m+k, m)$ -groupoids. It can be verified that the condition of associativity (3.10) is invariant under the matrix transformations (3.4), (3.5) and (3.6). Now we will consider some examples of projective groups.

Example 3.1. Let us consider the projective $\text{com}(3, 2)$ -groupoid which is given by the following matrix:

$$A = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}.$$

It can be verified by Theorem 3.3 that the associative law is satisfied, and thus it determines a projective $\text{com}(3, 2)$ -semigroup. Its characteristic polynomial is

$$\Delta^*(z) = \begin{vmatrix} -z^3 & z^2 & -z & 1 \\ -4 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{vmatrix} = -(z-1)^2(z-2).$$

Thus, the only singular points are 1 and 2, and by removing them we obtain a projective $\text{com}(3, 2)$ -group on $\mathbb{C}^* \setminus \{1, 2\}$. The singularity of the points 1 and 2 can directly be verified as follows. Let us consider the system

$$f(z_1, z_2, z) = (w_1, w_2)$$

where $z, w_1, w_2 \in \mathbb{C}$. By the change of variables $z_1 + z_2 = A$ and $z_1 z_2 = B$ this system is equivalent to the system

$$\begin{cases} (B + Az - 5)/(A + z - 4) = \alpha & (\alpha = w_1 + w_2) \\ (Bz - 2)/(A + z - 4) = \beta & (\beta = w_1 w_2) \end{cases} \quad (3.16)$$

i.e. to the system

$$\begin{cases} A(z - \alpha) + B = \alpha z - 4\alpha + 5 \\ A\beta - Bz = 4\beta - z\beta - 2 \end{cases} \quad (3.17)$$

Since the system (3.17) should have a unique solution for A and B, it

should be

$$\begin{vmatrix} z-\alpha & 1 \\ \beta & -z \end{vmatrix} = -z^2 + z\alpha - \beta \neq 0.$$

Let us suppose that z , α and β are chosen such that $z^2 - z\alpha + \beta = 0$. Then, $\beta = z(\alpha - z)$ and by replacing it in (3.14) one obtains the following system:

$$\begin{cases} A(z-\alpha) + B = \alpha(z-4) + 5 \\ Az(z-\alpha) + Bz = (z-4)z(\alpha-z) + 2. \end{cases}$$

This system has infinitely many solutions iff $z^3 - 4z^2 + 5z - 2 = 0$, i.e. $z_{1,2} = 1$ and $z_3 = 2$. Thus the points 1 and 2 are exactly the singular points. If $z \neq 1$ and $z \neq 2$ and $\beta = z(\alpha - z)$, then the system (3.17) does not have solutions in \mathbb{C} , but on the other hand the system (3.16) has the unique solution (A, B) in $\mathbb{C}P^2$ which is given by $A \rightarrow \infty$, $B \rightarrow \infty$ and $B/A = \alpha - z$. Thus there exists a unique pair $(z_1, z_2) \in (\mathbb{C}^*)^{(2)}$ such that $z_1 + z_2 = A$ and $z_1 z_2 = B$, namely $z_1 = \infty$ and $z_2 = \alpha - z$, or $z_1 = \alpha - z$ and $z_2 = \infty$. Similarly one can discuss the case when some of the elements z , w_1 and w_2 are ∞ . Further we choose the following bilinear transformation $\Psi(z) = 1/(z-1)$, such that $\varphi(1) = \infty$, and $\varphi(2) = 1$. One can verify that this transformation induces an affine $\text{com}(3,2)$ -group on $\mathbb{C} \setminus \{1\}$, and which is given by the following matrix:

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

In fact this is the affine $\text{com}(3,2)$ -group from Example 1.5.4. Its characteristic polynomial is

$$\Delta^*(z) = \begin{vmatrix} -z^3 & z^2 & -z & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = z-1$$

and hence the singular points are $z_{1,2} = \infty$ and $z_3 = 1$.

Example 3.2. Now we will consider a class of projective $\text{com}(m+k, m)$ -groups, whose construction appears very naturally.

Let us denote by $\text{PGL}(m; \mathbb{C})$ the set of all non-singular linear transformations on $\mathbb{C}P^m$. In fact, $\text{PGL}(m; \mathbb{C}) \cong \text{GL}(m+1; \mathbb{C}) / \approx$ where \approx is the equivalence relation on $\text{GL}(m+1; \mathbb{C})$ defined by

$$A \approx B \iff (\exists \lambda \in \mathbb{C} \setminus \{0\}) A = \lambda B.$$

The group $PGL(m; \mathbb{C})$ is a complex Lie group and its complex dimension is $(m+1)^2 - 1 = m(m+2)$.

Now let us suppose that A is an arbitrary $(m+1) \times (m+1)$ complex matrix, and let

$$S(A) = S = \{ \lambda \in \mathbb{C}^* \mid A + \lambda I \in PGL(m; \mathbb{C}) \text{ is a non-singular matrix} \},$$

i.e. $S = \mathbb{C}^* \setminus \{ \lambda_1, \dots, \lambda_t \}$ where $\lambda_1, \dots, \lambda_t$ are the eigenvalues of the matrix A . Specially $\omega \in S$ because $A + \omega I \approx I$ is a non-singular matrix. Suppose that the minimal polynomial for A is

$$P(t) = t^{r+1} + a_r t^r + \dots + a_0.$$

It means that the matrices I, A, \dots, A^r are linearly independent, and each of the matrices A^{r+1}, A^{r+2}, \dots can be represented as a linear combination of I, A, \dots, A^r . This leads to a polynomial mapping from the set of all polynomials to the set of polynomials of degree $\leq r$. Besides that, each of the polynomials of degree $\leq r$ is a fixed point for that mapping. This mapping determines a $\text{com}(r+k, r)$ -semigroup on \mathbb{C}^* and $\text{com}(r+k, r)$ -group on S . This $\text{com}(r+k, r)$ -semigroup (group) is induced by $\text{com}(r+1, r)$ -semigroup (group). Thus it is sufficient to study only such $\text{com}(r+1, r)$ -semigroups (groups).

In the case of $\text{com}(r+1, r)$ -semigroup (group), the polynomial

$$b_0 t^{r+1} + b_1 t^r + \dots + b_{r+1}$$

is mapped to

$$\begin{aligned} & -b_0(a_r t^r + \dots + a_0) + b_1 t^r + \dots + b_{r+1} \\ & = t^r(b_1 - b_0 a_r) + t^{r-1}(b_2 - b_0 a_{r-1}) + \dots + (b_{r+1} - b_0 a_0). \end{aligned}$$

Hence, this $\text{com}(r+1, r)$ -semigroup (group) is projective, and it is given by the following matrix:

$$\begin{bmatrix} -a_r & 1 & 0 & 0 & \dots & 0 \\ -a_{r-1} & 0 & 1 & 0 & \dots & 0 \\ -a_{r-2} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \quad (3.18)$$

It is also easy to check the associative law using Theorem 3.3. We notice that this projective $\text{com}(r+1, r)$ -semigroup (group) depends only on the minimal polynomial $P(t)$ but not on the chosen matrix A . Herewith we

obtain a natural class of projective groups which are parametrized by $r+1$ parameters. Now the following question arises. Whether each projective $\text{com}(r+1,r)$ -group is isomorphic to any group of the above class of projective $\text{com}(r+1,r)$ -groups? The answer is "no". Indeed each of these groups have the following property. The polynomial

$$0t^{r+1} + b_1t^r + \dots + b_{r+1}$$

of degree $r+1$ is mapped to the polynomial

$$b_1t^r + \dots + b_{r+1}$$

of degree r . Thus the first polynomial has roots ∞ and z_1, \dots, z_r , while the second has roots z_1, \dots, z_r . Hence there exists an element ξ ($\xi = \infty$ in this case) such that

$$f(\xi, z_1^r) = z_1^r$$

for each $z_1^r \in S^{(r)}$. Then the unit in the universal covering group is (ξ, \dots, ξ) but this is not always the case (see the examples in 1.5.). Conversely, let us suppose that there exists an element ξ in a projective $\text{com}(r+1,r)$ -group such that

$$f(\xi, z_1^r) = z_1^r$$

for each $z_1^r \in S^{(r)}$. Then we choose a bilinear transformation Ψ such that $\Psi(\xi) = \infty$. According to Theorem 3.1 the induced $\text{com}(r+1,r)$ -group is also projective and for it

$$f'(\infty, w_1^r) = w_1^r.$$

Hence each polynomial of degree $\leq r$ is a fixed point for that polynomial mapping, and the corresponding matrix has the form (3.18).

The group G_h was introduced in 11.2 in order to study the affine $\text{com}(n,m)$ -group structures. Now in the projective case we shall introduce a similar group. In Example 3.2 above the group $\text{PGL}(m; \mathbb{C})$ was introduced.

Let a non-zero vector $h = h_0^m \in \mathbb{C}P^{m+1}$ be given. Let us denote by G_h the subset of $\text{PGL}(m; \mathbb{C})$ which consists of all the non-singular matrices

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \dots & \alpha_{0m} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mm} \end{bmatrix} \quad (3.19)$$