Example 8. One can prove that:

$$F_i(a_1, \dots, a_{m+k}) = a_i, \quad \text{for } 1 \le i \le s, \text{ and}$$
$$F_{s+i}(a_1, \dots, a_{m+k}) = a_{s+k+i}, \quad \text{for } s+1 \le i \le m-s.$$

determine a com(m+k,m)-semigroup on $\mathbb{C}$. If $s=m$ we obtain a com(m+k,m)-group on $\mathbb{C}$ ([23]), and we call it additive. If s=0 we obtain com(m+k,m)-group on $\mathbb{C}\setminus\{0\}$ ([23]) and we call it multiplicative. If $0<s<m$ one can verify that the above functions determine a com(m+k,m)-group on $\mathbb{C}\setminus\{0\}$. It lies between the additive and the multiplicative group. Indeed the additive and the multiplicative com(m+k,m)-groups generalize the usual additive and multiplicative groups $(\mathbb{C},+)$ and $(\mathbb{C}\setminus\{0\},\cdot)$, which can be obtained in this way for $m=k=1$.

The examples 9, 10 and 11 give com(3,2)-semigroups on $\mathbb{C}$.

Example 9. $F_1(a_1,a_2,a_3) = a_1+a_2+a_3$, $F_2(a_1,a_2,a_3) = \lambda(a_1+a_2+a_3+1)$.

Example 10. $F_1(a_1,a_2,a_3) = a_3$, $F_2(a_1,a_2,a_3) = \lambda a_3$.

Example 11. $F_1(a_1,a_2,a_3) = a_1+\lambda$, $F_2(a_1,a_2,a_3) = a_1$.

These three examples determine singular com(3,2)-semigroups on $\mathbb{C}$, because the functions F_1 and F_2 are functionally dependent.

Example 12. Let $F_1(a_1,a_2,a_3) = a_1^2-2a_2$. One can verify that in order to obtain a com(3,2)-semigroup it should be

$$F_2(a_1,a_2,a_3) = \frac{1}{2} F_1(a_1,a_2,a_3)(F_1(a_1,a_2,a_3)-1).$$

Since F_1 and F_2 are functionally dependent, this com(3,2)-semigroup on $\mathbb{C}$ is singular. We note in this case that $F_1(a_1,a_2,a_3)$ and $F_2(a_1,a_2,a_3)$ are not affine transformations, while in all of the previous examples $F_1(a_1,a_2,a_3)$ and $F_2(a_1,a_2,a_3)$ were affine transformations.

The following two examples show that if the function $F_1(a_1,a_2,a_3)$ is given, then it is not always possible to find a function $F_2(a_1,a_2,a_3)$ such that they do define a com(3,2)-semigroup on $\mathbb{C}$.

Example 13. Let $F_1(a_1,a_2,a_3) = a_2^2$. Then,

$$F_1^* = (F_2(a_1,a_2,a_3) + z_4a_2^2)^2.$$

In order for this function to be invariant under each permutation of z_1,z_2,z_3 and z_4, it follows that $F_2(a_1,a_2,a_3)+z_4a_2^2$ should also be invariant under these permutations. Since $F_2(a_1,a_2,a_3)$ does not depend on z_4, the expression $F_2(a_1,a_2,a_3) + z_4a_2^2$ should be linear function of each argument z_1,z_2,z_3 and z_4. For $z_4=0$ we obtain that $F_2(a_1,a_2,a_3)$ is a linear function of each argument z_1,z_2 and z_3. Thus, $F_2(a_1,a_2,a_3) + z_4a_2^2$ is not a linear function of each argument z_1,z_2,z_3, z_4. This, leads to a contradiction.

Example 14. Let $F_1(a_1,a_2,a_3) = a_1^2-a_2$. Then

$$F_1^* = (a_1-a_2)^2 + z_4(a_1^2-a_2) + z_4^2 - F_2(a_1,a_2,a_3).$$

In order for F_1^* to be invariant under each permutation of z_1,z_2,z_3 and z_4, it follows that

$$F_1^* = z_1^2 + z_2^2 + z_3^2 + z_4^2 + \phi(z_1,z_2,z_3,z_4)$$

where ϕ is a linear function of each of the arguments z_1,z_2,z_3 and z_4. Thus:

$$\begin{aligned}-F_2(a_1,a_2,a_3) &= z_1^2+z_2^2+z_3^2 - (z_1^2+z_2^2+z_3^2+z_1z_2+z_2z_3+z_3z_1)^2 \\ &- z_4(z_1^2+z_2^2+z_3^2+z_1z_2+z_2z_3+z_3z_1) + \phi(z_1,z_2,z_3,z_4).\end{aligned}$$

Since $F_2(a_1,a_2,a_3)$ does not depend on z_4, $\phi(z_1,z_2,z_3,z_4)$ must contain $z_4(z_1^2+z_2^2+z_3^2+z_1z_2+z_2z_3+z_3z_1)$ as a summand. For the third partial derivative we have:

$$\frac{\partial^3\phi(z_1,z_2,z_3,z_4)}{\partial z_4\,\partial z_1^2} = 1.$$

But, since $\phi(z_1,z_2,z_3,z_4)$ is a linear function of z_1, it follows that for the second partial derivative we have:

$$\frac{\partial^2\phi(z_1,z_2,z_3,z_4)}{\partial z_1^2} = 0\ .$$

This implies that

$$\frac{\partial^3\phi(z_1,z_2,z_3,z_4)}{\partial z_4\,\partial z_1^2} = 0,$$

which leads to a contradiction. Thus, if $F_1(a_1,a_2,a_3)=a_1^2-a_2$, it follows that there does not exist a function $F_2(a_1,a_2,a_3)$ such that it determines a com(3,2)-semigroup on **C**.

II. AFFINE AND PROJECTIVE COM(n,m)-SEMIGROUPS AND GROUPS

II.1. Affine com(m+k,m)-semigrops on $\mathbb{C}$

In this section we will examine a class of com(m+k,m)-semigroups over the field of complex numbers $\mathbb{C}$, obtained from (m+k,m)-groupoids over $\mathbb{C}$.

Let $(\mathbb{C},f)$ be a com(m+k,m)-groupoid over $\mathbb{C}$ obtained from an (m+k,m)-groupoid $(\mathbb{C},g)$ via the symmetric functions I (4.1), as in (4.5), i.e. where $g:\mathbb{C}^{m+k}\to\mathbb{C}^m$ is defined by $g(x)=A\cdot x+b$, via the affine map given by two matrices $A=[\alpha_{ij}]_{m\times(m+k)}$ and $b=[\alpha_{i0}]_{m\times1}$ over $\mathbb{C}$.

Definition 1.1. A com(m+k,m)-groupoid $(\mathbb{C},f)$ as above is said to be an **affine** com(m+k,m)-groupoid over $\mathbb{C}$.

The next Proposition shows that affine com(m+1,m)-semigroups induce affine com(m+k,m)-semigroups.

Proposition 1.1. Let $(\mathbb{C},f)$ be an affine com(m+1,m)-semigroup. Then the induced com(m+k,m)-semigroup is also affine.

Proof. The proof is by induction on k. For $k=1$ $(\mathbb{C},f)$ is the induced com(m+k,m)-semigroup, and so it is affine.

Next, we assume that the induced com(m+k,m)-semigroup $(\mathbb{C},g)$ is affine, and let $(\mathbb{C},h)$ be the induced com(m+k+1,m)-semigroup. Let $uv\in\mathbb{C}^{(m+k+1)}$, for $u\in\mathbb{C}^{(m+k)}$ and $v\in\mathbb{C}$, and let $h(uv)=w$ and $g(u)=z$. Then, $f(zv)=w$, and because $(\mathbb{C},f)$ is affine, it follows that for each $1\le i\le m$, $\varphi^i(w)$ is a linear function of $\varphi^1(zv)$, $\varphi^2(zv)$, ..., $\varphi^{m+1}(zv)$. The inductive assumption implies that for each $1\le i\le m$, $\varphi^i(z)$ is a linear function of $\varphi^1(u)$, $\varphi^2(u)$, ..., $\varphi^{m+k}(u)$, which together with the fact that for each $1\le i\le m$, $\varphi^i(zv)=\varphi^i(z)+v\varphi^{i-1}(z)$, implies that for each $1\le i\le m$, $\wp^i(w)$ is a linear function of $\wp^1(u)$, $\wp^2(u)$, ..., $\wp^{m+k}(u)$, v, $v\wp^1(u)$, $v\wp^2(u)$, ..., and $v\wp^{m+k}(u)$. So, let

$$\wp^i(w) = a_0 + [a_1\wp^1(u) + b_1v] + [a_2\wp^2(u) + b_2v\wp^1(u)] + \dots + \\ + [a_{m+k}\wp^{m+k}(u) + b_{m+k}v\wp^{m+k-1}(u)] + b_{m+k+1}v\wp^{m+k}(u).$$

For $1\le i\le m+k$, let $r_i(uv) = a_i\wp^i(u) + b_iv\wp^{i-1}(u)$. Because $(\mathbb{C},h)$ is a com(m+k+1,m)-semigroup, it follows that $\wp^i(w)$ does not depend on the ordering of u and v, and so, r_i does not depend on the ordering of u and v. If $u=(1,1,\dots,1)$ and $v=0$, then $r_i(uv)=a_i\binom{m+k}{i}$, and $r_i(vu)=a_i\binom{m+k-1}{i} + b_i\binom{m+k-1}{i-1}$. But, since $r_i(uv) = r_i(vu)$, it follows that for each $1\le i\le m+k$, $a_i\binom{m+k}{i} = a_i\binom{m+k-1}{i} + b_i\binom{m+k-1}{i-1}$, i.e. $a_i\binom{m+k-1}{i-1} = b_i\binom{m+k-1}{i-1}$. Hence, for each $1\le i\le m+k$, $a_i=b_i$. This implies that for each $1\le i\le m$, $\wp^i(w)$ is a linear function of $\wp^1(uv), \wp^2(uv), \dots, \wp^{m+k+1}(uv)$, which is equivalent to $(\mathbb{C},h)$ being affine. ∎

We recall some details from the definition of affine com(n,m)-groupoids over $\mathbb{C}$. Let $(\mathbb{C},f)$ be an affine com(n,m)-groupoid, $n-m=k\ge 1$, obtained from $(\mathbb{C},g)$ via the symmetric functions I (4.1), where $g(x)=A\cdot x+b$, as above. Then, $f(x)=y$ iff $\wp_m(y)=g\circ\wp_n(x)$. If we denote the component maps of g by $g_1, g_2, \dots, g_m$, where each $g_i:\mathbb{C}^n\to\mathbb{C}$, then the fact that $g(x)=A\cdot x+b$ implies that for $a_1^n\in\mathbb{C}^n$,

$$g_i(a_1^n) = \alpha_{i0} + \sum_{1\le r\le n} \alpha_{ir}a_{ir}. \tag{1.1}$$

Above, α_{ij} are given complex numbers for $1\le i\le m$ and $0\le j\le n$. It is technically useful to have α_{ij} defined for some other indices. So we define $\alpha_{00}=1$, and $\alpha_{ij}=0$ for all other indices. These new numbers do not play any role in the definition of the map g, i.e. of the maps g_i, and so on to the map f. We recall the Theorem I.4.4, which says that the affine com(n,m)-groupoid $(\mathbb{C},f)$ is a com(n,m)-semigroup iff for each $1\le i\le m$, $1\le s\le k$ and $1\le j\le n$,

$$\sum_{0\le t\le m} \alpha_{i(s+t)}(\alpha_{t(j-1)} - \alpha_{(t+1)j}) = 0, \tag{1.2}$$

i.e. iff for each $1\le s\le k$,

$$A_s\cdot A_0 = A_s\cdot A_{m+1}, \tag{1.3}$$

where $A_s=[\alpha_{i(s+j)}]_{m\times(m+1)}$, $1\le i\le m$, $0\le j\le m$; $A_0=[\alpha_{rt}]_{(m+1)\times n}$ and $A_{m+1}=[\alpha_{(r+1)(t+1)}]_{(m+1)\times n}$, $0\le r\le m$, $0\le t\le n-1$.

We denote the vector rows of the **matrices** A_s by u_{is}, i.e., for each $1 \le i \le m$ and $1 \le s \le k$,

$$u_{is} = (\alpha_{is}, \alpha_{i(s+1)}, \dots, \alpha_{i(s+m)}). \tag{1.4}$$

The vector columns of the matrix A_0 are denoted by v_i, **and** the vector columns of the matrix A_{m+1} are denoted by w_i. Thus,

$$v_i = (\alpha_{0(i-1)}, \alpha_{1(i-1)}, \dots, \alpha_{m(i-1)}), \quad 1 \le i \le n, \tag{1.5}$$

$$w_i = (\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{mi}, 0), \quad 1 \le i \le n. \tag{1.6}$$

With the above notations, all of the vectors u_{ij}, v_i and w_i are $(m+1)$-dimensional, i.e. $u_{ij}, v_i, w_i \in \mathbb{C}^{m+1}$. Moreover,

$$A_s = [(u_{1s})^T (u_{2s})^T \dots (u_{ms})^T]^T,$$

$$A_0 = [v_1 \; v_2 \dots v_m] \text{ and } A_{m+1} = [w_1 \; w_2 \dots w_m],$$

where the upper script T, denotes the transpose of the matrix under consideration. Thus, the transpose $(u_{ij})^T$ of a vector row, is a vector column. The condition (1.3) can now be expressed as:

$$u_{is} \cdot v_j = u_{is} \cdot w_j, \text{ i.e. as } u_{is} \cdot (v_j - w_j) = 0, \tag{1.7}$$

where $\cdot$ denotes the scalar products of vectors. In other words, the affine com(n,m)-groupoid is a com(n,m)-semigroup iff all of the vectors u_{is} are orthogonal to all of the vectors $v_j - w_j$. Because of this, we will examine the vector subspaces of $\mathbb{C}^{m+1}$, generated by these vectors. Denote by $\mathcal{T}$ the vector subspace of $\mathbb{C}^{m+1}$ generated by the $m \cdot k$ vectors u_{is}, $1 \le i \le m$, $1 \le s \le k$, and by $\mathcal{L}$ the vector subspace of $\mathbb{C}^{m+1}$ generated by the $m+k$ vectors $v_j - w_j$, $1 \le j \le n$. With this notation, the associativity condition (1.3) **can be expressed as: the subspaces** $\mathcal{T}$ and $\mathcal{L}$ are orthogonal, i.e. for each $x \in \mathcal{T}$ and each $y \in \mathcal{L}$, $x \cdot y = 0$, where $x \cdot y$ denotes the scalar product of x and y, i.e. if $x = (a_0, a_1, \dots, a_m)$ and $y = (b_0, b_1, \dots, b_m)$, $a_i, b_i \in \mathbb{C}$, then $x \cdot y = \sum_{1 \le j \le m} a_j \cdot b_j$.

We point out that $\mathcal{L}$ is not a zero subspace of $\mathbb{C}^{m+1}$. Namely, if $\mathcal{L} = \{0\}$, then all of the $m+k$ vectors $v_i - w_j = 0$. Because the first m vectors $v_i - w_i = 0$, $1 \le i \le m$, it follows that $\alpha_{00} - \alpha_{11} = 0$, $\alpha_{10} - \alpha_{21} = 0$, $\dots$, $\alpha_{m0} - \alpha_{(m+1)1} = 0$, $\dots$, $\alpha_{0(n-1)} - \alpha_{1n} = 0$, $\alpha_{1(n-1)} - \alpha_{2n} = 0$, $\dots$, $\alpha_{m(n-1)} - \alpha_{(m+1)n} = 0$, which together with the convention that $\alpha_{00} = 1$, implies that $1 = \alpha_{00} = \alpha_{11}$

$= \alpha_{22} = \dots = \alpha_{mm}$. But then the $(m+1)$-th vector $v_{(m+1)}-w_{(m+1)}$ is not equal to the 0 vector, because $\alpha_{mm}-\alpha_{(m+1)(m+1)} = 1-0 \neq 0$ and $v_{(m+1)}-w_{(m+1)} = (\alpha_{om}-\alpha_{1(m+1)}, \dots, \alpha_{mm}-\alpha_{(m+1)(m+1)})$.

The above discussion shows that $\dim\mathcal{L}\geq 1$ and $\dim\mathcal{T}\leq m$, where dim is the vector-space dimension. In accordance with the dimension of $\mathcal{T}$ being $\leq m$ or being $= m$ we have the following definition:

Definition 1.2. A given affine com(n,m)-semigroup is called **regular**, if the associated subspace $\mathcal{T}$ has $\dim\mathcal{T}=m$, and is called **irregular** if $\dim\mathcal{T}\leq m-1$.

For us, the regular affine com(n,m)-semigroup are of special interest, and because of this, we will examine them in details. If $(\mathbb{C},f)$ is regular, then $\dim\mathcal{L}\leq 1$, and because always $\dim\mathcal{L}\geq 1$, we have that $\dim\mathcal{L}=1$. Because $\dim\mathcal{L}=1$, it follows that there exists a non-zero vector

$$h=(h_0, h_1, \dots, h_m)\in\mathbb{C}^{m+1}, \tag{1.8}$$

unique up to a multiplication with a non-zero scalar, such that the subspace $\mathcal{L}$ is generated by h.

Definition 1.3. We say that the vector h defined above, is the **characteristic vector** for the regular affine com(n,m)-semigroup $(\mathbb{C},f)$.

For a given affine com(n,m)-semigroup $(\mathbb{C},f)$ determined via the $m\times n$ matrix $A=[\alpha_{ij}]$ and each $1\leq s\leq k=n-m$, we define a vector $h^s\in\mathbb{C}^{m+1}$ by:

$$h^s=(\det D_{s0}, -\det D_{s1}, \dots, (-1)^m\det D_{sm}), \tag{1.9}$$

where det denotes the **determinant**, and D_{sj} is the $m\times m$ matrix obtained from the matrix A_s by removing the j-th column, i.e. $D_{sj}=[\alpha_{i(s+t)}]_{m\times m}$, $1\leq i\leq m$, $0\leq t\leq m$ and $t\neq j$.

Proposition 1.2. For an affine regular com(n,m)- semigroup $(\mathbb{C},f)$, the vectors $h, h^0, h^1, h^2, \dots, h^m$ are collinear.

Proof. It is enough to show that for each s, h and h^s are collinear. If h^s is the zero vector then h^s and h are collinear. So we assume that h^s is a non-zero vector. If we denote the standard orthonormal basis vectors for the complex vector space $\mathbb{C}^{m+1}$, by e_1, e_2, $\dots$, e_{m+1}, then h^s can be expressed as $\det E_s$, where E_s is the

$(m+1)\times(m+1)$ matrix obtained from A_s by adding a new row $e_1\ e_2\ \dots\ e_{m+1}$ as the first row, i.e.

$$h^s = \det\begin{bmatrix} e_1 & e_2 & \cdots & e_{m+1} \\ \alpha_{1s} & \alpha_{1(s+1)} & \cdots & \alpha_{1(s+m)} \\ \alpha_{2s} & \alpha_{2(s+1)} & \cdots & \alpha_{2(s+m)} \\ \dots & \dots & \dots & \dots \\ \alpha_{ms} & \alpha_{m(s+1)} & \cdots & \alpha_{m(s+m)} \end{bmatrix}. \tag{1.10}$$

For each $1\le i\le m$, the scalar product $h^s\cdot u_{is}$, where u_{is} is defined by (1.4), can be expressed as

$\alpha_{is}\det D_{s0}-\alpha_{i(s+1)}\det D_{s1}+\dots+(-1)^m\alpha_{i(s+m)}\det D_{sm}$, i.e. as

$$h^s\cdot u_{is} = \det\begin{bmatrix} \alpha_{is} & \alpha_{i(s+1)} & \cdots & \alpha_{i(s+m)} \\ \alpha_{1s} & \alpha_{1(s+1)} & \cdots & \alpha_{1(s+m)} \\ \alpha_{2s} & \alpha_{2(s+1)} & \cdots & \alpha_{2(s+m)} \\ \dots & \dots & \dots & \dots \\ \alpha_{ms} & \alpha_{m(s+1)} & \cdots & \alpha_{m(s+m)} \end{bmatrix}. \tag{1.11}$$

This determinant is 0 because it has two equal rows. Hence, h^s is orthogonal to each of the vectors u_{is}, and so it belongs to the vector subspace $\mathcal{L}$. Because the **com(n,m)-semigroup** is regular, it follows that $\mathcal{L}$ is generated by the vector h, which implies that h and h^s are collinear. ∎

The following example shows that, in general, in a regular affine com(n,m)-semigroup it is possible for all the vectors $h^1, \dots, h^k$ to be the zero vectors.

Example. Let $(\mathbb{C},f)$ be the affine com(5,3)-groupoid defined by the matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \alpha_{10} \\ \alpha_{20} \\ \alpha_{30} \end{bmatrix}$$

For this affine com(5,3)-groupoid we have:

$u_{11}=(1,0,0,0)$, $u_{12}=u_{31}=(0,0,0,0)$, $u_{21}=(0,1,1,1)$, $u_{22}=(1,1,1,0)$, $u_{32}=(0,0,0,1)$, $v_1=(1,\alpha_{10},\alpha_{20},\alpha_{30})$, $v_2=(0,1,0,0)$, $v_3=(0,0,1,0)$, $v_4=(0,0,1,0)$, $v_5=(0,0,1,0)$, $w_1=(1,0,0,0)$, $w_2=(0,1,0,0)$, $w_3=(0,1,0,0)$, $w_4=(0,1,0,0)$, $w_5=(0,0,1,0)$,

$$A_1=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_2=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$h^1 = \det\begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (0,0,0,0) = \det\begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = h^2 .$$

In order for this affine com(5,3)-groupoid to be a com(5,3)-semigroup, it is necessary for all of the vectors u_{ij} to be orthogonal to all of the vectors v_t-w_t, which is satisfied for $\alpha_{10}=-\alpha_{20}$ and $\alpha_{30}=0$. But then, $\dim\mathcal{T}=3$ and $\dim\mathcal{L}=1$, where $\mathcal{L}$ is generated by the characteristic vector $h=(0,1,-1,0)$. Hence $(\mathbb{C},f)$ is a regular affine com(5,3)-semigroup such that $h^1=h^2=(0,0,0,0)$.

In the above example $k=2$. It is possible to construct such examples for all $k\geq 2$, but not for $k=1$ which is shown by the following Proposition:

Proposition 1.3. In the case $k=1$, $n=m+1$, an affine com(n,m)-semigroup is regular iff the unique h^1 is a non-zero vector.

Proof. In the case $k=1$, we have only the vectors $u_{i1}=(\alpha_{i1}, \alpha_{12}, \dots, \alpha_{1(m+1)})$, $1\leq i\leq m$. The affine com(m+1,m)-semigroup is regular iff $\dim\mathcal{T}=m$ i.e. iff the vectors $u_{11}, u_{21}, \dots, u_{m1}$ are linearly independent. They are linearly independent iff

$$\det E_1 = \begin{vmatrix} e_1 & e_2 & \cdots & e_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{vmatrix} = h^1 \neq 0 \quad . \blacksquare$$

For an affine com(n,m)-semigroup $(\mathbb{C},f)$ defined by the matrices $A_{m\times n}$ and $b_{m\times 1}$ we introduce the following notions:

(1) For each $x\in\mathbb{C}^{(k)}$, let $F(x)$ be the $n\times m$ matrix

$$F(x) = \left[\ \wp^{j-s}(x)\right], \quad 1\leq j\leq n, \quad 1\leq s\leq m, \qquad (1.12)$$

where $\wp^j(x)=0$ for negative j; and

(2) For each $x \in \mathbb{C}^{(k)}$ let $F_0(x)$ be the $n \times 1$ matrix

$$F_0(x) = \left[\wp^j(x) \right], \quad 1 \le j \le n . \tag{1.13}$$

Until the end of this section we will be concerned mainly with the transformations φ_x defined in I.(3.1).

Proposition 1.4. If $(\mathbb{C}, f)$ is an affine com(n,m)-semigroup, $n=m+k$, then for each $r \in Q^{(k)}$, the transformation $\bar{\varphi}_r$ is an affine transformation on $\mathbb{C}^m$ given by:

$$\bar{\varphi}_r(x) = A \cdot (F(r) \cdot x + F_0(r)) + b . \tag{1.14}$$

We denote $A \cdot F(r)$ by $A(r)$ and $b + A \cdot F_0(r)$ by $b(r)$. With this notation,

$$\bar{\varphi}_r(x) = A(r) \cdot x + b(r) . \tag{1.15}$$

Proof. From the definitions of φ_r and $\bar{\varphi}_r$ we have:

$$\bar{\varphi}_r(x) = \mathcal{P}_m \circ \varphi_r \circ \mathcal{P}_m^{-1}(x) = \mathcal{P}_m \circ f(r, \mathcal{P}_m^{-1}(x)) = \mathcal{P}_m \circ \mathcal{P}_m^{-1}(A \cdot \mathcal{P}_n(r, \mathcal{P}_m^{-1}(x)) + b) =$$
$$= A \cdot \mathcal{P}_n(r, \mathcal{P}_m^{-1}(x)) + b.$$

Next, we will express the $n \times 1$ vector column $\mathcal{P}_n(r, \mathcal{P}_m^{-1}(x))$ as the product $F(r) \cdot x + F_0(r)$. The j-th component of this vector is:

$$\wp^j(r, \mathcal{P}_m^{-1}(x)) = \sum_{0 \le s \le m} \wp^{j-s}(r) \wp^s \mathcal{P}_m^{-1}(x)) = \sum_{1 \le s \le m} \wp^{j-s}(r) \cdot x_s + \wp^j(r),$$

which is exactly the j-th component of $F(r) \cdot x + F_0(r)$. ∎

Theorem 1.5. Let $(\mathbb{C}, f)$ be a com(m+1,m)-semigroup such that for each $z \in \mathbb{C}$, $\bar{\varphi}_z$ is an affine transformation on $\mathbb{C}^m$. Then, $(\mathbb{C}, f)$ is an affine com(m+1,m)-semigroup.

Proof. For each $z \in \mathbb{C}$ let $\bar{\varphi}_z(x) = A(z) \cdot x + b(z)$, for $x \in \mathbb{C}^m$, where $A(z) = \left[a_{ij}(z) \right]$ is an $m \times m$ matrix and $b(z) = \left[b_{i0}(z) \right]$, is an $m \times 1$ matrix, whose entries depend on z. Let $g: \mathbb{C}^{m+1} \to \mathbb{C}^m$ be the transformation defined by $g = \mathcal{P}_m \circ f \circ \mathcal{P}_m^{-1}$. We will show that g is an affine transformation, which will imply that $(\mathbb{C}, f)$ is an affine com(m+1,m)-semigroup.

First we will show that $a_{ij}(z)$ and $b_{i0}(z)$ are linear functions on z. For each $2 \le s \le m$, let $v_s \in \mathbb{C}^{m-1}$ be the element $v_s = (-1, 1, -1, 1, \ldots, (-1)^{s-1}, 0, 0, \ldots, 0)$, i.e $v_s = ((-1)^1, (-1)^2, \ldots, (-1)^{s-1}, 0, 0, \ldots, 0)$, and let $v_1 = (0, 0, 0, \ldots, 0)$. For each $1 \le s \le m$, let

$u_s = \mathcal{S}_{m-1}^{-1}(v_s)$. Then, for each $1 \le i \le m$, and each $x \in \mathbb{C}$, $\mathcal{S}^i(x, u_s) = x \cdot \mathcal{S}^{i-1}(u_s) + \mathcal{S}^i(u_s)$, which implies that for each $1 \le s \le m$,

$$\mathcal{S}_m(1, u_s) = (\underbrace{0,\ 0,\ 0,\ \dots,\ (-1)^{s-1}}_{s},\ 0,\ \dots,\ 0), \quad \text{and}$$

$$\mathcal{S}_m(x, u_s) = (\underbrace{x-1, -(x-1), x-1, \dots, (-1)^{s-2}(x-1), (-1)^{s-1} \cdot x}_{s}, 0, \dots, 0).$$

We denote the i-th component of $\bar{\varphi}_z$ by $\bar{\varphi}_z^i$. Then for each $z \in \mathbb{C}$,

$$\bar{\varphi}_z^i(\mathcal{S}_m(1, u_s)) = \bar{\varphi}_z^i(0, \dots, 0, (-1)^{s-1}, 0, \dots, 0) =$$
$$= (-1)^{s-1} \cdot a_{is}(z) + b_{io}(z).$$

On the other side, since $\bar{\varphi}_z = \mathcal{S}_m \cdot \varphi_z \cdot \mathcal{S}_m^{-1}$ and $\varphi_z(1, u_s) = f(z, 1, u_s) = f(1, z, u_s) = \varphi_1(z, u_s)$, we have that, $\bar{\varphi}_z(\mathcal{S}_m(1, u_s)) = \mathcal{S}_m \circ \varphi_z \circ \mathcal{S}_m^{-1} \circ \mathcal{S}_m(1, u_s) = \mathcal{S}_m(\varphi_z(1, v_s)) = \mathcal{S}_m(\varphi_1(z, v_s) = \mathcal{S}_m \circ \varphi_1 \circ \mathcal{S}_m^{-1}(\mathcal{S}_m(z, u_s) = \bar{\varphi}_1 \circ \mathcal{S}_m(z, u_s)$. Thus, for each $1 \le i, s \le m$, $(-1)^{s-1} \cdot a_{is}(z) + b_{io}(z) = \bar{\varphi}_z^{\ i}\ (\mathcal{S}_m(1, u_s)) = \bar{\varphi}_1 \circ \mathcal{S}_m(z, u_s)$ $= \sum_{1 \le j \le s-1} (-1)^{j-1} a_{ij}(1) \cdot (z-1) + (-1)^{s-1} a_{is}(1) \cdot z + b_{i0}(1)$, i.e.

$$a_{is}(z) = \sum_{1 \le j \le s} (-1)^{j-1} a_{ij}(1)(z-1) + (-1)^{s-1} a_{is}(1) + b_{i0}(1) - b_{i0}(z). \tag{1.16}$$

Next, using the facts that
$\mathcal{S}_m(0,\ 0, \dots,\ 0) = (0,\ 0,\ \dots,\ 0)$ and $\mathcal{S}_m(z,\ 0, \dots,\ 0) = (z,\ 0, \dots,\ 0)$, we have that

$$b_{i0}(z) = \bar{\varphi}_z^i(0,\ 0, \dots,\ 0) = \bar{\varphi}_z^i(\mathcal{S}_m(0, \dots, 0))$$
$$= \bar{\varphi}_0^i(\mathcal{S}_m(z, 0, \dots, 0)) = \bar{\varphi}_0^i(z, 0, \dots, 0) = a_{i1}(0) \cdot z + b_{i0}(0).$$

This, together with (1.16) and the equality $b_{i0}(1) = a_{i1}(1) \cdot 1 + b_{i0}(0)$, implies that for each $z \in \mathbb{C}$ and each $1 \le i, s \le m$,

$$a_{is}(z) = \gamma_{is} \cdot (z-1) + a_{is}(1)$$

where γ_{is} are the constants

$$\gamma_{is} = \sum_{1 \le j \le s} (-1)^{s+j} a_{ij}(1) - a_{i1}(0)\ . \tag{1.17}$$

From the definition of γ_{ij} it follows that

$$\gamma_{i(s+1)} = -\gamma_{is} + a_{i(s+1)}(1). \tag{1.18}$$

Now, let $z_1^{m+1} \in \mathbb{C}^{m+1}$, let $\mathcal{S}_{m+1}^{-1}(z_1^{m+1}) = uv$, where $u \in \mathbb{C}$ and $v \in \mathbb{C}^{(m)}$. Let g_i be the i-th component of the transformation g. The equality $g(z_1^{m+1})=\mathcal{g}_m\circ f\circ\mathcal{g}_{m+1}^{-1}(z_1^{m+1})= \mathcal{g}_m(f(uv)) =\mathcal{g}_m(\varphi_u(v))=\bar{\varphi}_u(v)$, implies that for each $1\le i\le m$,

$$\begin{aligned}
g_i(z_1^{m+1}) &= \bar{\varphi}_u^{i}(\mathcal{g}_m(v)) = \sum_{1\le s\le m} a_{is}(u)\cdot\mathcal{g}^s(v) + b_{i0}(u) = \\
&= \sum_{1\le s\le m}(\gamma_{is}\cdot u - \gamma_{is} + a_{is}(1))\cdot\mathcal{g}^s(v) + b_{s0}(v) = \\
&= \sum_{1\le s\le m}\gamma_{is}\cdot u\cdot\mathcal{g}^s(v) - \sum_{1\le s\le m}\gamma_{is}\cdot\mathcal{g}^s(v) + \sum_{1\le s\le m} a_{is}(1)\cdot\mathcal{g}^s(v) + b_{i0}(u) = \\
&= \sum_{1\le s\le m}\gamma_{is}\cdot u\cdot\mathcal{g}^s(v) + \sum_{1\le s\le m}\gamma_{is}\cdot\mathcal{g}^{s+1}(v) - \sum_{1\le s\le m}\gamma_{is}\cdot\mathcal{g}^{s+1}(v) - \\
&\quad - \sum_{1\le s\le m}\gamma_{is}\cdot\mathcal{g}^s(v) + \sum_{1\le s\le m} a_{is}(1)\cdot\mathcal{g}^s(v) + b_{i0}(u) = \\
&= \sum_{1\le s\le m}\gamma_{is}\cdot(u\cdot\mathcal{g}^s(v)+\mathcal{g}^{s+1}(v)) - \sum_{1\le s\le m-1}(\gamma_{is}+\gamma_{i(s+1)})\cdot\mathcal{g}^{s+1}(v) - \\
&\quad - \gamma_{im}\cdot\mathcal{g}^{m+1}(v) - \gamma_{i1}\cdot\mathcal{g}^1(v) + \sum_{1\le s\le m-1} a_{i(s+1)}(1)\cdot\mathcal{g}^{s+1}(v) = \\
&= \sum_{1\le s\le m}\gamma_{is}\cdot\mathcal{g}^{s+1}(uv) - \sum_{1\le s\le m-1}(\gamma_{is}+\gamma_{i(s+1)}-a_{i(s+1)}(1))\cdot\mathcal{g}^{s+1}(v) \\
&\quad - (\gamma_{i1}-a_{i1}(1))\cdot\mathcal{g}^1(v) + a_{i0}(0)\cdot u + b_{i0}(0) = \\
&= \sum_{1\le s\le m}\gamma_{is}\cdot x_{s+1} + a_{i0}(0)\cdot x_1 + b_{i0}(0).
\end{aligned}$$

To get the last line above, we have used the facts that $\mathcal{g}^{s+1}(uv)=z_{s+1}$, $\gamma_{i1}=a_{i1}(1) - a_{i1}(0)$, and $\mathcal{g}^1(v)+u = \mathcal{g}^1(uv) = z_1$. If we set $\alpha_{ij} = \gamma_{i(j-1)}$ for $2\le j\le m+1$, $\alpha_{i1} = a_{i0}(0)$, and $\beta_{i0} = b_{i0}$, we obtain that for each $1\le i\le m$ and each $z_1^{m+1}\in\mathbb{C}^{m+1}$:

$$g_i(z_1^{m+1}) = \sum_{1\le j\le m+1}\alpha_{ij}\cdot z_j + \beta_{i0}. \qquad (1.19)$$

The equality (1.19) shows that the map g is an affine transformation on $\mathbb{C}^{m+1}$, and hence $(\mathbb{C},f)$ is an affine com(m+1,m)-semigroup. ■

The proof of the following Proposition follows directly from Proposition I.4.2 and Proposition 1.4.

Proposition 1.6. If $(\mathbb{C},f^s)$ is an affine com(m+sk,m)- semigroup induced from an affine com(m+k,m)-semigroup $(\mathbb{C},f)$, then for each $y=r_1^s\in\mathbb{C}^{(sk)}$, $r_i\in\mathbb{C}^{(k)}$,

$$A(y) = A(r_1)\cdot A(r_2)\cdot\ldots\cdot A(r_s). \text{ ■}$$

Next we consider the special case when $k=1$ and then examine the general case. So, we assume that $(\mathbb{C}, f)$ is an affine com(n,m)-groupoid, $n=m+1$, defined by the $m\times n$ matrix $A=[\alpha_{ij}]$ and $m\times 1$ matrix $b=|\alpha_{i0}|$. For elements $z\in\mathbb{C}$, $x\in\mathbb{C}^m$, the equation

$$\varphi_z(x)=y, \quad \text{i.e.} \quad f(zx)=y \tag{1.20}$$

is equivalent to the following system of equations:

$$\begin{cases} \alpha_{10}+\alpha_{11}(a_1+z)+\alpha_{12}(a_2+za_1)+\ldots+\alpha_{1n}(za_m)=b_1 \\ \alpha_{20}+\alpha_{21}(a_1+z)+\alpha_{22}(a_2+za_1)+\ldots+\alpha_{2n}(za_m)=b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{m0}+\alpha_{m1}(a_1+z)+\alpha_{m2}(a_2+za_1)+\ldots+\alpha_{mn}(za_m)=b_m \end{cases} \tag{1.21}$$

where $a_i=\mathcal{P}^i(x)$ and $b_i=\mathcal{P}^i(y)$, $1\le i\le m$.

The system (1.21) is obtained from the equation (1.20) using the facts that $\mathcal{P}_m(y)=\mathcal{P}_m\circ\varphi_z(x)=\mathcal{P}_m\circ\varphi_z\circ\mathcal{P}_m^{-1}\circ\mathcal{P}_m(x)=\bar{\varphi}_z\circ\mathcal{P}_m(x)$ and that $\bar{\varphi}_z\circ\mathcal{P}_m(x) = A(z)\cdot\mathcal{P}_m(x) + b(z)$.

We can think of (1.21) as a system of m linear equation on a_1, a_2, ..., a_m, whose main determinant is $\Delta(z)=\det A(z)$, i.e.

$$\Delta(z) = \det\begin{bmatrix} \alpha_{11}+z\alpha_{12} & \alpha_{12}+z\alpha_{13} & \cdots & \alpha_{1m}+z\alpha_{1n} \\ \alpha_{21}+z\alpha_{22} & \alpha_{22}+z\alpha_{23} & \cdots & \alpha_{2m}+z\alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1}+z\alpha_{m2} & \alpha_{m2}+z\alpha_{m3} & \cdots & \alpha_{mm}+z\alpha_{mn} \end{bmatrix}. \tag{1.22}$$

The determinant $\Delta(z)$ is a polynomial of z of degree m. Namely, by the standard **transformation of the determinant** in (1.22), we can express $\Delta(z)$ as

$$\Delta(z) = \det\begin{bmatrix} (-z)^m & (-z)^{m-1} & \ldots & (-z)^1 & 1 \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} & \alpha_{mn} \end{bmatrix}, \tag{1.23}$$

i.e. by

$$\Delta(z)=h_0^1z^m-h_1^1z^{m-1}+h_2^1z^{m-1}+\ldots+(-1)^{m-1}h_{m-1}^1z+(-1)^mh_m^1 \tag{1.24}$$

where $h^1=(h_0^1,h_1^1,h_2^1,\ldots,h_{m-1}^1,h_m^1)$ is the vector defined by (1.9) or (1.10).

Definition 1.4. The polynomial $\Delta(z)$ is said to be the **characteristic polynomial** for the affine com(m+1,m)-groupoid.

The description (1.24) of the polynomial $\Delta(z)$, implies directly that $\Delta(z)$ is equal to the zero polynomial, i.e. $\Delta(z)\equiv 0$, iff all the h^1_i are 0, i.e. iff the vector h^1 is the zero vector. This, together with Proposition 1.3, implies:

Proposition 1.7. An affine com(m+1,m)-semigroup is irregular iff its characteristic polynomial is the zero polynomial. ∎

In section I.3, we were interested in the elements z for which φ_z were bijections, i.e. we were interested in the non-singular elements. The equivalence of (1.20) and (1.21) implies that φ_z is a bijection iff the system (1.21) has a unique solution on $a_1, a_2, \dots, a_m$ for every $y\in\mathbb{C}^m$. But the system (1.21) has a unique solution for ever y iff the main determinant of the system is not 0, i.e. iff $\Delta(z)\neq 0$. Thus we have:

Proposition 1.8. Let $(\mathbb{C},f)$ be a given affine com(m+1,m)-semigroup. Then:

(i) An element z from $\mathbb{C}$ is singular iff $\Delta(z)=0$;

(ii) If $(\mathbb{C},f)$ is irregular, then $(\mathbb{C},f)$ is a singular com(m+1,m)-semigroup;

(iii) If $(\mathbb{C},f)$ is regular then the singular elements in $(\mathbb{C},f)$ are exactly the roots of the characteristic polynomial $\Delta(z)$, and they are at most m. ∎

Because of the above Proposition, and Theorem I.3.8, which says that if from a com(n,m)-semigroup we take out the singular set, then we obtain a com(n,m)-group, of special interest to us are the non-singular com(n,m)-semigroups which in the case of affine com(m+1,m)-semigroups coincide with the regular ones. The above discussion justifies the introduction of the notion for regular affine com(n,m)-semigroups. The affine com(3,2)-semigroups from examples 1 to 7 from section I.5, are non-singular, i.e. regular, while the affine com(3,2)-semigroups from examples 9 to 11 from section I.5, are singular, i.e. are irregular.

Now we consider an arbitrary affine com(n,m)-groupoid $(\mathbb{C},f)$, $n=m+k$, defined by the $m\times n$ matrix $A=[\alpha_{ij}]$ and $m\times 1$ matrix $b=[\alpha_{i0}]$. For elements $z\in\mathbb{C}^{(k)}$, $x\in\mathbb{C}^{(m)}$, the equation

$$\varphi_z(x)=y\ , \quad \text{i.e.} \quad f(zx)=y \tag{1.25}$$

is equivalent to the following system of equations:

$$\begin{cases} \sum_{0\le t\le n} \sum_{0\le r\le \min(t,m)} \alpha_{1t}\mathcal{G}^r(x)\mathcal{G}^{t-r}(z) = \mathcal{G}^1(y) \\ \dots\dots\dots\dots\dots\dots \\ \sum_{0\le t\le n} \sum_{0\le r\le \min(t,m)} \alpha_{mt}\mathcal{G}^r(x)\mathcal{G}^{t-r}(z) = \mathcal{G}^m(y) \end{cases} \quad . \tag{1.26}$$

The system (1.26) is obtained from the equation (1.25) using the facts that $\mathcal{G}_m(y)=\mathcal{G}_m\circ\varphi_z(x)=\mathcal{G}_m\circ\varphi_z\circ\mathcal{G}_m^{-1}\circ\mathcal{G}_m(x)=\bar{\varphi}_z\circ\mathcal{G}_m(x)$ and that $\bar{\varphi}_z\circ\mathcal{G}_m(x) = A(z)\cdot\mathcal{G}_m(x) + b(z)$.

If we denote $\mathcal{G}^r(x)$ by a_r and $\mathcal{G}^s(z)$ by β_s, we can think of (1.26) as a system of m linear equations on $a_1, a_2, \dots, a_m$, whose main determinant is $\Delta(z) = \det A(z)$, i.e.,

$$\Delta(z)=\det\begin{bmatrix} \sum_{0\le i\le k}\alpha_{1(i+1)}\beta_i & \cdots & \sum_{0\le i\le k}\alpha_{1(i+m)}\beta_i \\ \dots\dots\dots & & \dots\dots\dots \\ \sum_{0\le i\le k}\alpha_{m(i+1)}\beta_i & \cdots & \sum_{0\le i\le k}\alpha_{m(i+m)}\beta_i \end{bmatrix} . \tag{1.27}$$

For a given affine com(n,m)-groupoid the determinant $\Delta(z)$ is a polynomial of $z_1, z_2, \dots, z_k$, where $z=z_1^k\in\mathbb{C}^k$.

Definition 1.5. The polynomial $\Delta(z)$ is said to be the **characteristic polynomial** for the affine com(n,m)-groupoid.

We note that the characteristic polynomial defined by Definition 1.5 for $k=1$ is the same as the characteristic polynomial defined by Definition 1.4. The Definition of $\Delta(z)$ and Proposition 1.4 imply the following:

Theorem 1.9. Let $(\mathbb{C},f)$ be an affine com(n,m)-semigroup **determined** by the matrices $A=\left[\alpha_{ij}\right]$, $1\le i\le k$, $1\le j\le n$, and b. Then the **following** conditions are equivalent:

(i) $(\mathbb{C},f)$ is non-singular com(n,m)-semigroup;

(ii) The characteristic polynomial is not the 0 polynomial;

(iii) The rank of the matrix A is m, i.e. $\operatorname{rank}A=m$;

Moreover, in the case $k=1$, (i), (ii) and (iii) are equivalent to

(iv) $(\mathbb{C},f)$ is regular.

Proof. (i) iff (ii): Since the characteristic polynomial $\Delta(z)$ is defined as the determinant of the linear transformation $A(z)$, it follows that φ_z is a bijection iff $\bar{\varphi}_z$ is a bijection iff $A(z)$ is non-singular iff $\Delta(z)\neq0$. Hence, $(\mathbb{C},f)$ is non-singular iff there exists a $z\in\mathbb{C}^{(k)}$ such that φ_z is a bijection iff the polynomial $\Delta(z)\neq0$ for some $z\in\mathbb{C}^{(k)}$.

(i) implies (iii); Let $\operatorname{rank}A<m$ and let $f(zx)=y_1^m$, for $z\in\mathbb{C}^{(k)}$, $x\in\mathbb{C}^{(m)}$, $y_i\in\mathbb{C}$. The definition of f via the symmetric functions and the affine transformation determined by the matrices A and b, implies that $y_1, y_2, \ldots, y_m$ are the roots of the equation

$$t^m + a_1t^{m-1}+ \ldots + a_{m-1}t + a_m = 0, \tag{1.28}$$

where $a_1,a_2, \ldots, a_m$ are functionally dependent linear functions of $\varphi^i(zx)$ as variables, $1\leq i\leq n$. Thus, for each z, φ_z is not a bijection, i.e. $(\mathbb{C},f)$ is a singular com(n,m)-semigroup.

(iii) implies (i): Let $\operatorname{rank}A=m$, and let $f(zx)=y_1^m$. By the same discussion as above, it follows that $y_1, y_2, \ldots, y_m$ are the roots of the equation (1.28), where $a_1,a_2, \ldots, a_m$ are functionally independent linear functions of $\varphi^i(zx)$ as variables, $1\leq i\leq n$. Thus, there exists a $z\in\mathbb{C}^{(k)}$, such that φ_z is a bijection, i.e. $(\mathbb{C},f)$ is a non-singular com(n,m)-semigroup.

For $k=1$, (ii) and (iv) are equivalent by Proposition 1.7. ■

Proposition 1.10. Let $(\mathbb{C},f^s)$ be the affine com(m+sk,m)-semigroup induced from an affine com(m+k,m)-semigroup $(\mathbb{C},f)$. Then for each $y=r_1^s\in\mathbb{C}^{(sk)}$, $r_i\in\mathbb{C}^{(k)}$,

$$\Delta(y) = \Delta(r_1)\cdot\Delta(r_2)\cdot\ldots\cdot\Delta(r_s). \ ■$$

Next we consider the question: For a given affine com(m+k,m)-semigroup $(\mathbb{C},f)$ is there a polynomial $\Delta'(r)$ of one variable such that for each $r_1^k\in\mathbb{C}^k$, $\Delta(r_1^k)=\Delta'(r_1)\cdot\Delta'(r_2)\cdot\ldots\cdot\Delta'(r_k)$? An affirmative answer to this question gives the following theorem:

Theorem 1.11. Let $(\mathbb{C},f)$ be a regular affine com(m+k,m)-semigroup Then there exists a polynomial $\Delta'(r)$ of one variable such that for each $r_1^k\in\mathbb{C}^k$, $\Delta(r_1^k)=\Delta'(r_1)\cdot\Delta'(r_2)\cdot\ldots\cdot\Delta'(r_k)$, and the degree of $\Delta'(r)$ is smaller than or equal to m.

Proof. If $\Delta(r_1^k)$ is identically equal to 0, then the proof is trivial. So, we assume that $\Delta(r_1^k)\not\equiv0$, i.e. the affine com(m+k,m)-semigroup is nonsingular. Because $(\mathbb{C},f)$ is regular, there exists the characteristic vector $h\in\mathbb{C}^{m+1}$. We will show that for arbitrary

$1 \le i \le k$, $\Delta(r_1^k)$ as a polynomial of a variable r_i with carefully chosen constants $r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k$ has the following form:

$$\Delta(r_1^k) = C(h_0 r_i^m - h_1 r_i^{m-1} + h_2 r_i^{m-2} - \ldots + (-1)^m h_m), \qquad (1.29)$$

where C is a non-zero constant. Without loss of generality we may take $i=k$, i.e. $r_i = r_k$. In this case, $\Delta(r_1^k)$ can be expressed as:

$$\Delta(r_1^k) = \det \begin{bmatrix} \gamma_{11}+r_k\gamma_{12} & \gamma_{12}+r_k\gamma_{13} & \cdots & \gamma_{1m}+r_k\gamma_{1(m+1)} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{m1}+r_k\gamma_{m2} & \gamma_{m2}+r_k\gamma_{m3} & \cdots & \gamma_{mm}+r_k\gamma_{m(m+1)} \end{bmatrix}, \qquad (1.30)$$

where $\gamma_{ij} = \sum_{0 \le s \le k-1} \alpha_{i(j+s)} \varphi^s(r_1^{k-1})$, for $1 \le i \le m$, $1 \le j \le m+1$. Let $\gamma_{00}=1$, $\gamma_{0j}=0$, for $1 \le j \le m+1$, and $\gamma_{i0} = \sum_{0 \le s \le k-1} \alpha_{is} \varphi^s(r_1^{k-1})$ for $1 \le i \le m$. With these notations, each of the following m vectors

$$x_i = (\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{i(m+1)}), \quad 1 \le i \le m, \qquad (1.31)$$

is a linear combination of the vectors u_{ij} defined by (1.4). The definition of γ_{ij} implies that

$$\gamma_{ij} - \gamma_{(i+1)(j+1)} = \sum_{0 \le s \le k-1} (\alpha_{i(j+s)} - \alpha_{(i+1)(j+s+1)}) \varphi^s(r_1^{k-1}),$$

for $0 \le i, j \le m$. Thus, each of the following $m+1$ vectors,

$$y_i = (\gamma_{0i} - \gamma_{1(i+1)}, \ldots, \gamma_{mi} - \gamma_{(m+1)(i+1)}), \quad 0 \le i \le m, \qquad (1.32)$$

is a linear combination of the vectors $v_j - w_j$ defined by (1.5) and (1.6). Because $(\mathbb{C}, f)$ is an affine com(m+k, m)-semigroup, it follows that each of the vectors u_{ij} is orthogonal to each of the vectors $v_t - w_t$. Thus, each of the vectors x_i is orthogonal to each of the vectors y_j, which implies that the $(m+1)\times(m+2)$ matrix $[\gamma_{ij}]$, $0 \le i \le m$, $0 \le j \le m+1$, determines an affine com(m+1, m)-semigroup, $(\mathbb{C}, h)$ whose characteristic polynomial $\Delta(r_k)$ is equal to the characteristic polynomial $\Delta(r_1^k)$. The assumption that $\Delta(r_1^k) \ne 0$, implies that $(\mathbb{C}, h)$ is non-singular, and so it is regular. The characteristic vector h for $(\mathbb{C}, f)$ is also a characteristic vector for $(\mathbb{C}, h)$. But, by Proposition 1.1, the vector h is collinear with the vector

$$h^1 = \det \begin{bmatrix} e_1 & e_2 & \cdots & e_{m+1} \\ \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1(m+1)} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2(m+1)} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{m(m+1)} \end{bmatrix}$$

defined by (1.9) i.e. by (1.10). Now, $\Delta(r_1^k)$ is of the form (1.24).

Because the polynomial $\Delta'(r_k) = h_0 r_k^m - h_1 r_k^{m-1} + h_2 r_k^{m-2} - \ldots + (-1)^m h_m$ depends only on r_k and the characteristic vector for $(\mathbb{C},f)$, it follows that

$$\Delta(r_1^k) = \Delta'(r_1)\cdot\Delta'(r_2)\cdot \ldots \cdot\Delta'(r_k). \blacksquare$$

Let us remark that if two affine com(m+k,m)-semigroups have the same characteristic polynomial, they do not have to be isomorphic. Thus, the com(3,2)-groups in Examples 1 and 2 from I.5 have the same characteristic polynomial $\Delta(z)$=const, but they are not isomorphic. Namely, the unit of the com(3,2)-group from Example 1 is (0,0), i.e. of form (x,x), while the unit of the com(3,2)-group from Example 2 is $(\sqrt{2},-\sqrt{2})$, i.e of form (x,y), where $x\neq y$; hence they are not isomorphic.

In the case of a non-singular affine com(n,m)-semigroup $(\mathbb{C},f)$, by throwing out the singular elements, we obtain the largest affine com(n,m)-group which is embedded in the given affine com(n,m)-semigroup. This result is not true in general, which can be seen from the following example:

Example. Let $(\mathbb{C}\setminus\{0\},f)$ be an arbitrary affine com(n,m)-group. We define a map $g:\mathbb{C}^{(n)}\to\mathbb{C}^{(m)}$ by:

$$g(x_1^n) = \begin{cases} f(x_1^n) & \text{if } x_i\neq 0 \text{ for each } 1\leq i\leq n \\ (0,0,\ldots,0)\in\mathbb{C}^{(m)} & \text{if } x_i=0 \text{ for some } 1\leq i\leq n \end{cases}.$$

It is easy to check that $(\mathbb{C},g)$ is a com(n,m)-semigroup. This com(n,m)-semigroup is singular, because for each $z\in\mathbb{C}^{(k)}$, $(1,0,\ldots,0)\in\mathbb{C}^{(m)}$ is not in the image of φ_z, i.e. φ_z is not a bijection. But on the other hand, the affine com(n,m)-group $(\mathbb{C}\setminus\{0\},f)$ is a com(n,m)-subsemigroup of $(\mathbb{C},g)$.

II.2. On some clases of commutative Lie subgroups of $A(m;\mathbb{C})$ and their application

Using the results of the previous section, in this section we will prove Theorems 2.1 and 2.2. As an important consequence of Theorem 2.2 we will prove Theorem 2.4 about the affine semigroups and groups. Namely, for a given affine com(m+k,m)-semigroup (or group) with a non-zero characteristic polynomial, we will be able to find (the number of) those affine com(m+1,m)-semigroups (groups) which induce the given semigroup (group). It is interesting that practically it is impossible to prove Theorems 2.1 and 2.2 without using the results of II.1, but on the other hand these two theorems do not say anything about the affine com(n,m)-semigroups and groups, alone.

Let us denote by $A(m;\mathbb{C})$ the set of all the affine complex transformations on $\mathbb{C}^m$, i.e. of all the non-singular complex matrices of the form

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix} . \tag{2.1}$$

Definition 2.1. Let a non-zero vector $h=h_o^m\in\mathbb{C}^{m+1}$ be given. We denote by G_h (or simply by G) the subset of all matrices of $A(m;\mathbb{C})$ such that the following m vectors

$$\begin{array}{l} (1-\alpha_{11},\ \alpha_{10}-\alpha_{21},\ \ldots\ ,\alpha_{(m-1)0}-\alpha_{m1},\ \alpha_{m0}) \\ (-\alpha_{12},\ \alpha_{11}-\alpha_{22},\ \ldots\ ,\alpha_{(m-1)1}-\alpha_{m2},\ \alpha_{m1}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (-\alpha_{1m},\ \alpha_{1(m-1)}-\alpha_{2m},\ldots,\alpha_{(m-1)(m-1)}-\alpha_{mm},\alpha_{m(m-1)}) \end{array} \tag{2.2}$$

are collinear with the vector h.

Theorem 2.1. Let a non-zero vector $h=h_0^m\in\mathbb{C}^{m+1}$ be given. The set of matrices G_h with the matrix multiplication is a closed commutative Lie subgroup of $A(m;\mathbb{C})$ with complex dimension m.

Proof. First we suppose that $h_m\neq 0$. We choose a matrix from $G_h=G$ as follows: We choose arbitrary numbers $\alpha_{10}, \alpha_{20}, \ldots, \alpha_{m0}$, and then the rest of the elements α_{ij} are uniquely determined by the condition that the vectors (2.2) are collinear with h and the fact that $h_m\neq 0$. Specially if we choose $\alpha_{10}=\alpha_{20}=\ldots=\alpha_{m0}=0$ then we would obtain A=I. We note that some choices of $\alpha_{10},\ldots,\alpha_{m0}$ will produce singular matrices. The matrix A can be extended by one vector column $(0, \alpha_{1(m+1)},\ldots,\alpha_{m(m+1)})^T$ such that the vector

$$(-\alpha_{1(m+1)},\ \alpha_{1m}-\alpha_{2(m+1)},\ \ldots\ ,\alpha_{mm})$$

is also collinear with the vector h. This extension is unique since $h_m\neq 0$. We denote the extended matrix by A'. This matrix A' determines an affine com(m+1,m)-semigroup whose characteristic polynomial is

$$h_0t^m - h_1t^{m-1} + \ldots +(-1)^mh_m.$$

Since this is a non-zero polynomial, it follows that the set $\mathcal{R}$ of singular points consists of at most m points, and this com(m+1,m)-semigroup forms an affine com(m+1,m)-group on $\mathbb{C}\setminus\mathcal{R}$ This affine com(m+1,m)-group induces an affine com(2m,m)-group on $\mathbb{C}\setminus\mathcal{R}$. Let us suppose that it is determined by the following $(m+1)\times(2m+1)$ matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ \beta_{10} & \beta_{11} & \beta_{12} & \ldots & \beta_{1(2m)} \\ \beta_{20} & \beta_{21} & \beta_{22} & \ldots & \beta_{2(2m)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{m0} & \beta_{m1} & \beta_{m2} & \ldots & \beta_{m(2m)} \end{bmatrix} \qquad (2.3)$$

According to the results of II.1, the following 2m vectors

$$(\beta_{0i}-\beta_{1(i+1)},\ \beta_{1i}-\beta_{2(i+1)},\ \ldots,\ \beta_{mi})\ (0\le i\le 2m-1) \qquad (2.4)$$

where $\beta_{0i}=\delta_{0i}$, are also collinear with the vector h. This affine com(2m,m)-group on $\mathbb{C}\setminus\mathcal{R}$ can be considered as an ordinary commutative group on $(\mathbb{C}\setminus\mathcal{R})^{(m)}$, and we will denote this group by G^*. In fact G^* is the derived group $(G^{(m)},*)$ for the affine com(m+1,m)-group.

The group G^* consists of all the transformations φ_r where

$r\in(\mathbb{C}\backslash\mathcal{R})^{(m)}$ (which were introduced in I.3), with the operation composition. The corresponding transformation $\bar{\varphi}_r$ in the case of an affine com(2m,m)-group is an affine transformation and we know from the previous section that the corresponding matrix is

$$\bar{\varphi}_r = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \sum_{i=0}^{m} \beta_{1i}\varphi^i & \sum_{i=0}^{m} \beta_{1(i+1)}\varphi^i & \dots & \sum_{i=0}^{m} \beta_{1(i+m)}\varphi^i \\ \sum_{i=0}^{m} \beta_{2i}\varphi^i & \sum_{i=0}^{m} \beta_{2(i+1)}\varphi^i & \dots & \sum_{i=0}^{m} \beta_{2(i+m)}\varphi^i \\ \dots & \dots & \dots & \dots \\ \sum_{i=0}^{m} \beta_{mi}\varphi^i & \sum_{i=0}^{m} \beta_{m(i+1)}\varphi^i & \dots & \sum_{i=0}^{m} \beta_{m(i+m)}\varphi^i \end{bmatrix} \tag{2.5}$$

where $r=(r_1,\dots,r_m)$ and $\varphi^i=\varphi^i(r_1^m)$ is the i-th symmetric function of the numbers $r_1,\dots,r_m$ and $\varphi^0=1$. Since all of the vectors (2.4) are collinear with the vector h, it is easy to see that the analogous condition holds for the columns of the matrix (2.5), and since it is non-singular it belongs to G. We will prove that $G=G^*$, and since G^* is a commutative group, it will imply that G is a group with respect to the matrix multiplication.

We have just proved that each matrix in G^* belongs to G, i.e. $G^*\subseteq G$. Now we will prove that $G\subseteq G^*$. Let

$$C = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \dots & \gamma_{1m} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \dots & \gamma_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ \gamma_{m0} & \gamma_{m1} & \gamma_{m2} & \dots & \gamma_{mm} \end{bmatrix}, \tag{2.6}$$

be an arbitrary matrix in G. We need to prove that there exist $r\in(\mathbb{C}\backslash\mathcal{R})^{(m)}$ such that $C=\bar{\varphi}_r$. Since $C, \bar{\varphi}_r\in G$ and $h_m\neq 0$, this equality will hold, if the first columns of the matrices C and $\bar{\varphi}_r$ are identical, i.e. if

$$\sum_{0\le i\le m} \beta_{ji}\varphi^i(r_1^m) = \gamma_{j0} \qquad (1\le j\le m). \tag{2.7}$$

This is a linear system of m equations with m unknown quantities $\varphi^i(r_1^m)$ $(1\le i\le m)$, whose main determinant is

$$\Delta = \begin{vmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{vmatrix} .$$

We will prove that $\Delta \neq 0$.

The affine com(2m, m)-group is induced from an affine com(m+1, m)-group, and thus,

$$\varphi_r = \varphi_{r_1} \circ \varphi_{r_2} \circ \ \cdots \ \circ \varphi_{r_m} \tag{2.8}$$

where $r=(r_1,\dots,r_m)$ and $\bar{\varphi}_{r_i}$ $(1 \le i \le m)$ are affine transformations on $\mathbb{C}^m$. It can be verified that the matrix which corresponds to $\bar{\varphi}_r$ is

$$\bar{\varphi}_r = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_{10}+r\alpha_{11} & \alpha_{11}+r\alpha_{12} & \alpha_{12}+r\alpha_{13} & \cdots & \alpha_{1m}+r\alpha_{1(m+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{m0}+r\alpha_{m1} & \alpha_{m1}+r\alpha_{m2} & \alpha_{m2}+r\alpha_{m3} & \cdots & \alpha_{mm}+r\alpha_{m(m+1)} \end{bmatrix}. \tag{2.9}$$

It follows from (2.8) that

$$\bar{\varphi}_r = \bar{\varphi}_{r_1} \circ \bar{\varphi}_{r_2} \circ \cdots \circ \bar{\varphi}_{r_m} \tag{2.10}$$

where $r=r_1^m$, and by taking $r_1=\dots=r_m=0$ in the matrix equality we obtain

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \beta_{10} & \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{20} & \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{m0} & \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix}^m . \tag{2.11}$$

Hence $\Delta=(\det A)^m \neq 0$, because $A \in G$. Thus the system (2.7) has a unique solution for $\varphi^i(r_1^m)$ for $i \in \{1,\dots,m\}$, and hence the system (2.7) has unique solution $(r_1,\dots,r_m) \in \mathbb{C}^{(m)}$. We will prove that r_i $(1 \le i \le m)$ are not singular points. Indeed, if $r_i \in \mathcal{R}$, then the matrix $\bar{\varphi}_{r_i}$ is singular, and from (2.10) it follows that the matrix $\bar{\varphi}_r$ is singular which is a contradiction. So we have proved that $G=G^*$ and G is a commutative subgroup of $A(m;\mathbb{C})$, assuming that $h_m \neq 0$.

Now we will assume that a non-zero vector $h=(h_0,h_1,\dots,h_m)$ is given such that $h_m=0$. Let us suppose that the last t coordinates of this vector are zeros, i.e. $h_m=h_{m-1}=\dots=h_{m-s+1}=0$ but $h_{m-s} \neq 0$. For the sake of simplicity, we will assume further that s=3. Using the definition of the matrices in G, one can conclude that in this case (s=3) each matrix in G must have the following form

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \alpha_{10} & \alpha_{11} & \dots & \alpha_{1(m-3)} & \alpha_{1(m-2)} & \alpha_{1(m-1)} & \alpha_{1m} \\ \alpha_{20} & \alpha_{21} & \dots & \alpha_{2(m-3)} & \alpha_{2(m-2)} & \alpha_{2(m-1)} & \alpha_{2m} \\ \dots & & & \dots & \dots & & \dots \\ \alpha_{(m-3)0} & & \dots & \alpha_{(m-3)(m-3)} & \alpha_{(m-3)(m-2)} & * & \alpha_{(m-3)m} \\ 0 & 0 & \dots & 0 & x & y & z \\ 0 & 0 & \dots & 0 & 0 & x & y \\ 0 & 0 & \dots & 0 & 0 & 0 & x \end{bmatrix} \qquad (2.12)$$

and $x \neq 0$ since $\det A \neq 0$. Besides that, using the definition of G and the fact $h_{m-3} \neq 0$, one can verify that if we take arbitrary numbers $\alpha_{10}, \alpha_{20}, \dots, \alpha_{(m-3)0}, x, y$ and z, then the rest of the elements α_{ij} $(i \in \{1, \dots, m-3\},\ j \in \{1, \dots, m\})$ can be uniquely solved, so that the vectors (2.2) would become collinear with the vector h. We notice that it is possible for some choices of these m numbers to give a singular matrix A.

Now let us choose an arbitrary affine com(m+1,m)-semigroup on $\mathbb{C}$ whose characteristic polynomial is

$$h_0 t^m - h_1 t^{m-1} + \ \dots \ + (-1)^m h_m .$$

In order to prove that such a semigroup exists, it is sufficient to give such an example. Indeed, let the affine com(m+1,m)-groupoid $(\mathbb{C}, f)$ be determined by the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & -h_0/h_{m-3} & u_1 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & -h_1/h_{m-3} & u_2 & u_1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -h_{m-4}/h_{m-3} & u_{m-3} & u_{m-4} & u_{m-5} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & u_{m-3} & u_{m-4} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & u_{m-3} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $u_1, u_2, \dots, u_{m-3}$ are chosen in such a way that the vector

$$(-u_1,\ -h_0/h_{m-3}-u_2,\ \dots\ , -h_{m-5}/h_{m-3}-u_{m-3}, -h_{m-4}/h_{m-3}-1)$$

is collinear with the vector $(h_0, h_1, \dots, h_{m-3})$. Then the com(m+k,m)-groupoid is associative and its characteristic polynomial is

$$C(h_0 t^m - h_1 t^{m-1} + \ \dots \ + (-1)^m h_m) \qquad (C\neq 0).$$

Since the characteristic polynomial is non-zero, the set $\mathcal{R}$ of singular points consists of at most m elements, and this affine com(m+1,m)-semigroup is an affine com(m+1,m)-group on $\mathbb{C}\setminus\mathcal{R}$. This group induces an affine com(2m,m)-group on $\mathbb{C}\setminus\mathcal{R}$, which can be considered as a usual commutative group on $(\mathbb{C}\setminus\mathcal{R})^{(m)}$ and we will denote this group by G^* as in the case $h_m\neq 0$. The group G^* can be considered as a set of matrices with the matrix multiplication, and we will prove that the sets of matrices G and G^* are equal. This will imply that G with the matrix multiplication is a commutative subgroup of $A(m;C)$. The proof that $G=G^*$ is similar to the proof for $h_m\neq 0$ and so we will use the same notations as above.

Let us suppose that the matrix which determines the affine com(2m,m)-group is given by (2.3). Since the vectors (2.4) have to be collinear with the vector h and $h_m=h_{m-1}=h_{m-2}=0$, it can be proven that:

$$\begin{cases} \beta_{ij}=0 \quad \text{for} \quad i\in\{m-2,m-1,m\} \ \text{and} \ j<i+m \\ \beta_{(m-2)(2m-2)} = \beta_{(m-1)(2m-1)} = \beta_{m(2m)} \\ \beta_{(m-2)(2m-1)} = \beta_{(m-1)(2m)} \end{cases} . \qquad (2.13)$$

The group G^* consists of all matrices $\bar{\varphi}_r$ where $r\in(\mathbb{C}\setminus\mathcal{R})^m$ and these matrices are given by (2.5). Since the vectors (2.4) are collinear with the vector h, it is easy to verify that $\bar{\varphi}_r\in G$, i.e. $G^*\subseteq G$. Further we will prove that $G\subseteq G^*$.

Using the formulas (2.13), the last row of the matrix (2.5) becomes $(0,0,\ \dots\ ,0,\ \beta_{m(2m)}\varphi^m(r_1^m))$.

Since the matrix (2.5) is non-singular, it follows that $\beta_{m(2m)}\neq 0$, i.e. $\beta_{(m-2)(m-2)}\neq 0$.

Further let us assume that the chosen affine com(m+1,m)-semigroup is given by the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1(m+1)} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \dots & \alpha_{2(m+1)} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{m(m+1)} \end{bmatrix} . \qquad (2.14)$$

Then the vectors

$$(\alpha_{0i}-\alpha_{1(i+1)},\alpha_{1i}-\alpha_{2(i+1)}, \dots ,\alpha_{(m-1)i}-\alpha_{m(i+1)},\alpha_{mi}) \tag{2.15}$$

where $\alpha_{0i}=\delta_{0i}$, are collinear with the vector h. Using this and the fact that $h_m=h_{m-1}=h_{m-2}=0$, one can verify that

$$\alpha_{ij}=0 \quad \text{for} \quad i\in\{m-2,m-1,m\} \quad \text{and} \quad i\geq j,$$
$$\alpha_{m(m+1)}=\alpha_{(m-1)m}=\alpha_{(m-2)(m-1)} \text{ and } \alpha_{(m-1)(m+1)}=\alpha_{(m-2)m}. \tag{2.16}$$

Using the formulas (2.16) and the fact that (2.9) is a non-singular matrix we obtain that $\alpha_{m(m+1)}\neq 0$. Indeed, if $\alpha_{m(m+1)}=0$, then the last row of the matrix $\bar{\varphi}_r$ which is given by (2.9) will be zero. We know that $\det\bar{\varphi}_r$ gives the characteristic polynomial for t=r, i.e.

$$\det\bar{\varphi}_r = C(h_0r^m-h_1r^{m-1}+ \dots +(-1)^m h_m)$$

where $\bar{\varphi}_r$ is the matrix given by (2.9) and $C=\text{const}\neq 0$. Since $h_m=h_{m-1}=h_{m-2}=0$ we obtain

$$\det\bar{\varphi}_r = C(h_0r^{m-3}-h_1r^{m-4}+ \dots +(-1)^{m-3}h_{m-3})\cdot r^3 . \tag{2.17}$$

On the other hand, using the formulas (3.16) we obtain

$$\det\bar{\varphi}_r = M\begin{vmatrix} \alpha_{11}+r\alpha_{12} & \dots\alpha_{1(m-3)}+r\alpha_{1(m-2)} \\ \vdots & \vdots \\ \alpha_{(m-3)1}+r\alpha_{(m-3)2} & \dots\alpha_{(m-3)(m-3)}+r\alpha_{(m-3)(m-2)} \end{vmatrix} \tag{2.18}$$

where $M=r^3\alpha^3_{m(m+1)}$.

Now (2.17) and (2.18) imply that

$$\begin{vmatrix} \alpha_{11}+r\alpha_{12} & \dots & \alpha_{1(m-3)}+r\alpha_{1(m-2)} \\ \vdots & & \vdots \\ \alpha_{(m-3)1}+r\alpha_{(m-3)2} & \dots & \alpha_{(m-3)(m-3)}+r\alpha_{(m-3)(m-2)} \end{vmatrix}$$
$$=C_1\left[h_0r^{m-3} - h_1r^{m-4} + \dots + (-1)^{m-3}h_{m-3}\right].$$

This equality holds for each $p\in\mathbb{C}\setminus\mathcal{R}$, and thus it holds for each $p\in\mathbb{C}$. Specially for p=0 we obtain

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1(m-3)} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2(m-3)} \\ \dots & \dots & \dots & \dots \\ \alpha_{(m-3)1} & \alpha_{(m-3)2} & \dots & \alpha_{(m-3)(m-3)} \end{vmatrix} = C_1h_{m-3}(-1)^{m-3} \neq 0. \tag{2.19}$$

The equality (2.10) also holds in the case $h_m=0$, and as a consequence of it, using the equalities (2.13) and (2.16) we obtain $X = Y^m$ where X and Y are the matrices:

$$X = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \beta_{10} & \beta_{11} & \beta_{12} & \dots & \beta_{1(m-3)} \\ \beta_{20} & \beta_{21} & \beta_{22} & \dots & \beta_{2(m-3)} \\ \dots & \dots & \dots & \dots & \dots \\ \beta_{(m-3)0} & \beta_{(m-3)1} & \beta_{(m-3)2} & \dots & \beta_{(m-3)(m-3)} \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1(m-3)} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \dots & \alpha_{2(m-3)} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{(m-3)0} & \alpha_{(m-3)1} & \alpha_{(m-3)2} & \dots & \alpha_{(m-3)(m-3)} \end{bmatrix}$$

and hence

$$\begin{vmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1(m-3)} \\ \vdots & \vdots & & \vdots \\ \beta_{(m-3)1} & \beta_{(m-3)2} & \dots & \beta_{(m-3)(m-3)} \end{vmatrix} = \Delta^m \neq 0. \tag{2.20}$$

Using the fact that $\beta_{(m-2)(2m-2)} \neq 0$ and (2.20) we can prove that $G \subseteq G^*$.

Let us suppose that the matrix given by (2.6) is an arbitrary matrix of G. We want to find $r \in (\mathbb{C}\setminus\mathcal{R})^{(m)}$ such that $\bar{\varphi}_r = C$ where $\bar{\varphi}_r$ is given by (2.5). Using the fact that each matrix in G has the form (2.12), it is sufficient to prove that the following system

$$\begin{cases} \sum\limits_{0\le i\le m} \beta_{ji}\, \varphi^i(r_1^m) = \gamma_{j0} \quad (i \in \{1,\dots,m-3\}) \\ \sum\limits_{0\le i\le m} \beta_{(m-2)(i+m-2)}\, \varphi^i(r_1^m) = \gamma_{(m-2)(m-2)} \\ \sum\limits_{0\le i\le m} \beta_{(m-2)(i+m-1)}\, \varphi^i(r_1^m) = \gamma_{(m-2)(m-1)} \\ \sum\limits_{0\le i\le m} \beta_{(m-2)(i+m)}\, \varphi^i(r_1^m) = \gamma_{(m-2)m} \end{cases} \tag{2.21}$$

has a solution $(r_1,\dots,r_m) \in (\mathbb{C}\setminus\mathcal{R})^{(m)}$. The system (2.21) is linear with respect to the unknown functions $\varphi^i(r_1^m)$ $(i \in \{1,\dots,m\})$. Using the equalities (2.13) we obtain that the main determinant of the system (2.21) is

$$\begin{vmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{(m-3)1} & \beta_{(m-3)2} & \cdots & \beta_{(m-3)m} \\ \beta_{(m-2)(m-1)} & \beta_{(m-2)m} & \cdots & \beta_{(m-2)(2m-2)} \\ \beta_{(m-2)m} & \beta_{(m-2)(m+1)} & \cdots & \beta_{(m-2)(2m-1)} \\ \beta_{(m-2)(m+1)} & \beta_{(m-2)(m+2)} & \cdots & \beta_{(m-2)(2m)} \end{vmatrix} =$$

$=\det \begin{bmatrix} A & | & B \\ - & - & - \\ 0 & | & C \end{bmatrix} = -\Delta^m \cdot \beta^3_{(m-2)(2m-2)} \neq 0$, where A, B and C are the matrices:

$$A = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1(m-3)} \\ \cdots & \cdots & \cdots \\ \beta_{(m-3)1} & \cdots & \beta_{(m-3)(m-3)} \end{bmatrix}, \qquad B = \begin{bmatrix} \beta_{1(m-2)} & \cdots & \beta_{1m} \\ \cdots & \cdots & \cdots \\ \beta_{(m-3)(m-2)} & \cdots & \beta_{(m-3)m} \end{bmatrix}$$

$$\text{and } C = \begin{bmatrix} 0 & 0 & \beta_{(m-2)(2m-2)} \\ 0 & \beta_{(m-2)(2m-2)} & \beta_{(m-2)(2m-1)} \\ \beta_{(m-2)(2m-2)} & \beta_{(m-2)(2m-1)} & \beta_{(m-2)(2m)} \end{bmatrix}.$$

So the system (2.21) has a unique solution for $\mathcal{S}^i(r_1^m)$, for $1 \le i \le m$, and hence has a unique solution $(r_1,\ldots,r_m) \in \mathbb{C}^{(m)}$. If $r_i \in \mathcal{R}$ for some $i \in \{1,\ldots,m\}$ then $\bar{\varphi}_r$ is a singular matrix, and $\bar{\varphi}_r$ where $r=(r_1,\ldots,r_m)$ is also a singular matrix which is a contradiction. Thus $(r_1,\ldots,r_m) \in (\mathbb{C}\setminus\mathcal{R})^{(m)}$, and we have proved $G^* = G$ in the case $h_m = 0$.

Now, finally we have proved that for arbitrary non-zero vector h, the matrix set G with the matrix multiplication is a commutative subgroup of the Lie group $A(m;\mathbb{C})$. We recall that this group is a subgroup of the universal covering group of each affine com(m+1,m)-group whose characteristic polynomial is $h_0 t^m - h_1 t^{m-1} + \ldots + (-1)^m h_m$.

It is known from the theory of Lie groups that if a subset H of a Lie group G is an abstract subgroup of G and it is a closed subset of the topological space G, then H is a closed Lie subgroup of the Lie group G. Using this theorem it follows that G is a closed Lie subgroup of the Lie group $A(m;\mathbb{C})$, because G is a subgroup of $A(m;\mathbb{C})$, and from the definition of G it follows that G is a closed subset of $A(m;\mathbb{C})$. The dimension of the Lie group G is m, because in both cases $h_m \neq 0$ and $h_m = 0$ we saw that the matrices in G are uniquely determined by m mutually independent parameters. That finishes the proof of the theorem. ■

Now we are going to study the groups G_h up to homomorphism. In order to do that we introduce the following notations. Let C_q (q=1,2,...) denote the following group

$$C_p = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_p \\ 0 & x_1 & x_2 & \dots & x_{p-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & x_1 \end{bmatrix} \middle| x_1, \dots, x_q \in \mathbb{C}, x_1 \neq 0 \right\} .$$

with the matrix multiplication, and C_0 **denote the** additive complex group $(\mathbb{C},+)$. Specially, for $q=1$ we obtain **the** multiplicative complex group $(\mathbb{C},\cdot)$.

Before we give the next Theorem we recall the following results from the theory of the Lie groups. Each Lie group G has a unique simply-connected covering Lie group G^* which is determined up to isomorphism only by the Lie algebra of G, and each Lie group G is isomorphic to the factor-group G^*/D where D is a discrete subgroup of the center of G^*. In order to find all the commutative complex Lie groups with complex dimension m we notice that $[X,Y]=0$ for each X and Y from the Lie algebra. Thus $(\mathbb{C}^m,+)$ as a simply-connected m-dimensional Lie group is its covering group G^*. Furthermore each discrete subgroup of $(\mathbb{C}^m,+)$ is given by

$$D = \{\lambda_1 u_1 + \dots + \lambda_s u_s \mid \lambda, \dots, \lambda_s \in \mathbb{Z} \text{ and } u_1 \dots, u_s \text{ are linearly idependent vectors from } \mathbb{C}^m \}$$

where $0 \leq s \leq 2m$. On the other hand it is easy to verify that if H_1, H_2, H_3 and H_4 are commutative complex Lie groups which are homeomorphic to $\mathbb{R}\times S^1$, $\mathbb{R}\times\mathbb{R}$ and $S^1 \times S^1$ respectively, Then $H_1 \times H_2$ and $H_3 \times H_4$ are isomorphic Lie groups. Hence if two complex Lie groups have homeomorphic underlined manifolds, then they are isomorphic. Thus we obtain that up to isomorphism there are exactly 2m+1 **commutative** complex Lie groups with complex dimension m, and **their underlined manifolds are** homeomorphic to $(S^1)^s \times \mathbb{R}^{2m-s}$ $(0 \leq s \leq 2m)$. **Each of these groups** can be obtained as a direct product of m 1-dimensional complex Lie groups, and each 1-dimensional complex Lie group is isomorphic to one of the following three groups $(\mathbb{C},+)$, $(\mathbb{C}\setminus\{0\},\cdot)$ and $(\mathbb{C},+)/(\mathbb{Z}\times i\mathbb{Z})$, where $\mathbb{C}\setminus\{0\}$ is homeomorphic to $\mathbb{R}\times S^1$.

Theorem 2.2. Let a non-zero vector $h=h_0^m \in \mathbb{C}^{m+1}$ be given and let it be assumed that the polynomial

$$P(t) = h_m - h_{m-1}t + h_{m-2}t^2 - \dots +(-1)^m h_0 t^m \tag{2.22}$$

has exactly s different roots of arbitrary multiplicity. Then the corresponding group G_h satisfies

$$G_h \cong \underbrace{C_1 \times C_1 \times \dots \times C_1}_{s} \times \underbrace{C_0 \times C_0 \times \dots \times C_0}_{m-s}. \tag{2.23}$$

Proof. We will prove this Theorem by induction on s. First let us suppose that $s=0$, i.e. $h=(0,\dots,0,\lambda)$ where $\lambda\neq0$. It is easy to check that in this case

$$G_h = \left\{ \begin{bmatrix} 1 & 0 & \dots 0 & 0 \\ x_1 & 1 & \dots 0 & 0 \\ x_2 & x_1 & \dots 0 & 0 \\ \dots & \dots & \dots & \dots \\ x_m & x_{m-1} & \dots x_1 & 1 \end{bmatrix} \middle| x_1,\dots,x_m \in \mathbb{C} \right\}.$$

In this case G_h is a commutative Lie group on $\mathbb{C}^m$, and from the above discussion it follows that

$$G_h \cong (\mathbb{C}^m,+) \cong \underbrace{C_0 \times C_0 \times \dots \times C_0}_{m}.$$

Thus, the theorem holds for $s=0$.

Now let us suppose that the theorem holds for $s-1$ and arbitrary number m. In order to prove that the theorem holds for the number s let us suppose that a non-zero vector $h = (h_0,\dots,h_m)\in\mathbb{C}^{m+1}$ is given and the polynomial (2.22) has exactly s different roots, namely

$$P(t) = C(t-z_1)^{r_1}\cdot(t-z_2)^{r_2}\cdot\dots\cdot(t-z_s)^{r_s}$$

where $r_1,\dots,r_s\geq 1$, $r_1+\dots+r_s\leq m$ and $z_1,\dots,z_s$ are different complex numbers.

The group G_h coincides with the com(2m,m)-group considered as a usual (2.1)-group and such that the characteristic polynomial is given by (2.22). In the proof of Theorem 2.1 we denoted that (2.1)-group as G^*. Now let us define a mapping $\varphi:\mathbb{C}\to\mathbb{C}$ by $\varphi(z)=z+z_1$. We define a com(2m,m)-group structure f' by

$$f'(z_1^{2m})=w_1^m \iff f(\varphi(z_1),\dots,\varphi(z_{2m}))=(\varphi(w_1),\dots,\varphi(w_m))$$

where f denotes the group structure of the above com(2m,m)-group. It is easy to verify that this com(2m,m)-structure is affine and its characteristic polynomial is $P'(t) = P(\varphi(t)) = P(t+z_1)$, and it is obvious that these two com(2m,m)-groups are isomorphic. Hence we obtain that G_h is isomorphic to $G_{h'}$, where $h'=(h_0',h_1',\dots,h_m')\in\mathbb{C}^{m+1}$ is such that

$$P'(t) = h'_m - h'_{m-1}t + h'_{m-2}t^2 - \dots + (-1)^m h'_0 t^m.$$

Thus we should consider the polynomial $P'(t)$ instead of the polynomial $P(t)$. Since

$$P'(t) = Ct^{r_1}\cdot(t+z_1-z_2)^{r_2}\cdot\ldots\cdot(t+z_1-z_s)^{r_s}$$

we obtain that P' has s different roots and one of them is $t=0$. For the sake of **simplicity** we suppose that $r_1=3$. Hence we obtain that $h'_m = h'_{m-1} = h'_{m-2} = 0$ and $h'_{m-3} \neq 0$. We know from the proof of Theorem 2.1 that the general form of the matrices in $G_{h'}$ is

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ x_1 & * & * & \dots & * & * & * & * \\ x_2 & * & * & \dots & * & * & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ x_{m-3} & * & * & \dots & * & * & * & * \\ \hdashline 0 & 0 & 0 & \dots & 0 & x_{m-2} & x_{m-1} & x_m \\ 0 & 0 & 0 & \dots & 0 & 0 & x_{m-2} & x_{m-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & x_{m-2} \end{bmatrix} \qquad (2.24)$$

where the spaces denoted by "*" can uniquely be fulfilled by linear functions of $x_1,\dots,x_m$. The condition that $x_1,\dots,x_m$ should satisfy is that the matrix (2.24) is not singular. We rewrite this matrix (2.24) in the following form, i.e. as the following block matrix

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

Now we shall verify that the mapping $\tau: G_{h'} \to G_{(h^*)} \times C_3$ where $h^* = (h'_0, h'_1, \dots, h'_{m-3}) \in \mathbb{C}^{m-2}$, which is defined by

$$\tau\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right) = (A, D) \qquad (2.25)$$

is an isomorphism. First we notice that if $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in G_{h'}$, then $A \in G_{(h^*)}$ and $D \in C_3$. Conversely, let us suppose that $A \in G_{(h^*)}$ and $D \in C_3$. Then using the facts that $h'_m = h'_{m-1} = h'_{m-2} = 0$ and $h'_{m-3} \neq 0$, one can verify that the matrix B can uniquely be determined such that $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in G_{h'}$. Hence τ is a bijection. It is obvious that τ is a homeomorphism, and so τ is an isomorphism.

Since the polynomial

$$h'_{m-3} - h'_{m-4}t + h'_{m-5}t^2 - \dots + (-1)^{m-1}h'_0 = -P'(t)/t^3$$

has exactly $s-1$ different roots, it follows from the inductive assumption that

$$G_{(h^*)} \cong \underbrace{C_1 \times C_1 \times \dots \times C_1}_{s-1} \times \underbrace{C_0 \times C_0 \times \dots \times C_0}_{m-s-2} . \tag{2.26}$$

Further we will prove the following Lemma:

Lemma. The group C_r with matrix multiplication is commutative and for $r \geq 1$:

$$C_r \cong C_1 \times \underbrace{C_0 \times \dots \times C_0}_{r-1} . \tag{2.27}$$

Proof of the lemma. It is sufficient to prove that the group

$$C_r^T = \{A^T \mid A \in C_r\}$$

with matrix multiplication is commutative and that for $r \geq 1$:

$$C_r^T \cong C_1 \times \underbrace{C_0 \times \dots \times C_0}_{r-1} .$$

It is easy to verify that the mapping $\tau((z, A)) = zA$ defines an isomorphism between $C_1 \times G_h$ and C_r^T, where $h = (0, \dots, 0, 1) \in \mathbb{C}^r$. We saw that $G_h \cong (C_0)^{r-1}$, and hence $C_r^T \cong C_1 \times (C_0)^{r-1}$ and the group C_r^T is commutative. That finishes the proof of the lemma. ■

Using the above lemma we obtain $C_3 \cong C_1 \times C_0 \times C_0$. Using this isomorphism, the isomorphism (2.26) and τ we obtain that

$$G_h \cong \underbrace{C_1 \times \dots \times C_1}_{s} \times \underbrace{C_0 \times \dots \times C_0}_{m-s}$$

and the proof of Theorem 2.2 is finished. ■

Suppose that $h_0^m \in \mathbb{C}^{m+1}$ is a non-zero vector and that the polynomial

$$P(t) = h_m - h_{m-1}t + h_{m-2}t^2 - \dots + (-1)^m h_0 t^m$$

has exactly **s** different roots. Then from Theorem 2.2 we obtain that the derived **group** $(G^{(m)}, *)$ for each affine com(m+1,m)- group with characteristic polynomial $P(t)$ is isomorphic to

$$\underbrace{C_1 \times \dots \times C_1}_{s} \times \underbrace{C_0 \times \dots \times C_0}_{m-s} \quad (s \in \{0, 1, \dots, m+1\}). \tag{2.28}$$

Hence we have proved that there exist at least **m+1 non-isomorphic affine** com(m+1,m)-groups, because for different values **of s** the derived groups $(G^{(m)}, *)$ are non-isomorphic. As a consequence of the following Theorem 2.3, it follows that each affine com(m+k,m)-group is **induced by** an affine com(m+1,m)-group. Hence, we will prove that for each **k≥1** there exist at least m+1 non-isomorphic affine com(m+k,m)-groups.

As an important consequence of Theorem 2.2 we obtain the following Theorem:

Theorem 2.3. Let an affine com(m+k,m)-semigroup (group) be given such that its characteristic polynomial is non-zero, and considered as a polynomial of one variable has s different roots. Then there are exactly k^s different affine com(m+1,m)-semigroups (groups) which induce the given affine com(m+k,m)-semigroup (group). Specially, each affine com(m+k,m)-group on $\mathbb{C}$ is induced by one affine com(m+1,m)-group on $\mathbb{C}$.

Proof. Let us suppose that the characteristic polynomial as a function of one variable is

$$P(t) = h_m - h_{m-1}t + h_{m-2}t^2 - \dots +(-1)^m h_0 t^m.$$

We know that each matrix $\bar{\varphi}_r$ of each transformation φ_r $(r\in\mathbb{C}^{(k)})$ belongs to the group G_h where $h=h_0^m$. Besides that, the group H(G,f) which is generated by $\{\varphi_r \mid r\in\mathbb{C}^{(k)}\}$ has m independent parameters, and thus H(G,f) must coincide with G_h.

Now we can use Theorem I 3.7. The following equation

$$x^k = \bar{\varphi}_{a_1,\dots,a_1} \cdot \bar{\varphi}_{a_2,\dots,a_2} \cdot \dots \cdot \bar{\varphi}_{a_{k-1},\dots,a_{k-1}} \qquad (2.29)$$

in the group $H(G,f)=G_h$ has exactly k^s solutions because according to Theorem 2.2 (2.28) holds. Thus the given affine com(m+k,m)-semigroup (group) is induced by k^s com(m+k,m)-semigroups (groups). It remains to prove that each of these k^s com(m+1,m)-semigroups (groups) is affine. From the construction of the transformation φ'_r from the proof of Theorem I.3.6 we know that

$$\varphi'_r = \varphi_{r,a_1,\dots,a_{k-1}} \circ u^{-1}$$

where u is a solution of the equation

$$u^k = \varphi_{a_1,\dots,a_1} \circ \dots \circ \varphi_{a_{k-1},\dots,a_{k-1}}.$$

Hence

$$\bar{\varphi}'_r = \bar{\varphi}_{r,a_1,\dots,a_{k-1}} \circ X^{-1}$$

where X is a solution of the matrix equation (2.29), in G_h. The matrix $\bar{\varphi}_{r,a_1,\dots,a_{k-1}}$ represents an affine transformation, its elements are linear functions of r, and the matrix X^{-1} has constant elements and represents an affine transformation. Thus $\bar{\varphi}'_r$ represents an affine transformation whose elements depend linearly on r. We will prove that the matrix $\bar{\varphi}'_r$ has the form (2.9).