dorff topological space and each point $p \in M$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$. We will call these groups locally euclidean topological com $(m+k,m)$-groups. Specially, if $m=k=1$, then it is known [18] that each locally euclidean group is a Lie group, and hence M is a differentiable manifold.

Proposition 0.19. If (M,f) is a locally euclidean topological com-$(m+k,m)$-group, then $M^{(m)}$ is a manifold of dimension $m \cdot \dim M$. ∎

Proposition 0.20. If (M,f) is a connected locally euclidean topological com$(m+k,m)$-group, then the derived group $(M^{(m)},*)$ is a Lie group, and, moreover,

$$M^{(m)} \cong \mathbb{R}^t \times (S^1)^q. \ \blacksquare \tag{0.23}$$

Theorem 0.21. Let (M,f) be a connected locally euclidean topological com$(m+k,m)$-group. Then this group is induced by k^s connected locally euclidean topological com$(m+1,m)$-groups where s is the rank of the homology group $H_1(M^{(m)},\mathbb{Z})$. Specially if $M^{(m)}$ is simply-connected manifold, then the corresponding com$(m+k,m)$-group is induced by exactly one com-$(m+1,m)$-group.∎

Proposition 0.22. If $(\mathbb{R},f)$ is a topological com$(n,1)$-group, then $(\mathbb{R},f)$ is isomorphic to $(\mathbb{R},g)$ where $g(r_1^n)=r_1+r_2+\ldots+r_n$. ∎

Theorem 0.23. If (M,f) is a locally euclidean topological com-$(m+k,m)$-group for $m \ge 2$, then $\dim M=2$, M is an orientable manifold and M is not homeomorphic to the sphere S^2. ∎

In section III.4. we give a classification of locally euclidean topological com(n,m)-groups under certain conjectures, made under the influence of the discussion from the last section III.3.

Conjecture 1. Let M be a given topological manifold and let $m \ge 2$.

(i) If $0 \le t \le m$, and $M^{(m)} \cong \mathbb{R}^{2m-t} \times (S^1)^t$ which is equivalent to $M^{(m)} \cong \mathbb{C}^{m-t} \times (\mathbb{C}\setminus\{0\})^t$, then $M \cong \mathbb{C}\setminus\mathcal{A}_t$ where $\mathcal{A}_t$ is a set of t distinct complex numbers.

(ii) If $t \ge m$, then there does not exist a topological manifold such that $M^{(m)} \cong \mathbb{R}^{2m-t} \times (S^1)^t$

With this Conjecture we have the following Corollary.

Corollary 0.24. If M is a connected topological manifold and (M,f) is a topological com(n,m)-group for $m \ge 2$, and if Conjecture 1 holds, then M is homeomorphic to one of the manifolds $\mathbb{C}\setminus\mathcal{A}_t$, for $0\le t\le m$. ■

Theorem 0.25. If Conjecture 1 holds, then any two connected, locally euclidean com(n,m)-groups determining the same surface S, are isomorphic. ■

For a smooth surface S in $(C_1)^t\times(C_0)^q$, where $C_0 = (\mathbb{C},+)$ and $C_1 = (\mathbb{C}\setminus\{0\},\cdot)$, we state the following property.

Property P : For each $u\in(C_1)^t\times(C_0)^{m-t}$, there exists a unique element $v_1^m\in S^{(m)}$, such that $u = v_1 * v_2 * \ldots * v_m$, where $*$ is the operation in the group $(C_1)^t\times(C_0)^{m-t}$.

The examination of all the locally euclidean topological com(n.m)-groups can be reduced to the examination of all the submanifolds S in $(C_1)^t\times(C_0)^{m-t}$ of complex dimension 1, which satisfy Property P.

Conjecture 2. Each connected, locally euclidean topological com-(n,m)-group is isomorphic to affine com(n,m)-group.

I. COMMUTATIVE VECTOR VALUED GROUPS

I.1. Preliminaries

Vector valued groupoids, semigroups and groups are examined in detail in [2], [7], ... , [3]. One generalization of (usual, binary) commutative groupoids, semigroups and groups, introduced in [4], [5], is the notion of fully commutative vector valued groupoids, semigroups and groups. Here we will call such structures only "commutative" instead of "fully commutative".

We recall some basic definitions and facts about vector valued and commutative vector valued structures, although some of them are given in the introduction.

From now on, let n, m be two positive integers, such that $n-m=k\geq 1$, and let Q be a nonempty set.

Definition 1.1 A map $f:Q^n\to Q^m$ is called **(n,m)-operation**, and the pair (Q,f) is called **(n,m)-groupoid**. An (n,m)-groupoid (Q,f) is called an **(n,m)-semigroup** if the operation f is associative, i.e. for every $1\leq i\leq k$ and every $x_1^{n+k}\in Q^{n+k}$,

$$f(x_1^i f(x_{i+1}^{i+n})x_{i+n+1}^{n+k})=f(f(x_1^n)x_{n+1}^{n+k}) \ . \tag{1.1}$$

An (n,m)-semigroup is called an **(n,m)-group** if for each $a\in Q^k$, $b\in Q^m$, the equations

$$f(ax)=b=f(ya) \tag{1.2}$$

have solutions $x,y\in Q^m$.

In [10], [13] it is proven that if (Q,f) is an (m+1,m)-group with

$m \geq 2$ and if Q is a finite set, then $|Q|=1$, i.e. Q has only one element. A combinatorial description of free (n,m)-groups is given in [10], [12], showing that each infinite set is a carrier of an (n,m)-group.

In order to define the fully commutative analog of the above notions, we need the following definitions and conventions.

Let $Q^{+} = \cup_{p \geq 1} Q^{p}$ and let $Q^{(+)} = \cup_{p \geq 1} Q^{(p)}$. Then, it can be checked that Q^{+} i.e. $Q^{(+)}$ is a free i.e. a free commutative semigroup generated by the set Q. Let $\pi: Q^{+} \to Q^{(+)}$ be the natural projection, by considering $Q^{(+)}$ as a factor semigroup of Q^{+}.

Let $(G,\cdot)$ be a semigroup, and let $Q \subseteq G$. We define a family $\{Q_{\alpha} | \alpha \geq 1\}$ of subsets of G, inductively, by:

$$Q_{1}=Q, \quad Q_{\alpha+1}=Q_{\alpha} \cdot Q \tag{1.3}$$

where $M \cdot N = \{x \cdot y | x \in M, \ y \in N\}$ for $M, N \subseteq G$.

The following facts can be easily checked.

Proposition 1.1 Let $(G,\cdot)$ be a semigroup and $Q \subseteq G$. Then the subsemigroup $\langle Q \rangle$ of G generated by Q is the union of the Q_{α}, i.e.

$$\langle Q \rangle = \cup_{\alpha \geq 1} Q_{\alpha}, \tag{1.4}$$

and $\langle Q \rangle$ is a factor semigroup of Q^{+}. If G is a commutative semigroup, then $\langle Q \rangle$ is a factor semigroup of $Q^{(+)}$ and $\tau' \circ \pi = \tau$, where $\tau: Q^{+} \to \langle Q \rangle$ and $\tau': Q^{(+)} \to \langle Q \rangle$ are the natural projections.■

In the case $G=Q^{+}$, i.e. $G=Q^{(+)}$, the natural projection $\tau: Q^{+} \to \langle Q \rangle$ is the identity map, so we can identify Q^{p}, i.e. $Q^{(p)}$ with Q_{p}. Let $\pi_{p}: Q^{p} \to Q^{(p)}$ and $\pi: Q^{+} \to Q^{(+)}$ be the natural projections. Note that $\pi_{p}(a_{1}^{p}) = \pi_{p}(b_{1}^{p})$ iff $b_{1}, b_{2}, \ldots, b_{p}$ is a permutation of $a_{1}, a_{2}, \ldots, a_{p}$, i.e. $a_{1}^{p} = b_{1}^{p}$ in $Q^{(p)}$.

Definition 1.2. A map $f: Q^{(n)} \to Q^{(m)}$ is called a **commutative (n,m)-operation** on Q, and the pair (Q,f) is called a **commutative (n,m)-groupoid**. A commutative (n,m)-groupoid is called a **commutative (n,m)-semigroup**, if the operation f is associative, i.e. for each $1 \leq i \leq k$ and each $x_{1}^{n+k} \in Q^{(n+k)}$,

$$f(x_1^i f(x_{i+1}^{i+n}) x_{i+n+1}^{n+k}) = f(f(x_1^n) x_{n+1}^{n+k}) \ . \tag{1.5}$$

A commutative (n,m)-semigroup (Q,f) is called a **commutative (n,m)-group** if for each $a \in Q^{(k)}$, $b \in Q^{(m)}$, the equation

$$f(ax) = b \tag{1.6}$$

has solution $x \in Q^{(m)}$.

For shorter notations, we will write "com(n,m)-..." instead of "commutative (n,m)-....".

It is easy to check that a com(n,m)-groupoid is a com(n,m)-semigroup iff for each $x_1^{n+k} \in Q^{(n+k)}$,

$$f(f(x_1^n) x_{n+1}^{n+k}) = f(f(x_2^{n+1}) x_{n+2}^{n+k}, x_1). \tag{1.7}$$

The following theorem, shown in [5], is a generalization of the associative law for semigroups.

Theorem 1.2. Let (Q,f) be a com(n,m)-semigroup. For $s \geq 1$, let $f^{(s)}: Q^{(m+s+k)} \to Q^{(m)}$ be defined inductively by:

$$f^{(1)} = f; \quad f^{(s+1)}(xy) = f(f^{(s)}(x), y); \ x \in Q^{(m+sk)}, \ y \in Q^{(k)}. \tag{1.8}$$

Then:

(i) For each $s \geq 1$, $f^{(s)}$ is well defined;

(ii) For each $s \geq 1$, $(Q, f^{(s)})$ is a com(m+sk,m)-semigroup; and

(iii) For each $s,t \geq 1$, $x \in Q^{(m+sk)}$, $y \in Q^{(tk)}$,

$$f^{(t)}(f^{(s)}(x)\ y) = f^{(s+t)}(xy). \ \blacksquare$$

Definition 1.3. The com(m+sk,m)-semigroup $(Q, f^{(s)})$ from the above Theorem is said to be **induced by the com(m+k,m)-semigroup** (Q,f), i.e. (Q,f) **induces** $(Q, f^{(s)})$. In general, **a com(m+sk,m)-groupoid** (Q,g) is said to be **induced** by a com(m+k,m)-groupoid (Q,f), i.e. (Q,f) **induces** (Q,g), if for each $a_1, \ldots, a_s \in Q^{(k)}$, and each $b \in Q^{(m)}$,

$$g(a_1 a_2 \ldots a_{s-1} a_s b) = f(a_1 f(a_2 \ldots f(a_{s-1} f(a_s b)) \ldots)). \tag{1.9}$$

Theorem 1.2 shows that a com(m+k,m)-semigroup induces a com(m+sk,m)-semigroup for each $s \geq 1$. This is not the case for com(m+k,m)-groupoids, i.e. it is possible a com(m+k,m)-groupoid to induce a com(m+sk,m)-groupoid for some, but not for all s, as shown by the following example.

Example: Let $Q = \{a,b,c,d\}$ and let $f: Q^{(3)} \to Q^{(2)}$ be defined by:

$f(axy) = f(bcx) = ab$ for all $x,y \in Q$; $f(bbb) = f(bbd) = f(bdd) = bc$;

$f(ccc)=f(ccd)=f(cdd)=aa$; and $f(ddd)=cc$.

Then (Q,f) is a com(3,2)-groupoid which is not a com(3,2)-semigroup because $f(c(f(ddd))=f(ccc)=aa\neq ab=f(aad)=f(f(cdd)d)$. Hence, (Q,f) does not induce a com(4,2)-groupoid on Q. On the other hand, the com(5,2)-semigroup (Q,g) where g is defined by $g(xyzuv)=ab$ for each $xyzuv\in Q^{(5)}$, is induced by (Q,f). □

Because of Theorem 1.2, when f is an associative com(n,m)-operation, we will use the notation f instead of $f^{(s)}$, i.e. we will consider f as a map

$$f: \cup_{s\geq 1} Q^{(m+sk)} \to Q^{(m)}. \tag{1.10}$$

The following Proposition is shown in [5].

Proposition 1.3. Let (Q,f) be a com(n,m)-semigroup. Then the following conditions are equivalent.

(i) (Q,f) is a com(n,m)-group;

(ii) (Q,f) is a com(m+sk,m)-group for some $s\geq 1$; and

(iii) (Q,f) is a com(n,m)-group for each $s\geq 1$. ■

As in the non-commutative case, commutative (n,1)-groups coincide with commutative n-groups, and specially for n=2, they coincide with the usual, binary commutative groups.

In [5] it is shown that: if (Q,f) is a com(n,m)-group, $m\geq 2$, and Q is a finite set, then $|Q|=1$ or $|Q|=2$, and that each infinite set is a carrier of a com(n,m)-group.

Now we will introduce the notions of commutative vector valued substructures, homomorphisms and isomorphisms analogous to the same vector valued notions, given in [3]. We will give the definitions for these notions only in the case of com(n,m)-groupoids. These notions for com(n,m)-semigroups and com(n,m)-groups are defined similarly.

Definition 1.4. Let (Q,f) and (P,g) be com(n,m)-groupoids. We say that **(P,g) is a com(n,m)-subgroupoid of (Q,f)** if $P\subseteq Q$ and g is a restriction of f, i.e. if $P\subseteq Q$ and $g(x) = f(x)$ for each $x\in P^n$.

Definition 1.5. Let (Q,f) and (P,g) be com(n,m)-groupoids, and $h:Q\to P$ be a map. We say that h **is a homomorphism** if for each $x\in Q^n$,

$$h^{(m)}(f(x)) = g(h^{(n)}(x)), \tag{1.11}$$

where for each $t \geq 1$, the map $h^{(t)}: Q^{(t)} \to P^{(t)}$ is defined by:

$$h^{(t)}(q_1^t) = (h(q_1), h(q_2), \ldots, h(q_t)). \tag{1.12}$$

We say that h is an **isomorphism**, if it is a bijection and a homomorphism. In this case we say that (Q,f) and (P,g) are **isomorphic**.

The next Proposition gives a way for a construction of new com(n,m)-groupoids, -semigroups and -groups, from given ones.

Proposition 1.4. Let (Q,f) be a com(n,m)-groupoid, -semigroup or -group, and let $h: Q \to P$ be a bijection. Then (P,g), where $g: P^{(n)} \to P^{(m)}$ is defined by $g = h^{(m)} \circ f \circ (h^{(n)})^{-1}$, is a com(n,m)-groupoid, -semigroup, or -group respectively, where

$$(h^{(n)})^{-1}(x_1^n) = (h^{-1}(x_1), h^{-1}(x_2), \ldots, h^{-1}(x_n)).$$

Moreover, h is an isomorphism, i.e. (Q,f) and (P,g) are isomorphic.

Proof. The proof follows directly from the definition of the map g, the definitions of com(n,m)-groupoids, -semigroups and -groups, respectively, and the definition of isomorphisms. ∎

§.2. Universal covering commutative groups

In this section we will recall some results from [5], about the universal covering commutative groups.

For the rest of this section let (Q,f) be given com(n,m)-semigroup, $n-m=k\geq 1$, let p be the least non-negative integer, such that $m+p\equiv 0 (\mathrm{mod}\ k)$, and let s be the least non-negative integer such that $k(s-1) < m \leq ks$ and $m+p=ks$.

Definition 2.1. Define a binary operation $*$ on $Q^{(m)}$ by:

$$a*b=f(acb) \tag{2.1}$$

where $c\in Q^{(p)}$ for $p\geq 1$; and c is the empty symbol for p=0, i.e.

$$a*b=f(ab). \tag{2.2}$$

for p=0.

Proposition 1.3 implies that $*$ is well defined and that $(Q^{(m)},*)$ is a semigroup.

We say that $(Q^{(m)},*)$ is a **derived semigroup for** (Q,f).

The semigroup $(Q^{(m)},*)$ is unique for $p=0$. In the case $p\geq 1$, the operation $*$ depends on c. The following example shows that for $p\geq 1$ it is possible to have two derived semigroups for one com(n,m)-semigroup which are not even isomorphic.

Example. Let $Q = \{a,b\}$ be a two element set. Denote $a^5, b^5\in Q^{(5)}$ by α and β respectively. Let $f:Q^{(7)}\to Q^{(5)}$ be defined by $f(x_1^7)=\alpha$ if for some i, $x_i=a$, and $f(x_1^7)=\beta$ if for all i, $x_i\neq a$. It is easy to check that (Q,f) is a com(7,5)-semigroup. The semigroup $(Q^{(5)},*')$ defined via a is the constant semigroup $x*'y=\alpha$, while the semigroup $(Q^{(5)},*")$ defined via b is not a constant semigroup, because $\alpha*"\alpha=\alpha$ and $\beta*"\beta=\beta$. Hence $(Q^{(5)},*')$ and $(Q^{(5)},*")$ are not even isomorphic.

The following Proposition shows that a situation like the one in the example for com(n,m)-groups is not possible.

Proposition 2.1. Let (Q,f) be a com(n,m)-group. If $a,b\in Q^{(p)}$, then the groups $(Q^{(m)},*')$ and $(Q^{(m)},*'')$ defined by a and b respectively, are isomorphic.

Proof. Let $c\in Q^{(m)}$ be a fixed element. Then, it is easy to verify that the map $\rho:Q^{(m)}\to Q^{(m)}$ defined by $\rho(x)=y$ if $f(xac) = f(ybc)$ is an isomorphism from $(Q^{(m)},*)$ defined by a to $(Q^{(m)},*)$ defined by b. ∎

Because of the above Proposition, if (Q,f) is a com(n,m)-group, we will say that $(Q^{(m)},*)$ is **its derived group**, or that $(Q^{(m)},*)$ is **the derived group for** (Q,f).

Proposition 2.2. (Q,f) is a com(n,m)-group iff its derived semigroup $(Q^{(m)},*)$ is a commutative group.

Proof. Proposition 1.3 implies that if (Q,f) is a com(n,m)-group, then $(Q^{(m)},*)$ is a commutative group. Conversely, let $(Q^{(m)},*)$ be a commutative group. It is enough to show that the equation (1.6) has a solution in $Q^{(m)}$. Let $a\in Q^{(k)}$, $b\in Q^{(m)}$. Choose an arbitrary element $e\in Q$. If $s\geq 1$, then $ae^{k(s-1)}\in Q^{(ks)}=Q^{(m+p)}$, i.e. $ae^{k(s-1)}=(ad)c$, where $ad\in Q^{(m)}$, $c\in Q^{(p)}$ for $p\geq 1$, and c is the empty symbol for $p=0$. If $p\geq 1$, let $(Q^{(m)},*)$ be defined via c. Then the equation $(ad)*y=b$ has a solution $y\in Q^{(m)}$. This implies that, $f(adcy)=b=f(af(dcy))$, i.e. $f(dcy)=x$ is a solution of the equation $f(ax)=b$. If $p=0$, then the equation $ad*y=b$ has a solution $y\in Q^{(m)}$, which implies that $x=f(dy)$ is a solution of the equation $f(ax)=b$. ∎

Proposition 2.3. ([5]) If (Q,f) is a com(n,m)-group, then (Q,f) is cancellative, i.e. for each $a\in Q^{(k)}$ and $b,c\in Q^{(m)}$, $f(ab)=f(ac)$ implies $b=c$ in $Q^{(m)}$. ∎

Definition 2.2. (i) For $x\in Q^{(r)}\subseteq Q^{(+)}$, we say that **dimension** of x is r, and write $\dim(x)=r$.

(ii) Define a relation $\cong$ on $Q^{(+)}$ by:

$$u\cong v \text{ iff there is } a\in Q^{(+)} \text{ so that } f(au)=f(av). \tag{2.3}$$

Proposition 2.4. ([5]) (i) $u\cong v$ implies that $\dim(u)\equiv\dim(v) \pmod k$;

(ii) $\cong$ is a congruence on $Q^{(+)}$ and the factor structure $Q^{(+)}/\cong$ is a commutative semigroup. ∎

Definition 2.3. The commutative semigroup $Q^{(+)}/\cong$ is called a **universal commutative semigroup for** (Q,f) and is denoted by $Q^{(v)}$.

Proposition 2.5. ([5]) If (Q,f) is a com(n,m)-group, then:

(i) $Q^{(v)}$ is a commutative group;

(ii) $\dim(u)=\dim(v)\le m$ implies that: if $u\cong v$ then $u=v$;

(iii) $\dim(u)\le\dim(v)<\dim(u)+k$ implies that:

if $u\cong v$ then $\dim(u)=\dim(v)$;

(iv) If $r\ge 1$ and j is the smallest non-negative integer such that $r\equiv m+j (\mathrm{mod}\ k)$, then for each $u\in Q^{(r)}$, $v\in Q^{(j)}$, there exists a unique $w\in Q^{(m)}$ so that $u\cong vw$. In the case $j=0$, v is the "empty symbol", i.e. $u\cong w$; and

(v) $Q^{(m+p)}/\cong$ is a subgroup of $Q^{(v)}$. ∎

The proof of the following Proposition follows directly from Definition 2.2, and Propositions 2.4 and 2.5.

Proposition 2.6. Let $K=\cup_{t\ge 1}Q^{(kt)}\subseteq Q^{(+)}$, and let $\cong$ be the restriction of $\cong$ on K. Then:

(i) K is a subsemigroup of $Q^{(+)}$;

(ii) $\cong$ is a congruence on K, and $K/\cong$ (denoted by $K^{(v)}$) is a subsemigroup of $Q^{(v)}$; and

(iii) If (Q,f) is a com(n,m)-group, then $K^{(v)}$ is a subgroup of $Q^{(v)}$ and $Q^{(m+p)}/\cong$ is a subgroup of $K^{(v)}$. ∎

Definition 2.4. If (Q,f) is a com(n,m)-group, then we say that $Q^{(v)}$ is a **universal commutative group** for (Q,f).

The next Theorem gives another description of $Q^{(v)}$.

Theorem 2.7. ([5]) Let $(G,\cdot)$ be a commutative group and $Q\subseteq G$. Let $\tau_r:Q^{(r)}\to Q_r$ be the restriction of the natural projection $\tau:Q^{(+)}\to\langle Q\rangle$. (See Definition 1.2 and Proposition 1.1.) Suppose that the following conditions are satisfied:

(i) The map $\tau_m:Q^{(m)}\to Q_m$ is bijective;

(ii) For each $x\in Q_k$, $Q_m=x\cdot Q_m$ ($=\{x\}\cdot Q_m$);

(iii) If $0\le i\le j<k$ and $Q_{m+i}\cap Q_{m+j}\ne\emptyset$, then $i=j$; and

(iv) $G=\cup_{\alpha\ge 1}Q_\alpha$.

Then, (Q,g), where $g:Q^{(n)}\to Q^{(m)}$, is defined by:

$$g(a)=b \ \text{ iff } \ \tau_n(a)=\tau_m(b), \tag{2.4}$$

is a com(n,m)-group, whose universal covering commutative group $Q^{(v)}$ is G. ∎

The above Theorem together with Propositions 2.4, 2.5 and 2.6 implies the following facts.

Proposition 2.8 ([5]) If (Q,f) is a com(n,m)-group whose universal covering commutative group is G, then:

(i) Q_{m+p} is a subgroup of G, isomorphic to $Q^{(m+p)}/\cong$, and $G/Q_{m+p}=\{Q_m, Q_{m+1}, \dots, Q_{n-1}\}$ is a cyclic group of order k, with a generator Q_{m+p+1};

(ii) For every $r\geq 1$ and every $u\in Q_j$, $Q_r = uQ_m$, where j is the smallest non-negative integer such that $r\equiv m+j(\mathrm{mod}\ k)$, and in the case $j=0$, $Q_0=\{1\}$, where 1 is the neutral element in G;

(iii) If $a\in Q$, then $Q^{(v)}=Q_m \cup aQ_{m+1} \cup a^2 Q_{m+2} \cup \dots \cup a^{k-1} Q_{n-1}$;

(iv) For every $1\leq i\leq m$, the canonical map $\tau_i : Q^{(i)} \to Q_i$ is bijective; and

(v) $Q \subseteq Q_{m+p+1}$; $Q^{-1} \subseteq Q_{m+p-1}$ (in the case $p=0$, $Q^{-1} \subseteq Q_{n-1}$); and $QQ^{-1} \subseteq Q_{m+p}$, where

$Q^{-1}=\{a^{-1} \mid a\in Q,\ a^{-1}$ is the inverse of a in $G\}$. ■

The universal commutative group $Q^{(v)}$ for a given com(n,m)-group (Q,f) has the presentation $\langle Q;\Sigma\rangle$ in the category of commutative groups with Q as the set of generators and Σ the set of relations where

$$\Sigma = \{a_1\cdot a_2\cdot \dots a_n = b_1\cdot b_2\cdot \dots \cdot b_m \mid f(a_1^n) = b_1^m\}.$$

I.3. Transformations on $Q^{(m)}$ defined by (n, m)-operations

In this section we will give a description of com(n, m)-semigroups and com(n, m)-groups via transformations on $Q^{(m)}$. For the rest of this section let (Q,f) be a com(n, m)-groupoid, $n-m=k\geq 1$, $m+p=ks$, $k(s-1)<m\leq ks$, $p\geq 0$.

Definition 3.1. Let $Hom(Q^{(m)},Q^{(m)})=\{\varphi\,|\,\varphi:Q^{(m)}\to Q^{(m)}\}$ be the semigroup of all the maps from $Q^{(m)}$ to $Q^{(m)}$, with the operation composition of maps. For each $x\in Q^{(k)}$ we define $\varphi_x:Q^{(m)}\to Q^{(m)}$ by:

$$\varphi_x(a)=f(ax). \tag{3.1}$$

Let $H'(Q,f)=\{\varphi_x\,|\,x\in Q^{(k)}\}$, and let $H(Q,f)=\langle H'(Q,f)\rangle$ be the subsemigroup of $Hom(Q^{(m)},Q^{(m)})$, generated by $H'(Q,f)$.

Proposition 3.1. (Q,f) is a com(n, m)-semigroup iff $H(Q,f)$ is commutative.

Proof. If (Q,f) is a com(n, m)-semigroup, then Theorem 1.2 implies that for each $x,y\in Q^{(k)}$, $\varphi_x\circ\varphi_y=\varphi_y\circ\varphi_x$, which implies that $H(Q,f)$ is commutative.

Conversely, let $H(Q,f)$ be a commutative semigroup. Let $x,y,z\in Q$, $v\in Q^{(m-1)}$, and $u,w\in Q^{(k-1)}$. Then: $f(f(xuyv)zw) = \varphi_{zw}(f(xvyu)) = \varphi_{zw}(\varphi_{yu}(xv)) = (\varphi_{zw}\circ\varphi_{yu})(xv) = (\varphi_{yu}\circ\varphi_{zw})(xv) = \varphi_{yu}(\varphi_{zw}(xv)) = \varphi_{yu}(f(xvzw)) = \varphi_{yu}(f(zvxw)) = (\varphi_{yu}\circ\varphi_{xw})(zv) = (\varphi_{xw}\circ\varphi_{yu})(zv) = \varphi_{xw}(\varphi_{yu}(zv)) = \varphi_{xw}(f(zvyu)) = f(f(zvyu)xw)$. This, together with (1.7) implies that (Q,f) is a com(n, m)-semigroup. ■

A similar result holds for com(n, m)-groups.

Theorem 3.2. (Q,f) is a com(n, m)-group iff $H(Q,f)$ is a commutative group.

Proof. Let (Q,f) be a com(n, m)-group. Then, Proposition 3.1 implies that $H(Q,f)$ is a commutative semigroup. Next we will consider two cases, for $p\geq 1$ and $p=0$.

Case 1. Let $p\geq 1$. Then $(Q^{(m)},\bullet)$ is a group for each $c\in Q^{(p)}$, and hence for each $a\in Q^{(m)}$ there exists $x\in Q^{(m)}$ so that $a=a\bullet x=f(acx)$. This gives a map $\psi: Q^{(p)}\to Q^{(m)}$ such that for each $c\in Q^{(p)}$, $f(ac\psi(c))=a$, i.e. $\psi(c)$ is the neutral element for the operation $\bullet$ defined by c. Let $x=y_1y_2 \dots y_tz\in Q^{(k)}$ be given, where $y_i\in Q^{(p)}$, $z\in Q^{(r)}$, for $k=tp+r$, $0\leq r<p$. Let $y\in Q^{(p-r)}$ be an arbitrary element. Then $zy\in Q^{(p)}$. For each $a\in Q^{(m)}$, $f(ay_1\psi(y_1)y_2\psi(y_2) \dots\dots\dots\dots y_t\psi(y_t)zy\psi(zy))=a$, i.e. $f(ax\psi(y_1) \dots\psi(y_t)y\psi(zy))=a$. For the numbers m,n,k,s,p,r,t we have: $tm+p-r+m = t(ks-p)+p-r+m = tks-tp-r+p+m = tks-k+ks = k(ts+s-1) = kq$, for some $q\geq 1$. Thus, $\psi(y_1)\dots\dots\psi(y_t)y\psi(zy)\in Q^{(kq)}$, i.e. $\psi(y_1) \dots \psi(y_t)\psi(zy)y=x_1x_2 \dots x_q$ for some $x_i\in Q^{(k)}$. This implies that $f(xax_1x_2 \dots x_q)=a$ for each $a\in Q^{(m)}$, i.e. $\varphi_x\circ\varphi_{x_1}\circ \dots \circ\varphi_{x_q}=id$.

Case 2. Let $p=0$, i.e. $m=ks$. Then $(Q^{(m)},\bullet)$, where $x\bullet y=f(xy)$, is a commutative group. So there is a neutral element $e\in Q^{(m)}$, such that $f(ae)=a$ for each $a\in Q^{(m)}$. Since (Q,f) is a com(n,m)-group, for given $x\in Q^{(m)}$, there exists $y\in Q^{(m)}$ with $f(xy)=e$, and so, for each $a\in Q^{(m)}$, $f(xay)=a$. Let $y=y_1 \dots y_s$, for some $y_i\in Q^{(k)}$. Then, $f(xy_1 \dots y_sa)=a$ for each $a\in Q^{(m)}$, i.e. $\varphi_x\circ\varphi_{y_1}\circ \dots \circ\varphi_{y_s}=id$.

Since $H(Q,f)=\langle H'(Q,f)\rangle$ is a subsemigroup of $Hom(Q^{(m)},Q^{(m)})$, we have shown that $id\in H$ and that each $\varphi_x\in H'(Q,f)$ has an inverse in $H(Q,f)$. Thus, $H(Q,f)$ is a commutative group.

Conversely, let $H(Q,f)$ be a commutative group. Then Proposition 3.1 implies that (Q,f) is a com(n,m)-semigroup. Hence, it is enough to show that the equation $f(ax)=b$ has a solution $x\in Q^{(m)}$ for each $a\in Q^{(k)}$ and each $b\in Q^{(m)}$. But, since $\varphi_a\in H(Q,f)$ has an inverse, say $\varphi_{a_1}\circ\dots\circ\varphi_{a_t}$, it follows that $\varphi_a\circ\varphi_{a_1}\circ\dots\circ\varphi_{a_t}(b)=b$, i.e.

$$b=f(aa_1 \dots a_tb)=f(ax), \quad \text{where} \quad x=f(a_1 \dots a_tb). \ \blacksquare$$

The last part of the proof of the above Theorem leads to the following weaker Proposition, but whose proof is simpler.

Proposition 3.3. A com(n,m)-semigroup (Q,f) is a com(n,m)-group, iff for each $x\in Q^{(k)}$, φ_x is a bijection. ■

Proposition 3.4. Let (Q,f) be a com$(m+k,m)$-group. Then, the derived group $(Q^{(m)},\bullet)$ is isomorphic to the group $H(Q,f)$.

Proof. Let us suppose that $m+p=ks$. Next, we define a map $\tau: Q^{(m)} \to H(Q,f)$ by $\tau(x) = \varphi_{xa}$, where $a \in Q^{(p)}$. Then, $\tau(x*y) = \tau(f(xya)) = \varphi_{f(xya)a} = \varphi_{xaya} = \varphi_{xa} \circ \varphi_{ya} = \tau(x) \circ \tau(y)$. Hence, τ is an isomorphism from $(Q^{(m)}, *)$ to the subgroup $\tau(Q^{(m)})$ of $H(Q,f)$. It is enough to show that $\tau(Q^{(m)}) = H(Q,f)$. For this, it is enough to show that $H'(Q,f) \subseteq \tau(Q^{(m)})$. So, let $\varphi_b \in H'(Q,f)$, i.e $b \in Q^{(k)}$. We choose an arbitrary element $x \in Q^{(m)}$. Then there exists unique $y \in Q^{(m)}$ such that $f(bxac)=f(yac)$. This implies that $\varphi_{bxa} = \varphi_{ya}$, i.e. $\varphi_b \circ \varphi_{xa} = \varphi_{ya}$. So, $\varphi_b = \varphi_{ya} \circ \varphi_{xa}^{-1}$, i.e. $\varphi_b = \tau(y) \circ (\tau(x))^{-1}$, which shows that $H'(Q,f) \subseteq \tau(Q^{(m)})$. ∎

Next, we will examine under what conditions a given com(m+sk)-group is induced by a com(m+k, m)-group.

Proposition 3.5. A com(m+sk, m)-semigroup (Q,g) is induced by a com(m+k, m)-semigroup (Q,f) iff for each $x_1, \ldots, x_s \in Q^{(k)}$,

$$\varphi_y = \varphi_{x_1} \circ \ldots \circ \varphi_{x_s}, \tag{3.2}$$

where $y=x_1 \ldots x_s \in Q^{(sk)}$, $\varphi_{x_i} \in H'(Q,f)$ and $\varphi_y \in H'(Q,g)$. This is equivalent to $H(Q,g)$ being generated by $H'(Q,f)$, i.e. equivalent to $H(Q,g)$ being a subsemigroup of $H(Q,f)$.

Proof. The proof follows directly from the Definitions 1.3 and 3.1. Namely, for each $a \in Q^{(m)}$,

$$\varphi_y(a) = g(ya) = g(x_1 \ldots x_s a) = f(x_1 \ldots f(x_s a) \ldots)$$

$$= \varphi_{x_1}(\ldots \varphi_{x_s}(a) \ldots) = (\varphi_{x_1} \circ \ldots \circ \varphi_{x_s})(a). \ ∎$$

Theorem 3.6. For a com(m+sk, m)-group (Q,g), the following conditions are equivalent:

(i) (Q,g) is induced from a com(m+k, m)-group (Q,f);

(ii) There exist $a_1, \ldots, a_{s-1} \in Q^{(k)}$, such that the equation

$$x^s = x \circ \ldots \circ x = \varphi_{b_1} \circ \ldots \circ \varphi_{b_{s-1}} \tag{3.3}$$

has a solution $x \in Hom(Q^{(m)}, Q^{(m)})$, which commutes with every element from $H(Q,g)$, where $b_i = \underbrace{a_i \ldots a_i}_{s}$.

(iii) For each $a_1, \ldots, a_{s-1} \in Q^{(k)}$, the equation (3.3) has a solution $x \in Hom(Q^{(m)}, Q^{(m)})$, which is a bijection.

Proof: It is obvious that (iii) implies (ii). We will show that (i) implies (iii), and that (ii) implies (i).

(i) implies (iii). Let (Q,g) be induced from a $com(m+k,m)$-group (Q,f), and let a_i, b_i be as in (ii). Proposition 3.5 implies that for each i,

$$\varphi_{b_i} = \underbrace{\varphi_{a_i} \circ \dots \circ \varphi_{a_i}}_{s} .$$

Then, $$\varphi_{b_1} \circ \dots \circ \varphi_{b_{s-1}} = \underbrace{(\varphi_{a_1} \circ \dots \circ \varphi_{a_1})}_{s} \circ \dots \circ \underbrace{(\varphi_{a_{s-1}} \circ \dots \circ \varphi_{a_{s-1}})}_{s} =$$

$$= \underbrace{(\varphi_{a_1} \circ \dots \circ \varphi_{a_{s-1}}) \circ \dots \circ (\varphi_{a_1} \circ \dots \circ \varphi_{a_{s-1}})}_{s} = x^s,$$

where

$$x = \varphi_{a_1} \circ \dots \circ \varphi_{a_{s-1}} . \tag{3.4}$$

Since each $\varphi_{a_i} \in H(Q,f)$, $H(Q,g) \subseteq H(Q,f)$ and $H(Q,f)$ is commutative, it follows that $x \in Hom(Q^{(m)},Q^{(m)})$, and that it commutes with each element from $H(Q,g)$.

(ii) implies (i). Let a_i, b_i and x be as in (ii). For $a \in Q^{(k)}$ let $b=a\dots a \in Q^{(sk)}$. Then, $x^s \circ \varphi_b = \varphi_{b_1} \circ \dots \circ \varphi_{b_{s-1}} \circ \varphi_b$. Because (Q,g) is a $com(m+sk,m)$-group, it follows that $\varphi_{b_1} \circ \dots \circ \varphi_{b_{s-1}}$ is a bijection, which implies that x is also a bijection, and moreover, $\varphi_{b_1} \circ \dots \circ \varphi_{b_{s-1}} \circ \varphi_b = (\varphi_{y(a)})^s = x^s \circ \varphi_b$, where $y(a)=a_1 \dots a_{s-1} a \in Q^{(s+k)}$. Next, we define a map $f: Q^{(m+k)} \to Q^{(m)}$, by:

$$f(ac) = (x^{-1} \circ \varphi_{y(a)})(c) \tag{3.5}$$

for $a \in Q^{(k)}$, $c \in Q^{(m)}$, and x and $\varphi_{y(a)}$ as above.

From the definition of f, it follows that:

$$f(ac) = (x^{-1})(\varphi_y(c)) = (x^{-1})(g(yc)) = (x^{-1})(g(a_1 \dots a_{s-1} ac)),$$

which implies that f is well defined, and that $\varphi_a = x^{-1} \circ \varphi_{y(a)}$. The facts that x commutes with every element from $H(Q,g)$ and that $H(Q,g)$ is commutative, imply that (Q,f) is a $com(m+k,m)$-semigroup. This, together with the definition of f, implies that for each

$c_1,\dots,c_s \in Q^{(k)}$,

$$\varphi_{c_1}\circ\dots\circ\varphi_{c_s} = (x^{-1})^s\circ\varphi_{y(c_1)}\circ\dots\circ\varphi_{y(c_s)} =$$

$$= (x^{-1})^s\circ\varphi_{b_1}\circ\dots\circ\varphi_{b_{s-1}}\circ\varphi_{c_1}\circ\dots\circ\varphi_{c_s} = \varphi_w,$$

where $w = c_1\dots c_s \in Q^{(sk)}$. Hence, (Q,g) is induced from (Q,f). The fact that $H(Q,g)$ is a commutative group, implies that the equation

$$\varphi_a\circ z = \varphi_d, \quad \text{for } a,d\in Q^{(k)},$$

has a solution

$$z = x\circ(\varphi_{y(a)})^{-1}\circ x^{-1}\circ\varphi_{y(d)}=(\varphi_{y(a)})^{-1}\circ\varphi_{y(d)}=(\varphi_a)^{-1}\circ\varphi_d \in H(Q,f).$$

Hence $H(Q,f)$ is a commutative group, and by Theorem 3.2, (Q,f) is a com$(m+k,m)$-group. ■

Theorem 3.7. Let (Q,g) be a com$(m+sk,m)$-group, and $a_i\in Q^{(k)}$, $b_i = a_1\dots a_i\in Q^{(sk)}$, $i=1,2,\dots,s-1$, be given. If the equation (3.3) has r solutions for x as in Theorem 3.6 (ii), where $r\in\{0,1,2,\dots,\infty\}$, then there are exactly r com$(m+k,m)$-groups (Q,f), such that (Q,g) is induced from (Q,f). Moreover, r does not depend on the choice of $a_1,\dots,a_{s-1}$.

We note that some of these r com$(m+k,m)$-groups may be isomorphic.

Proof. Let $U = \{(Q,f) \mid (Q,g)$ is induced from $(Q,f)\}$ be the set of all the com$(m+k,m)$-groups such that (Q,g) is induced from (Q,f), and let $V = \{x \mid x$ is a solution of (3.3) as in Theorem 3.6 (ii)$\}$. The proof of Theorem 3.6, (i) **implies** (ii), and (3.4), imply that there exists a map $F: U\to V$, defined by: $F((Q,f)) = \varphi_{a_1}\circ\dots\circ\varphi_{a_{s-1}} \in H(Q,f)$. The proof of Theorem 3.6, (ii) implies (i), and (3.5) imply that there is a map $G: V\to U$, defined by: $G(x) = (Q,f)$, where f is defined by (3.5). The following calculations show that these two maps are inverse each to the other.

$$F\circ G(x) = F((Q,f)) = \varphi_{a_1}\circ\dots\varphi_{a_{s-1}} = x^{-1}\circ\varphi_{y(a_1)}\circ\dots\circ x^{-1}\circ\varphi_{y(a_{s-1})}$$

$$= (x^{-1})^{s-1}\circ(\varphi_{a_1})^s\circ\dots\circ(\varphi_{a_{s-1}})^s = (x^{-1})^{s-1}\circ x^s = x.$$

$G\circ F((Q,f)) = G(x) = (Q,f')$, where $x=\varphi_{a_1}\circ\dots\circ\varphi_{a_{s-1}}$.

Then:

$$f'(ac) = (x^{-1})(g(a_1\dots a_{s-1}ac) = ((x^{-1})\circ\varphi_{a_1}\circ\dots\circ\varphi_{a_{s-1}}\circ\varphi_a)(c)$$

$= \varphi_a(c) = f(ac).$

Thus, $f' = f$, i.e. $G \circ F((Q,f)) = (Q,f)$.

Since the set U does not depend on the choice of $a_1, \ldots, a_{s-1}$, it follows that r does not depend on the choice $a_1, \ldots, a_{s-1}$. ∎

Definition 3.2. Let $a \in Q$ and let $b = a^k \in Q^{(k)}$. We say that the element a is **singular** if the map φ_b, denoted by $\varphi(a)$, is not a bijection. The set of all the singular elements in Q will be called the **singular set** and will be denoted by $\mathcal{R}(Q)$ or simply by $\mathcal{R}$. If $\mathcal{R}=Q$, then we say that (Q,f) is a **singular** com(n,m)-semigroup, and if $\mathcal{R} \neq Q$, then we say that (Q,f) is **non-singular.**

Theorem 3.8. Let (Q,f) be a non-singular com(n,m)-semigroup. Then $(Q \backslash \mathcal{R}, g)$ is a com(n,m)-group, where $g(x)=f(x)$.

Proof. First we will show that $(Q \backslash \mathcal{R}, f)$ is a com(n,m)-groupoid. Let $a_1, \ldots, a_n \in Q \backslash \mathcal{R}$, i.e. let $\varphi(a_i)$ be a bijection for each i, and let $f(a_1^n) = b_1^m$. We will show that for each i, $\varphi(b_i)$ is a bijection. For arbitrary $c \in Q^{(m)}$ we have

$$\varphi(b_1) \circ \varphi(b_2) \circ \ldots \circ \varphi(b_m)(c) = f((b_1)^k (b_2)^k \ldots (b_m)^k c)$$

$$= f(\underbrace{(b_1^m) \ldots (b_1^m)}_{k} c) = f(\underbrace{f(a_1^n) \ldots f(a_1^n)}_{k} c) = f(\underbrace{a_1^n \ldots a_1^n}_{k} c)$$

$$= f((a_1)^k \ldots (a_n)^k c) = \varphi(a_1) \circ \ldots \circ \varphi(a_n)(c).$$

Hence, $\varphi(b_1) \circ \varphi(b_2) \circ \ldots \circ \varphi(b_m) = \varphi(a_1) \circ \ldots \circ \varphi(a_n)$, which implies that $\varphi(b_1) \circ \varphi(b_2) \circ \ldots \circ \varphi(b_m)$ is a bijection. This, together with the fact that the (Q,f) is a com(n,m)-semigroup implies that each $\varphi(b_i)$ is a bijection.

Since $(Q \backslash \mathcal{R}, g)$ is a com(n,m)-subgroupoid of (Q,f) it follows that it is a com(n,m)-semigroup.

For each $x = a_1^k \in (Q \backslash \mathcal{R})^{(k)}$, $\varphi_x \circ \ldots \circ \varphi_x = \varphi(a_1) \circ \ldots \circ \varphi(a_k)$, which implies that for each $x \in (Q \backslash \mathcal{R})^{(k)}$, φ_x is a bijection. This, together with Proposition 3.3 implies that $(Q \backslash \mathcal{R}, g)$ is a com(n,m)-group. ∎

I.4. A construction of commutative (n,m)-semigroups

In [23] several examples of com(n,m)-semigroups and com(n,m)-groups are obtained, all of them via a specific construction. Here we will generalize this construction.

In the rest of this section, let $(Q,+,\cdot)$ be a commutative ring with a multiplicative unit 1.

Definition 4.1. Let $\{\varphi^t \mid t\geq 0\}$ be a sequence of maps $\varphi^t:Q^{(+)}\to Q$ which satisfies the following conditions:

(i) $\varphi^0(x) = 1$;

(ii) $\varphi^r(x) = 0$ for $x\in Q^{(t)}$ and $t<r$;

(iii) $\varphi^r(xy) = \sum_{0\leq s\leq r} \varphi^{r-s}(x)\varphi^s(y)$;

where the sum and the product are from the given ring.

The maps φ^t are called **symmetric functions.**

By a simple inductive argument, using the condition (iii) from Definition 4.1, it can be shown that if φ^r are symmetric functions, then they are uniquely determined by φ^1.

For given symmetric functions φ^t let $\varphi_r:Q^{(r)}\to Q^r$ be defined by $\varphi_r(a^r)=(\varphi^1(a_1^r), \varphi^2(a_1^r), \ldots, \varphi^{r-1}(a_1^r), \varphi^r(a_1^r))$.

The simplest examples of symmetric functions are the functions where φ^1 is the identity transformation on Q, i.e. $\varphi^1(x)=x$ for all $x\in Q$. For the following proposition which describes such functions, we need the following notations. For a positive integer t, let $M_t=\{1,2,\ldots, t\}$ be the set of the first t positive integers. For a given set A and a non-negative integer t, we denote by $\mathcal{P}_t(A)$ the set of all subsets X of A having t elements, i.e.

$$\mathcal{P}_t(A)=\{ X \mid X\subseteq A,\ |X|=t \}.$$

Proposition 4.1. Let φ^t be symmetric functions on $(Q,+,\cdot)$ such that $\varphi^1(x)=x$ for all $x\in Q$. Then, for all s and $1\leq t\leq s$,

$$\varphi^t(a_1^s) = \sum_{\{i_1,\dots,i_t\}\in\mathcal{P}_t(M_s)} a_{i_1}\cdot a_{i_2}\cdot \dots \cdot a_{i_t} . \tag{4.1}$$

Proof. The proof is by a double induction on t and s, and by Definition 4.1, (iii).

For $t=1$, the condition (4.1) is satisfied trivially, because by definition, $\varphi^1(x)=x$.

Assume that (4.1) is satisfied for all $t\le r$ for some $r\ge 1$ and all s. Then, by the inductive hypothesis and Definition 4.1.(iii),

$$\varphi^s(a_1^s)= \varphi^{r+1}(a_1^{r+1}) = \sum_{0\le s\le r+1} \varphi^{r+1-s}(a_1^r)\varphi^s(a_{r+1}) = \varphi^r(a_1^r)\cdot a_{r+1}$$

$$= \Big(\sum_{\{i_1,\dots,i_r\}\in\mathcal{P}_r(M_r)} a_{i_1}\cdot a_{i_2}\cdot \dots \cdot a_{i_r}\Big)\cdot a_{r+1}$$

$$= a_1\cdot a_2\cdot \dots \cdot a_{r+1}, \quad \text{i.e. (4.1) is satisfied.}$$

Next, assume that (4.1) is satisfied for all $r+1\le s\le q$ for some $q\ge r+1$. Then by the inductive hypothesis and Definition 4.1 (iii),

$$\varphi^s(a_1^s)= \varphi^{r+1}(a_1^{r+1}) = \sum_{0\le s\le r+1} \varphi^{r+1-s}(a_1^q)\varphi^s(a_{q+1}) = \varphi^r(a_1^p)\cdot a_{r+1}$$

$$= \varphi^{r+1}(a_1^q)\cdot\varphi^0(a_{q+1}) + \varphi^r(a_1^q)\cdot\varphi^1(a_{q+1})$$

$$= \varphi^{r+1}(a_1^q)\cdot 1 + \varphi^r(a_1^q)\cdot a_{q+1}$$

$$= \sum_{\{i_1,\dots,i_{r+1}\}\in\mathcal{P}_{r+1}(M_q)} a_{i_1}\cdot a_{i_2}\cdot \dots \cdot a_{i_r}\cdot a_{r+1} +$$

$$+ \Big(\sum_{\{i_1,\dots,i_r\}\in\mathcal{P}_r(M_q)} a_{i_1}\cdot a_{i_2}\cdot \dots \cdot a_{i_r}\Big)\cdot a_{q+1}$$

$$= \sum_{\{i_1,\dots,i_{r+1}\}\in\mathcal{P}_{r+1}(M_{q+1})} a_{i_1}\cdot a_{i_2}\cdot \dots \cdot a_{i_{r+1}} . \ \blacksquare$$

The symmetric functions from Proposition 4.1 have the following form:

$$\varphi^t(a_1^s) = \begin{cases} 0 & t>s \\ 1 & t=0 \\ \displaystyle\sum_{\{i_1,\dots,i_t\}\in\mathcal{P}_t(M_s)} a_{i_1}\cdot a_{i_2}\cdot \dots \cdot a_{i_t} & \text{otherwise.} \end{cases}$$

Next, assume that (Q,g) is an (n,m)-groupoid, and φ^t are given symmetric functions on $(Q,+,\cdot)$.

Definition 4.2. If there is a map $f:Q^{(n)}\to Q^{(m)}$ such that the diagram

$$\begin{array}{ccc} Q^n & \xrightarrow{\;g\;} & Q^m \\ \uparrow{\scriptstyle \mathscr{P}_n} & & \uparrow{\scriptstyle \mathscr{P}_m} \\ Q^{(n)} & \xrightarrow{\;f\;} & Q^{(m)} \end{array}$$

is commutative, we say that the com(n,m)-groupoid (Q,f) is **obtained** from the (n,m)-groupoid (Q,g) via the symmetric functions $\mathscr{P}^t$.

In the case when $\mathscr{P}_m$ is a bijection, such an f always exists, i.e. we can take $f = \mathscr{P}_m^{-1}\circ g\circ \mathscr{P}_n$. In such a case, for each $p\in Q^k$, we will denote the transformation $\mathscr{P}_m\circ\varphi_p\circ\mathscr{P}_m^{-1}: Q^m\to Q^m$ by $\bar{\varphi}_p$.

Proposition 4.2. Let (Q,f) be a com(m+k,m)-semigroup obtained from an (m+s,m)-groupoid (Q,g) via symmetric functions $\mathscr{P}^t$ where each $\mathscr{P}_t$ is a bijection and $f = \mathscr{P}_m^{-1}\circ g\circ\mathscr{P}_{m+k}$. Then the com(m+sk,m)-semigroup (Q,f^s) induced by (Q,f) is obtained from an (m+sk,m)-groupoid (Q,h) via the symmetric functions $\mathscr{P}^t$, i.e. $f^s = \mathscr{P}_m^{-1}\circ h\circ\mathscr{P}_{m+sk}$, and for each $y = p_1^s$, $p_i\in Q^{(k)}$,

$$\bar{\varphi}_y = \bar{\varphi}_{p_1}\circ\bar{\varphi}_{p_2}\circ\ldots\circ\bar{\varphi}_{p_s}.$$

Proof. The (m+sk,m)-groupoid (Q,h) is defined by $h=\mathscr{P}_m\circ f^s\circ\mathscr{P}_{m+sk}^{-1}$. For the second part, we use Proposition 3.5.

$$\begin{aligned}\bar{\varphi}_y &= \mathscr{P}_m\circ\varphi_p\circ\mathscr{P}_m^{-1} = \mathscr{P}_m\circ\varphi_{p_1}\circ\varphi_{p_2}\circ\ldots\circ\varphi_{p_s}\circ\mathscr{P}_m^{-1} = \\ &= \mathscr{P}_m\circ\varphi_{p_1}\circ\mathscr{P}_m^{-1}\circ\mathscr{P}_m\circ\varphi_{p_2}\circ\mathscr{P}_m^{-1}\circ\mathscr{P}_m\circ\ldots\circ\mathscr{P}_m^{-1}\circ\mathscr{P}_m\circ\varphi_{p_s}\circ\mathscr{P}_m^{-1} = \\ &= \bar{\varphi}_{p_1}\circ\bar{\varphi}_{p_2}\circ\ldots\circ\bar{\varphi}_{p_s}. \ \blacksquare\end{aligned}$$

The next proposition gives examples of symmetric functions with $\mathscr{P}_q$ being bijections for all $q\geq 1$.

Proposition 4.3. Let $(Q,+,\cdot)$ be an algebraically closed field, and let $\mathscr{P}^t$ be the symmetric functions on Q satisfying the condition (4.1). Then, for every $q\geq 1$, $\mathscr{P}_q:Q^{(q)}\to Q^q$ is a bijection. Moreover,

$$\mathscr{P}_q(a_1^q)=b_1^q \quad \text{iff} \quad (x+a_1)\ldots(x+a_q)=b_1+b_2x+\ldots+b_qx^{q-1}+x^q, \tag{4.2}$$

for each $x\in Q$, i.e. the second equality is between two polynomials in $Q[x]$, the ring of polynomials over Q with one variable x.

Proof. We define two maps $\alpha: Q^{+} \to Q[x]$ and $\beta: Q^{(+)} \to Q[x]$, by:

$$\alpha(a_1^q) = a_1 + a_2 x + \ldots + a_q x^{q-1} + x^q, \tag{4.3}$$

$$\beta(b_1^q) = (x+b_1) \cdot \ldots \cdot (x+b_q). \tag{4.4}$$

If we multiply $(x+b_1) \cdot \ldots \cdot (x+b_q)$ we get

$$\beta(b_1^q) = (x+b_1) \cdot \ldots \cdot (x+b_q) = c_1 + c_2 \cdot x + \ldots + c_q \cdot x^{q-1} + x^q = \alpha(c_1^q)$$

for some $c_1, c_2, \ldots, c_q \in Q$. It is obvious that α is a bijection, and the fact that Q is an algebraically closed field implies that β is a bijection. So we have two bijections $\alpha^{-1} \circ \beta$ and $\beta^{-1} \circ \alpha$. From the definitions of α and β, and the facts that they are bijections, it follows that for each $q \geq 1$, $\alpha^{-1} \circ \beta$ and $\beta^{-1} \circ \alpha$ restrict to bijections $\alpha^{-1} \circ \beta_| : Q^{(q)} \to Q^q$ and $\beta^{-1} \circ \alpha_| : Q^q \to Q^{(q)}$.

Next, by a simple induction, using Proposition 4.2., it can be shown that:

$$(x+a_1) \cdot \ \ldots \ \cdot (x+a_q) = x^q + \mathcal{S}^1(a_1^q) \cdot x^{q-1} + \ldots + \mathcal{S}^q(a_1^q).$$

This, together with the definitions of α and β, implies that $\alpha^{-1} \circ \beta_| : Q^{(m)} \to Q^m$ coincides with $\mathcal{S}_q$ for every $q \geq 1$, and moreover,

$$\mathcal{S}_q(a_1^q) = b_1^q \ \text{ iff } (x+a_1) \cdot \ldots \cdot (x+a_q) = x^q + b_1 \cdot x^{q-1} + \ldots + b_q \ . \ \blacksquare$$

In the case when (Q,f) is a com(n,m)-groupoid obtained from an (n,m)-groupoid (Q,g) via symmetric functions, it is possible from some known facts about (Q,g) to get some information about (Q,f). The most complicated condition to be checked almost always is the associativity. In general, almost nothing can be said about the associativity of (Q,f) even if we have plenty of information about (Q,g), but there are cases when certain conditions on (Q,g) imply the associativity of (Q,f). Next, we will set up this situation.

Let $(Q,+,\cdot)$ be an algebraically closed field, and let $g: Q^n \to Q^m$ be defined by:

$$g(x) = A \cdot x + b, \tag{4.5}$$

where $A = [\alpha_{ij}]_{m \times n}$ is an $n \times m$-matrix over Q, $b = [\alpha_{i0}]$ is an $m \times 1$-matrix over Q, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, and where we identify the elements (i.e. the vectors) from Q^t with vector columns, i.e. with $t \times 1$-matrices. Using the component representation $(g_1, \ldots g_m)$ of g, where for each i, $g_i : Q^n \to Q$ is the i-th component of g, the definition (4.5) can be restated as:

$$g_i(a_1^n) = \alpha_{i0} + \sum_{1\le r\le n} \alpha_{ir}\cdot a_r \tag{4.6}$$

where α_{ij}, $a_r\in Q$, $i=1,2,\dots,n$, $j=1,2,\dots,m$, $r=1,2,\dots,n$, and the multiplication and the addition are in $(Q,+,\cdot)$.

Theorem 4.4. Let (Q,g) be as above, and let (Q,f) be a com-(n,m)-groupoid obtained from (Q,g) via the symmetric functions of type (4.1). Then, (Q,f) is a com(n,m)-semigroup iff for each $i\in\{1,2,\dots,m\}$, $s\in\{1,2,\dots,k\}$ and $r\in\{1,2,\dots,m+k\}$,

$$\sum_{0\le j\le m} \alpha_{i(s+j)}(\alpha_{j(r-1)} - \alpha_{(j+1)r}) = 0 \tag{4.7}$$

where we use the following conventions:

$$\alpha_{00}=1,\quad \alpha_{0i} = \alpha_{si} = 0 \quad \text{for each } 1\le i\le m+k \text{ and } s>m. \tag{4.8}$$

Proof. We will show that the (Q,f) satisfies the condition (1.5). For shorter notations let $x_1^{m+k} = xa$, and $x_{m+k+1}^{m+2k} = yb$, where $x,y\in Q$, $a\in Q^{(n-1)}$, and $b\in Q^{(k-1)}$. From the definition of f, via g and the symmetric functions, we have: $f(xa)=z$, $f(ay)=w$, $f(f(xa)yb)=f(zyb)=u$ and $f(f(ay)bx)=f(wbx)=v$ where $z=\varphi_m^{-1}\circ g\circ\varphi_{m+k}(xa)$, $w=\varphi_m^{-1}\circ g\circ\varphi_{m+k}(ay)$, $u=\varphi_m^{-1}\circ g\circ\varphi_{m+k}(zyb)$ and $v=\varphi_m^{-1}\circ g\circ\varphi_{m+k}(wbx)$. Because φ_m is a bijection, it follows that $u=v$ iff $g\circ\varphi_{m+k}(zyb)=g\circ\varphi_{m+k}(wxb)$. From the definition of φ_{m+k} it follows that $\varphi_{m+k}(zub)=a_1^n$ and $\varphi_{m+k}(wxb)=b_1^n$, where for each $i\in\{1,2,\dots,n\}$, $a_i=\varphi^i(zub)$, and $b_i=\varphi^i(wxb)$. Then, from the definition of g , it follows that $u=v$ iff for each $i\in\{1,2,\dots,m\}$, $g_i(\varphi_{m+k}(zyb))=g_i(\varphi_{m+k}(wxb))$, or equivalently,

$$\sum_{0\le t\le n} \alpha_{it}\varphi^t(zyb) = \sum_{0\le t\le n} \alpha_{it}\varphi^t(wxb). \tag{4.9}$$

Next, we will compute $\varphi^t(zyb)$ and $\varphi^t(wxb)$. Because of the symmetry, we will give a detailed computation only of $\varphi^t(zyb)$, and state the final result for $\varphi^t(wxb)$. The condition (iii) from Definition 4.1, implies that $\varphi^t(zyb) = \sum_{0\le s\le t} \varphi^{t-s}(yb)\varphi^s(z)$. From the definition of the symmetric functions it follows that for given $c_1^m\in Q^{(m)}$, $\varphi^s(\varphi_m^{-1}(c_1^m)) = c_s$, because $\varphi_m^{-1}(c_1^m)=d_1^m$ iff $\varphi_m(d_1^m)=c_1^m$ iff for each s, $\varphi^s(d_1^m)=c_s$. This, together with the fact that $z=\varphi_m^{-1}\circ g\circ\varphi_{m+k}(xa)$, implies that $\varphi^s(z)g_s(\varphi_{m+k}(xa))$. Now, again, the definition of g implies that

$\varphi^s(z) = \sum_{0\le p\le n} \alpha_{sp}\varphi^p(xa)$. So, $\varphi^t(zyb) = \sum_{0\le s\le t} \varphi^{t-s}(yb)\cdot \sum_{0\le p\le n} \alpha_{sp}\varphi^p(xa)$.
Similarly,

$$\varphi^t(wxb)= \sum_{0\le s\le t} \varphi^{t-s}(xb)\cdot \sum_{0\le p\le n} \alpha_{sp}\varphi^p(ya).$$

Next, using Definition 4.1, we get the following:

$$\varphi^{t-s}(yb)=\varphi^{t-s}(b) + \varphi^{t-s-1}(b)\cdot y \quad \text{and} \quad \varphi^p(xa)=\varphi^p(a) + \varphi^{p-1}(a)\cdot x.$$

Thus,

$$\varphi^t(zyb) = \sum_{0\le s\le t} (\varphi^{t-s}(b) + \varphi^{t-s-1}(b)\cdot y)\cdot \sum_{1\le p\le n} \alpha_{sp}(\varphi^p(a) + \varphi^{p-1}(a)\cdot x)$$

$$= \sum_{0\le s\le t}\ \sum_{0\le p\le n} \alpha_{sp}\varphi^{t-s}(b)\ \varphi^p(a) + \sum_{0\le s\le t}\ \sum_{1\le p\le n} \alpha_{sp}\varphi^{t-s}(b)\ \varphi^{p-1}(a)x$$

$$+ \sum_{0\le s\le t}\ \sum_{0\le p\le n} \alpha_{sp}\varphi^{t-s-1}(b)\varphi^p(a)y$$

$$+ \sum_{0\le s\le t}\ \sum_{1\le p\le n} \alpha_{sp}\varphi^{t-s-1}(b)\varphi^{p-1}(a)xy.$$

Similarly,

$$\varphi^t(wxb) = \sum_{0\le s\le t}\ \sum_{0\le p\le n} \alpha_{sp}\varphi^{t-s}(b)\ \varphi^p(a) + \sum_{0\le s\le t}\ \sum_{1\le p\le n} \alpha_{sp}\varphi^{t-s}(b)\ \varphi^{p-1}(a)y$$

$$+ \sum_{0\le s\le t}\ \sum_{0\le p\le n} \alpha_{sp}\varphi^{t-s-1}(b)\varphi^p(a)x +$$

$$+ \sum_{0\le s\le t}\ \sum_{1\le p\le n} \alpha_{sp}\varphi^{t-s-1}(b)\varphi^{p-1}(a)yx.$$

Now, (Q,f) is a com(n,m)-semigroup iff (4.9) holds for every x_1^{m+2k}. The above presentations of $\varphi^t(zyb)$ and $\varphi^t(wxb)$ imply that (4.9) holds iff for each $i=1,2,\dots,m$, $A=B$, where

$$A = \sum_{0\le t\le n}\ \sum_{0\le s\le t}\ \sum_{1\le p\le n} \alpha_{it}\alpha_{sp}\varphi^{t-s}(b)\ \varphi^{p-1}(a) \quad \text{and}$$

$$B = \sum_{0\le t\le n}\ \sum_{0\le s\le t}\ \sum_{0\le p\le n-1} \alpha_{it}\alpha_{sp}\varphi^{t-s-1}(b)\ \varphi^p(a).$$

In the sum over p in B we have taken $0\le p\le n-1$, because the conditions on the symmetric functions and the fact that $a\in Q^{(n-1)}$, imply that $\varphi^n(a)=0$.

If we change the indices in A by $t-s=q$ and $p=r$, and in B by $t-s-1=q$ and $p=r-1$, we get:

$$A = \sum_{0\le t\le n} \sum_{0\le q\le k-1} \sum_{1\le r\le n} \alpha_{it}\alpha_{(t-q)r}\wp^{q}(b)\ \wp^{r-1}(a) \quad \text{and}$$

$$B = \sum_{0\le t\le n} \sum_{0\le q\le k-1} \sum_{1\le r\le n} \alpha_{it}\alpha_{(t-q-1)(r-1)}\wp^{q}(b)\ \wp^{r-1}(a)\ .$$

Next, if we change the indices again, by $t-q=j+1$ and $q=s-1$, we get:

$$A = \sum_{0\le j\le m} \sum_{1\le s\le k} \sum_{1\le r\le n} \alpha_{i(s+j)}\alpha_{(j+1)r}\wp^{s-1}(b)\ \wp^{r}(a) \quad \text{and}$$

$$B = \sum_{0\le j\le m} \sum_{1\le s\le k} \sum_{1\le r\le n} \alpha_{i(s+j)}\alpha_{j(r-1)}\wp^{s-1}(b)\ \wp^{r}(a)\ .$$

From this presentation, we have that $A = B$ for each $1\le i\le m$, and each $a\in Q^{(n-1)}$, $b\in Q^{(k-1)}$ iff for each $1\le i\le m$, each $1\le s\le k$ and each $1\le r\le n$,

$$\sum_{0\le j\le m} \alpha_{(s+j)}(\alpha_{(j+1)r} - \alpha_{j(r-1)}) = 0,$$

i.e. iff (4.7) is satisfied. ∎

With the convention as in (4.8), for each $1\le s\le k$, let $A_s=[\alpha_{i(s+j)}]_{m\times(m+1)}$, let $A_0=[\alpha_{j(q-1)}]_{(m+1)\times n}$ and $A_{m+1}=[\alpha_{(j+1)q}]_{(m+1)\times n}$, where $1\le i\le m$, $0\le j\le m$, and $1\le q\le n$. Then the condition (4.7) can be restated as

$$A_s(A_0-A_{m+1})=0. \tag{4.10}$$

I.5. Examples

In this section we will give **several** examples of com(m+k, m)-semigroups and groups on $\mathbb{C}$ or on a subset of $\mathbb{C}$. In all of these examples we will define $\varphi^i(w_1^m)$, $i\in\{1,\dots,m\}$, as functions F_i of $\varphi^j(z_1^{m+k})$ $j\in\{1,\dots,m+k\}$, where $f(z_1^{m+k}) = w_1^m$, and (G,f) ($G\subseteq\mathbb{C}$) is the considered com(m+k, m)-groupoid. In other words, $f(z_1^{m+k}) = w_1^m$ iff

$$\varphi^i(w_1^m) = F_i(\varphi^1(z_1^{m+k}),\ \varphi^2(z_1^{m+k}),\ \dots,\ \varphi^{m+k}(z_1^{m+k}))$$

where $F_i:G^{m+k}\to G$, i.e. $F:G^{m+k}\to G^m$, for $F = (F_1,\ F_2,\ \dots,F_m)$. For the sake of simplicity we will denote $\varphi^j(z_1^{m+k})$ by a_j for $j\in\{1,\dots,m+k\}$, and $\varphi^i(w_1^m)$ by F^i for $i\in\{1,\dots,m\}$. Most of the examples are for the special case m=2 and k=1. It is easy to verify in this case that the associative law is satisfied iff

$$F_i^* = F_i(F_1(a_1^3)+z_4,\ F_2(a_1^3)+z_4F_1(a_1^3),\ z_4F_2(a_1^3))$$

is invariant under each permutation of the numbers z_1, z_2, z_3 and z_4, for $i\in\{1,2\}$.

The next four examples were given in [23].

Example 1. $F_1(a_1,a_2,a_3)=a_1$, $F_2(a_1,a_2,a_3)=a_2$ and $G=\mathbb{C}$.

Example 2. $F_1(a_1,a_2,a_3)=a_1$, $F_2(a_1,a_2,a_3)=a_2+1$ and $G=\mathbb{C}$.

Example 3. $F_1(a_1,a_2,a_3)=a_2$, $F_2(a_1,a_2,a_3)=a_3$ and $G=\mathbb{C}\setminus\{0\}$.

Example 4. $F_1(a_1,a_2,a_3)=a_1$, $F_2(a_1,a_2,a_3)=a_2-a_3$ and $G=\mathbb{C}\setminus\{1\}$.

Example 5. $F_1(a_1,a_2,a_3)=a_1+\lambda$ and $F_2(a_1,a_2,a_3)=a_3$ where λ is an arbitrary complex number. We will prove that $(\mathbb{C}\setminus\{0\},f)$ is a com(3,2)-group. We note that

$$f(z_1,z_2,z_3)=(w_1,w_2) \iff \begin{cases} z_1+z_2+z_3+\lambda=w_1+w_2 \\ z_1z_2z_3=w_1w_2 \end{cases}$$

and hence $w_1, w_2 \neq 0$ if $z_1, z_2, z_3 \neq 0$. Thus $(\mathbb{C}, f)$ is a com(3,2)-groupoid. Further,

$$F_1^* = a_1+z_4+2\lambda = z_1+z_2+z_3+z_4+2\lambda \quad \text{and} \quad F_2^* = z_1z_2z_3z_4$$

are invariant under each permutation of the numbers z_1, z_2, z_3, z_4, and the associative law is satisfied. For $z_3, w_1, w_2 \neq 0$, the system

$$\begin{cases} z_1+z_2+z_3+z_4+\lambda = w_1+w_2 \\ z_1z_2z_3 = w_1w_2 \end{cases}$$

has a unique solution $(z_1, z_2) \in \mathbb{C}^{(2)}$, and moreover $z_1, z_2 \neq 0$ because $z_1z_2 = w_1w_2/z_3 \neq 0$. Thus the axiom for solvability is satisfied, and $(\mathbb{C}\setminus\{0\}, f)$ is a com(3,2)-group.

The induced com(4,2)-group is given by

$$f(z_1, z_2, z_3, z_4) = (w_1, w_2) \iff \begin{cases} z_1+z_2+z_3+z_4 = w_1+w_2-2\lambda \\ z_1z_2z_3z_4 = w_1w_2 \end{cases},$$

and one can verify that the unit in its derived group $(G^{(2)}, *)$ is $(-\lambda+(\lambda^2-1)^{1/2}, -\lambda-(\lambda^2-1)^{1/2})$. Now we will prove that for $\lambda=1$ this com-(3,2)-group is not isomorphic to any one of the groups of examples 1-4. Moreover it was proved in [23] that any two groups of examples 1-4 are not isomorphic.

If $\lambda=1$, the unit in the derived group is $(-1,1)$. The units in the derived groups of examples 1-4 are, respectively $(0,0)$, $(2^{1/2}, -2^{1/2})$, $(i,-i)$ and $(0,0)$. Thus the considered com(3,2)-group for $\lambda=1$ is not isomorphic to any one of the com(3,2)-groups of examples 2 and 3. Further we note that

$$f(x,x,x) = (y,y) \iff (2y=3x+1,\ y^2=x^3),$$

and the equation $((3x+1)/2)^2 = x^3$ has three different roots, x_1, x_2 and x_3. On the other hand the equation $f(x,x,x)=(y,y)$ in the group of Example 1 has one solution for x, and in the group of Example 4 has two solutions for x. Thus the considered com(3,2)-group is not isomorphic to any one of the groups of Examples 1-4, and hence we have obtained five non-isomorphic com(3,2)-groups.

Example 6. Let $F_1(a_1, a_2, a_3) = a_1+a_3$ and $F_2(a_1, a_2, a_3) = a_2$. We will prove that $(\mathbb{C}\setminus\{-1,1\}, f)$ is a com(3,2)-group. In order to prove that

$(\mathbb{C}\setminus\{-1,1\},f)$ is a com(3,2)-groupoid we note that

$$f(z_1,z_2,z_3) = (w_1,w_2) \iff \begin{cases} z_1 + z_2 + z_3 + z_1z_2z_3 = w_1 + w_2 & (5.1) \\ z_1z_2 + z_2z_3 + z_3z_1 = w_1w_2 & (5.2) \end{cases}$$

By adding the equations (5.1) and (5.2) we obtain

$$(w_1+1)(w_2+1) = (z_1+1)(z_2+1)(z_3+1)$$

and by subtracting (5.1) from (5.2) we obtain

$$(1-w_1)(1-w_2) = (1-z_1)(1-z_2)(1-z_3).$$

Hence, if $z_1,z_2,z_3\neq\pm1$, then $w_1,w_2\neq\pm1$, and $(\mathbb{C},f)$ is a com(3,2)-groupoid. Further,

$$F_1^{\bullet} = (z_1+z_2+z_3+z_4)+(z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4)$$

and

$$F_2^{\bullet} = (z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+ z_3z_4)+z_1z_2z_3z_4$$

are invariant under each permutation of z_1,z_2,z_3,z_4. So, the associative law is satisfied. In order to verify the axiom for solvability, we suppose that $z_3,w_1,w_2\neq\pm1$ are given, and consider the system (5.1) and (5.2). It is a linear system on z_1+z_2 and z_1z_2, and it has a unique solution for z_1+z_2 and z_1z_2 because

$$\begin{vmatrix} 1 & z_3 \\ z_3 & 1 \end{vmatrix} = 1 - z_3^2 \neq 0,$$

and thus there exists a unique solution $(z_1,z_2)\in\mathbb{C}^{(2)}$. Moreover $z_1,z_2\neq\pm1$ because $z_1\in\{1,-1\}$ or $z_2\in\{1,-1\}$ implies $w_1\in\{1,-1\}$ or $w_2\in\{1,-1\}$ as we saw previously. Hence $(\mathbb{C}\setminus\{-1,1\},f)$ is a com(3,2)-group.

The same result can be obtained by using Theorems 4.4 and 3.8. Theorem 4.4 implies that $(\mathbb{C},f)$ is a com(3,2)-semigroup. From the system (5.1) and (5.2) we obtain that $\bar{\varphi}_z$ is an affine transformation on $\mathbb{C}^2$ and it is given by

$$(\bar{\varphi}_z) \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} z \\ 0 \end{bmatrix}.$$

Hence $\bar{\varphi}_z$ is not a bijection on $\mathbb{C}^2$ iff $1-z^2 = 0$. Thus by Theorem 3.8, $(\mathbb{C}\setminus\{-1,1\},f)$ is a com(3,2)-group.

Example 7. $F_1(a_1,a_2,a_3) = a_1+a_2+1$ and $F_2(a_1,a_2,a_3) = a_1+a_3$. One can verify that the associative law is satisfied, and $(\mathbb{C},f)$ is a com-

(3,2)-semigroup. We shall prove that $(\mathbb{C}\setminus\{\lambda_1,\lambda_2\},f)$ is a com(3,2)-group, where λ_1 and λ_2 are the roots of the equation $\lambda^2+\lambda-1=0$. From the definition of f it follows that:

$$f(z_1,z_2,z_3)=(w_1,w_2) \iff \begin{cases} w_1+w_2=1+z_1+z_2+z_3+z_1z_2+z_2z_3+z_3z_1 \\ w_1w_2=z_1+z_2+z_3+z_1z_2z_3 \end{cases}.$$

Let $\lambda\in\{\lambda_1,\lambda_2\}$. Then,

$$\begin{aligned}(z_1-\lambda)(z_2-\lambda)(z_3-\lambda) &= z_1z_2z_3 - \lambda(z_1z_2 + z_2z_3 + z_3z_1) + \\ &+ \lambda^2(z_1+z_2+z_3)-\lambda^3= z_1z_2z_3 + z_1 + z_2 + z_3- \\ &- \lambda(z_1z_2 + z_2z_3 + z_3z_1 + z_1 + z_2 + z_3 + 1) \\ &+ (\lambda^2-1+\lambda)(z_1+z_2+z_3) + \lambda(-\lambda^2+1) \\ &= w_1w_2 - \lambda(w_1+w_2) + \lambda^2= (w_1-\lambda)(w_2-\lambda),\end{aligned}$$

and hence $(\mathbb{C}\setminus\{\lambda_1,\lambda_2\},f)$ is a com(3,2)-semigroup. In order to verify the axiom for solvability, we suppose that $z_3,w_1,w_2\in\mathbb{C}\setminus\{\lambda_1,\lambda_2\}$ are given. Then the system

$$\begin{cases} 1 + z_1+z_2+z_3 + z_1z_2+z_2z_3+z_3z_1 = w_1+w_2 \\ z_1+z_2+z_3 + z_1z_2z_3 = w_1w_2 \end{cases}$$

i.e.

$$\begin{cases} (z_1+z_2)(1+z_3) + z_1z_2 = w_1+w_2-z_3 \\ (z_1+z_2) + z_3(z_1z_2) = w_1w_2 - z_3 \end{cases}$$

has ~~a~~ unique solution for z_1+z_2 and z_1z_2 because

$$\begin{vmatrix} 1+z_3 & 1 \\ 1 & z_3 \end{vmatrix} = z_3^2+z_3-1\neq 0.$$

Thus there exists a unique solution $(z_1,z_2)\in\mathbb{C}^{(2)}$ and besides that $z_1,z_2\notin\{\lambda_1,\lambda_2\}$ because in the opposite case it follows that $w_1\in\{\lambda_1,\lambda_2\}$ or $w_2\in\{\lambda_1,\lambda_2\}$. Hence $(\mathbb{C}\setminus\{\lambda_1,\lambda_2\},f)$ is a com(3,2)-group.

This also can be verified by using Theorems 4.4 and 3.8. The transformation $\bar{\varphi}_z$ in $\mathbb{C}^2$ is affine, and it is given by

$$(\bar{\varphi}_z) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+z & 1 \\ 1 & z \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} z \\ z \end{bmatrix}.$$

Hence, z is a singular element iff $\begin{vmatrix} 1+z & 1 \\ 1 & z \end{vmatrix} = z^2+z-1=0.$