The map $\varphi$ has the same form, i.e.,

$$\varphi(z) = C^{-1}z + A \qquad \text{or} \qquad \varphi(z) = C^{-1}\bar{z} + A.$$

(i.2) Considering the second equation of $\mathbf{g}'(\psi(z)) = \mathbf{g}''(z)$, we obtain that $\psi(z)$ as well as $\varphi(z)$ is a linear function of $z$ or $\bar{z}$.

We note that $C_1 = C_2 = \cdots = C_t$,

$$\alpha'_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and $\alpha_1 = \cdots = \alpha_t \in \{1, -1\}$. Considering the other equations, we obtain that $r_1 = s_1, r_2 = s_2, \ldots, r_t = s_t$ (hence $u = v$), $D_1 = \cdots = D_{q-v} \equiv 0$, $\gamma'_{jr} \equiv 0$. Moreover,

— if $\psi(z) = Cz + K$ ($K = a_i - Cb_i = const.$), then

$$(\delta_{11}, \ldots, \delta_{q-v,q-v}) = \big(C^{-1}, C^{-2}, \ldots, C^{-r_1+1}, C^{-1}, C^{-2}, \ldots, C^{-r_2+1}, \ldots, C^{-1}, C^{-2}, \ldots, C^{-r_t+1}\big), \tag{5.6}$$

$$D_{q-v+i} = K^i, \ \ \delta_{q-v+i,q-v+j} = \binom{i}{j} K^{i-j} C^j \qquad (1 \leq i \leq j \leq v), \tag{5.7}$$

and the other $\delta_{ij}$ are zeros, and all of the $\delta'_{ij}$ are zeros.

— If $\psi(z) = C\bar{z} + K$ ($K = a_i - C\bar{b}_i = const.$), then

$$(\delta'_{11}, \ldots, \delta'_{q-v,q-v}) = \big(C^{-1}, C^{-2}, \ldots, C^{-r_1+1}, C^{-1}, C^{-2}, \ldots, C^{-r_2+1}, \ldots, C^{-1}, C^{-2}, \ldots, C^{-r_t+1}\big), \tag{5.8}$$

$$D_{q-v+i} = K^i, \ \ \delta'_{q-v+i,q-v+j} = \binom{i}{j} K^{i-j} C^j \qquad (1 \leq i \leq j \leq v), \tag{5.9}$$

and the other $\delta'_{ij}$ are zeros, and all of the $\delta_{ij}$ are zeros.

(ii) In this case, the system $\mathbf{g}'(\psi(z)) = \mathbf{g}''(z)$ becomes

$$(\psi(z))^j = D_j + \sum_{j=1}^{m} (\delta_{ij} z^j + \delta'_{ij} \bar{z}^j) \qquad (1 \leq j \leq m).$$

Hence, for $j = 1$, it follows that $\psi(z) = P(z) + Q(\bar{z})$, $\deg P, \deg Q \leq m$. Also,

$$(\psi(z))^m = P_1(z) + Q_1(\bar{z}),$$

where $\deg P_1$, $\deg Q_1 \leq m$. So, $\deg P \leq 1$ and $\deg Q \leq 1$. It is not possible to have $\deg P = \deg Q = 1$, because $(\psi(z))^j$ ($j \geq 2$) does not contain terms of the form $z^i \bar{z}^j$. Thus,

$$\psi(z) = Cz + K \qquad \text{or} \qquad \psi(z) = C\bar{z} + K. \quad ∎ \tag{5.10}$$

From this theorem and its proof we obtain information about the constants $D_j$, $C_j$, $\alpha'_{ij}$, etc.

Now we will give some important consequences of this theorem.

COROLLARY 5.2. *Each isomorphism between affine or projective com$(m+k, m)$-groups is induced by a complex map having one of the following forms:*

$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{or} \quad \varphi(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \quad (ad-bc \neq 0). \tag{5.11}$$

Note that each of these maps sends a projective $com(m+k, m)$-group into the projective $com(m+k, m)$-group.

In the proof of Theorem 5.1 we have proved that $r_1 = s_1, r_2 = s_2, \ldots, r_t = s_t, u = v$, where $r_1, r_2, \ldots, r_t$ are the multiplicities of the singular elements $a_1, \ldots, a_t$; $s_1, s_2, \ldots, s_t$ are the multiplicities of the singular elements $b_1, \ldots, b_t$; and $\varphi(a_i) = b_i$ for $i = 1, \ldots, t$. Note that the multiplicities of the infinity point $\infty$ of the first and the second affine $com(m+k, m)$-groups of Theorem 5.1 are $u+1$ and $v+1$, respectively, and $u+1 = v+1$. Next we will prove a more general statement.

THEOREM 5.3. *Each isomorphism between two affine or projective com$(m+k, m)$-groups sends each singular point into a singular point of the same multiplicity.*

*Proof.* It is sufficient to consider only the projective $com(m+k, m)$-groups. Let $(M, f)$ and $(M', f')$ be isomorphic projective $com(m+k, m)$-groups, $M \subseteq \mathbf{C}^*$. According to Corollary 5.2 the isomorphism is induced by a map $\varphi(z) = (az+b)/(cz+d)$ or $\varphi(z) = (a\bar{z}+b)/(c\bar{z}+d)$ $(ad-bc \neq 0)$. The set of these maps with respect to the composition forms a Lie group $G$ with two components of connection. One component contains the identical map $\epsilon$ and the other contains the map $\varphi_-$ defined by $\varphi_-(z) = \bar{z}$. It is obvious that the theorem is true for $\varphi = \epsilon$ and $\varphi = \varphi_-$. If $\varphi$: $\mathbf{C}^* \to \mathbf{C}^*$ induces the isomorphism between $(M, f)$ and $(M', f')$, then there exists a path in the Lie group $G$ which connects $\varphi$ to $\epsilon$ or $\varphi_-$. Indeed, there exists a continuous family of isomorphisms $\varphi_\lambda$, $\lambda \in [0, 1]$, such that $\varphi_0 = \epsilon$ or $\varphi_0 = \varphi_-$ and $\varphi_1 = \varphi$. Since the theorem is true for $\varphi = \epsilon$ and $\varphi = \varphi_-$ and both of the two isomorphic projective $com(m+k, m)$-groups have the same number of singular elements, we obtain that each map $\varphi_\lambda$ preserves the multiplicities of the singular elements. ∎

The Lie group $G$ mentioned in the proof of Theorem 5.3 is generated by the maps

(1) $$z \to 1/z,$$

(2) $$z \to z/\lambda \ (\lambda \neq 0, \infty),$$

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(3) $$z \to z - \lambda \ (\lambda \neq \infty),$$

(4) $$z \to \bar{z}.$$

Note that each projective $com(m+k,m)$-group is uniquely determined by the corresponding $(m+1) \times (m+k+1)$ matrix $A = [\alpha_{ij}]$, $i \in \{0,1,\dots,m\}$, $j \in \{0,1,\dots,m+k\}$. In [28] (Theorem 3.1, Chapter 2) it is shown that the first three maps induce the matrix transformations

(1) $$\alpha'_{ij} = \alpha_{m-i,\,m+k-j}, \tag{5.12}$$

(2) $$\alpha'_{ij} = \lambda^{j-i}\alpha_{ij}, \tag{5.13}$$

(3) $$\alpha'_{ij} = \sum_{0 \le r \le i} (-\lambda)^r \binom{r+m-i}{r} \times \sum_{0 \le s \le m+k-j} \alpha_{i-r,\,s+j}\lambda^s \binom{m+k-j}{s} \tag{5.14}$$

for $i \in \{0,1,\dots,m\}$, $j \in \{0,1,\dots,m+k\}$, respectively. Similarly, for the fourth transformation we have now the matrix transformation

(4) $$\alpha'_{ij} = \bar{\alpha}_{ij}, \quad i \in \{0,1,\dots,m\}, \ j \in \{0,1,\dots,m+k\}. \tag{5.15}$$

Using these four matrix transformations, we are able to deduce whether two projective $com(m+k,m)$-groups are isomorphic or not.

According to Theorem 5.1, Corollary 5.2, and the canonical form of the affine $com(m+k,m)$-groups we obtain the following corollary.

COROLLARY 5.4. *For each $t \in \{0,1,\dots,m\}$,*

(i) *the set of different affine $com(m+k,m)$-groups on $\mathbf{C}\backslash\mathcal{A}_t$ can be parameterized by $m+t$ independent complex parameters;*

(ii) *the set of nonisomorphic affine $com(m+k,m)$-groups on $\mathbf{C}\backslash\mathcal{A}_t$ can be parameterized by $m+t-2$ independent complex parameters;*

(iii) *the set of different projective $com(m+k,m)$-groups on $\mathbf{C}^*\backslash\mathcal{A}_{t+1}$ can be parameterized by $m+t+1$ independent complex parameters; and*

(iv) *the set of nonisomorphic projective $com(m+k,m)$-groups on $\mathbf{C}^*\backslash\mathcal{A}_{t+1}$ can be parameterized by $m+t-2$ independent complex parameters.*

EXAMPLE. Let us consider the special case $m = 2$ and $k = 1$. It is proved in [28] that up to isomorphism there are only two affine $com(3, 2)$-groups (and hence $com(2 + k, 2)$-groups) on $\mathbf{C}$ ($t = 0$), and they are given by

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow \quad \begin{cases} z_1 + z_2 + z_3 = w_1 + w_2 \\ z_1 z_2 + z_2 z_3 + z_3 z_1 = w_1 w_2 \end{cases}, \tag{5.16}$$

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow \quad \begin{cases} z_1 + z_2 + z_3 = w_1 + w_2 \\ z_1 z_2 + z_2 z_3 + z_3 z_1 = 1 + w_1 w_2 \end{cases}. \tag{5.17}$$

If $t = 1$, then it can be proved that each of the affine $com(3, 2)$-groups is isomorphic to one of the groups

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow \quad \begin{cases} z_1 \cdot z_2 \cdot z_3 = w_1 \cdot w_2 \\ z_1 + z_2 + z_3 = \lambda + w_1 + w_2 \end{cases} (\lambda \in \mathbf{C}). \tag{5.18}$$

The parameter $\lambda$ in (5.18) cannot vanish or be changed to a constant, for example, $\lambda = 0$ or $\lambda = 1$, but note that $\lambda$ and $\bar{\lambda}$ give isomorphic groups. If $t = 2$, then without loss of generality we can assume that the singular points are 0 and 1 (and $\infty$ too). So each affine $com(3, 2)$-group on $\mathbf{C} \backslash \{\alpha, \beta\}$ ($\alpha \neq \beta$) is isomorphic to one of the groups

$$f(z_1^3) = w_1^2 \Leftrightarrow \begin{cases} z_1 \cdot z_2 \cdot z_3 = \lambda \cdot w_1 \cdot w_2 \\ (z_1 - 1)(z_2 - 1)(z_3 - 1) = \mu(w_1 - 1)(w_2 - 1) \end{cases} (\lambda, \mu \in \mathbf{C}_1), \tag{5.19}$$

and some of the groups in (5.19) are isomorphic, but the parameters $\lambda$ and $\mu$ cannot vanish.

## APPENDIX

The affine and projective $com(m + k, m)$-groups can be generalized to topological $com(m + k, m)$-groups. In this section we do not present new results. We will only refer to the results in [28] in order to make a connection and to give motivation for further research.

DEFINITION A.1. Let $X$ be a topological space and $n$ be a positive integer. The $n$-fold symmetric product of $X$ is the topological factor space $X^{(n)} = X^n / \approx$, where $X^n$ is the $n$-fold topological product of $X$ and $\approx$ is as previously defined in Section 0.

DEFINITION A.2. Let $Q$ be a topological space and let $(Q, f)$ be a $com(m + k, m)$-groupoid. We say that $(Q, f)$ is a topological (continuous) $com(m + k, m)$-groupoid, if the map $f\colon Q^{(n)} \to Q^{(m)}$ is continuous. If $(Q, f)$ is a topological $com(m + k, m)$-groupoid and $(Q, f)$ is a $com(m + k, m)$-semigroup, then we say that $(Q, f)$ is a topological (continuous) $com(m + k, m)$-semigroup.

DEFINITION A.3. Let $(Q, f)$ be a $com(m+k,m)$-group and let $f'\colon Q^{(k)} \times Q^{(m)} \to Q^{(m)}$ be a map defined by

$$f'(ab) = \varphi_a^{-1}(b), \qquad a \in Q^{(k)},\ b \in Q^{(m)}, \tag{A.1}$$

where $\varphi_a$ is as defined in Section 1. We say that $(Q, f)$ is a topological $com(m+k,m)$-group if $(Q, f)$ is a topological $com(m+k,m)$-semigroup and $f'$ is a continuous map.

Studies of the symmetric products from a topological viewpoint have been done in [20–25, 28, 30]. The most important result is that if $M$ is a manifold, then $M^{(m)}$ $(m > 1)$ is manifold if only if $\dim M = 2$ [30]. Moreover, the following is proved in [28].

THEOREM A.1 [28, Theorem 3.5 and Prop. 3.2, Chap. III]. *If $(M, f)$ is a locally euclidean topological $com(m+k,m)$-group for $m \geq 2$, then* $\dim M = 2$, *$M$ is an oriented manifold not homeomorphic to the sphere $S^2$, and*

$$M^{(m)} \cong \mathbf{R}^u \times (S^1)^v. \tag{A.2}$$

We recall that all known examples of $com(m+k,m)$-groups, except on the discrete set $Q$, are constructed on subsets of $\mathbf{C}$ and $\mathbf{C}^*$ with real dimension 2. Theorem A.1 implies that each projective $com(m+k,m)$-semigroup on $\mathbf{C}^* = S^2$ has at least one singular element. Many theorems for affine and projective $com(m+k,m)$-groups can be generalized here. For example, the following theorem is a generalization of Theorem 1.5.

THEOREM A.2 [28, Theorem 3.3, Chap. III]. *Let $(M, f)$ be a connected locally euclidean topological $com(m+k,m)$-group. Then this group is induced by $k^s$ connected locally euclidean topological $com(m+1,m)$-groups where $s$ is the rank of the homology group $H_1(M^{(m)}, \mathbf{Z})$. Specially, if $M^{(m)}$ is a simply-connected manifold, then the corresponding $com(m+k,m)$-group is induced by exactly one $com(m+1,m)$-group.*

Note that the concept of the complex lines which is basic in this paper can also be introduced for topological $com(m+k,m)$-groups. Moreover, Theorems 1.6, 1.7, and 2.1 also hold for locally euclidean topological $com(m+k,m)$-groups.

The theorems and corollaries presented in this paper and in [28] give deep results about the description of the affine and projective $com(m+k,m)$-groups. We do not know examples of smooth $com(m+k,m)$-groups which are not isomorphic to affine $com(m+k,m)$-groups. The following conjecture is stated in [28].

*Conjecture.* Each connected, locally euclidean topological $com(m+k,m)$-group is isomorphic to an affine $com(m+k,m)$-group.

If this conjecture is true, then all the surfaces $S \subseteq \mathbf{C}_1^t \times \mathbf{C}_0^q$ which have the property $P$ are those described for the affine $com(m+k, m)$-groups. Indeed, if we change the parameterization of the surface $S$, then we obtain a new $com(m+k, m)$-group which is isomorphic to the former $com(m+k, m)$-group.

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