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# VECTOR VALUED GROUPOIDS INDUCED BY VARIETIES OF SEMIGROUPS

Ǵ. Čupona and S. Markovski

Abstract. Vector valued groupoids induced by semigroups are considered in [3]. Here we consider vector valued groupoids induced by (nontrivial) varieties of semigroups.

Preliminaries. First we state some definitions and results concerning vector valued groupoids induced by semigroups, considered in [3].

Let $\underline{S}=(S;\cdot)$ be a semigroup, and Q a nonempty subset of S. Define a collection of subsets $(Q_\alpha \mid \alpha \geq 1)$ of S by: $Q_1=Q$, $Q_{\alpha+1} = \{xy \mid x\in Q_\alpha,\ y\in Q\}$. If n and m are positive integers and $f:Q_n \to Q_m$ a mapping from $Q_n$ into $Q_m$, then the ordered pair $(Q;f)$ is called an $(\underline{S};n,m)$-groupoid. Then, a nonempty subset P of Q is said to be a subgroupoid of $(Q;f)$ if $f(P_n)\subseteq P_m$, and P is called a strong subgroupoid of $(Q;f)$ iff

$$(\forall a_i\in P, b_j\in Q)(f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m \Longrightarrow b_1,\ldots,b_m\in P)$$

If $(Q;f)$ is an $(\underline{S};n,m)$-groupoid and $(Q';f')$ is an $(\underline{S}';n,m)$-groupoid, then a mapping $\phi:Q\to Q'$ is said to be a homomorphism from $(Q;f)$ into $(Q';f')$ if for every $a_i,b_j\in Q$ the equation $f(a_1\cdot\ldots\cdot a_n) = b_1\cdot\ldots\cdot b_m$ implies $f'(\phi(a_1)*\ldots*\phi(a_n)) = =\phi(b_1)*\ldots*\phi(b_m)$, where $\underline{S}'=(S';*)$. If, moreover, $\phi$ is bijective and $\phi^{-1}$ is a homomorphism then $\phi$ is called an isomorphism.

We state now some results, proved in [3].

(i) A nonempty intersection of strong subgroupoids is a strong subgroupoid as well, but a nonempty intersection of subgroupoids is not necesseraly a subgroupoid.

(ii) A bijective homomorphism is not necesseraly an isomorphism.

(iii) A homomorphic image of a subgroupoid is a subgroupoid, but a homomorphic image of a strong subgroupoid is not necessarily a strong subgroupoid.

(iv) A complete nonempty homomorphic inverse image of a strong subgroupoid is a strong subgroupoid, but this is not true, in general, for subgroupoids.

Assume now that $\mathbb{V}$ is a nontrivial variety of semigroups. (By "a nontrivial" we mean that $\mathbb{V}$ contains objects with more than one element.) If Q is a nonempty set then we denote by $\mathbb{V}(Q)$ a free semigroup in $\mathbb{V}$ with a basis Q. Every $(\mathbb{V}(Q);n,m)$-groupoid is called a $(\mathbb{V};n,m)$-groupoid. Here, we will write $\mathbb{V}_p(Q)$ instead of $Q_p$.

All mentioned "positive" properties for semigroup (n,m)-groupoids are, certainly, true for $(\mathbb{V};n,m)$-groupoids; nevertheless, some properties hold in the class of $(\mathbb{V};n,m)$-groupoids, which do not hold in the general case. Below we state some properties of this kind.

(i') A nonempty intersection of subgroupoids of a $(\mathbb{V};n,m)$-groupoid (Q;f) is a subgroupoid as well. If P is a subgroupoid of (Q;f) and if P is not a strong one, then the strong subgroupoid generated by P coincides with Q.

(ii') A bijective homomorphism is an isomorphism. (When we say that $\phi:(Q;f) \to (Q';f')$ is a homomorphism then we assume that both (Q;f) and (Q';f') are $(\mathbb{V};n,m)$-groupoids.)

The corresponding "negative" properties stated in (iii) and (iv) remains "negative", in general, in the class of $(\mathbb{V};n,m)$-groupoids as well.

It is given (in Pr. 2.6) a description of the set of varieties $\mathbb{V}$ for which every subgroupoid of a $(\mathbb{V};n,m)$-groupoid is a strong subgroupoid too.

In the last part of the paper, some connections between $(\mathbb{W};n,m)$-groupoids and $(\mathbb{V};n,m)$-groupoids are described, where $\mathbb{W}$ is a nontrivial subvariety of $\mathbb{V}$.

Consider some examples.

Example 1. If $\mathcal{V}$=Sem is the variety of all semigroups then a ($\mathcal{V}$;n,m)-groupoid is a usual (n,m)-groupoid ([2]).

Example 2. The class of fully commutative groupoids ([4]) is obtained in the case when $\mathcal{V}$=Comsem is the variety of commutative semigroups.

Example 3. Let $\mathcal{V}$=Sl be the variety of semilattices, i.e. idempotent and commutative semigroups, and let Q be a nonempty set. As it is well known, the semigroup Sl(Q) can be interpreted as the semigroup F(Q) of all finite nonempty subsets of Q, where the operation is the usual (set theoretical) union. Then an (Sl;n,m)-groupoid can be considered as a mapping $f:X \to Y=f(X)$ from $\{X \in F(Q) \mid 1 \le |X| \le n\}$ into $\{Y \in F(Q) \mid 1 \le |Y| \le m\}$. ($|A|$ denotes the cardinal number of the set A.)

Example 4. Let $\mathcal{V}$=RB be the variety of rectangular bands, i.e. idempotent semigroups satisfying the law $xyz = xz$. Then, $\mathcal{V}_\alpha(B) = B \times B$, for every $\alpha \ge 2$, where an element $a \in Q$ is identified by the pair (a,a) (=a·a). If $1 \le n,m \le 2$ then an (RB;n,m)-groupoid is the same as an (n,m)-groupoid, according to Ex. 1. If $n \ge 3$, $m = 2$, then the class of (RB;n,m)-groupoids coincides with the class of all (n,m)-groupoids which satisfy all the identities of the form

$$f(xz_1 \ldots z_{n-2}y) = f(xu_1 \ldots u_{n-2}y).$$

We also note that in the first three examples there are not any distinctions between subgroupoids and strong subgroupoids, but, if $m \ge 3$, Q is the unique strong subgroupoid of an (RB;n,m)-groupoid (Q;f).

1. Contents in $\mathcal{V}$(Q). Further on we assume that $\mathcal{V}$ is a given nontrivial variety of semigroups, and Q is a given nonempty set. We will introduce here a notion of a p-content $c_p(u)$ of an element $u \in \mathcal{V}_p(Q)$.

First, let us make some remarks.

(i) Let $a_1, a_2, \ldots$ be a sequence of different elements of Q, and $i_\lambda$, $j_\nu$ positive integers. Then

$$a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_p} = a_{j_1} \cdot a_{j_2} \cdot \ldots \cdot a_{j_q}$$

is an equality in $\mathcal{V}(Q)$ iff

$$x_{i_1} x_{i_2} \ldots x_{i_p} = x_{j_1} x_{j_2} \ldots x_{j_q}$$

is an identity in $\mathcal{V}$.

(ii) Let $u \in \mathcal{V}_p(Q)$, where $p \geq 1$. We define a family $[u;p]$ of subsets of Q as follows.

$A \in [u;p]$ iff there exist $a_1, a_2, \ldots, a_p \in Q$ such that $u = a_1 \cdot a_2 \cdot \ldots \cdot a_p$ and $A = \{a_1, a_2, \ldots, a_p\}$. (We note that $\{a_1, \ldots, a_p\}$ has the usual meaning, i.e. $a \in \{a_1, \ldots, a_p\} \iff (\exists j)\ a = a_j$.)

Clearly we have

$$u \in \mathcal{V}_p(Q) \implies [u;p] \neq \emptyset \ \&\ 0 < |A| \leq p$$

for every $A \in [u;p]$.

(iii) If $u \in \mathcal{V}_p(Q)$ then $[u;p]$ is a family of finite subsets of Q, and thus for every $L \in [u;p]$ there is at least one minimal element $M \in [u;p]$.

Suppose that $M'$ and $M''$ are two different minimal elements of $[u;p]$, and let

$$u = a_1 \cdot a_2 \cdot \ldots \cdot a_p = b_1 \cdot b_2 \cdot \ldots \cdot b_p,$$

where $M' = \{a_1, \ldots, a_p\}$, $M'' = \{b_1, \ldots, b_p\}$. Assume that $b_j \notin M'$ and that $|M''| \geq 2$. Choose an element $b_r \in M''$, such that $b_r \neq b_j$. Define $c_1, \ldots, c_p \in Q$ by:

$$c_i = \begin{cases} b_i & \text{if } i \neq j \\ b_r & \text{if } i = j \end{cases}$$

Then we have $u = c_1 \cdot c_2 \cdot \ldots \cdot c_p$ and $M = \{c_1, \ldots, c_p\}$ is a proper subset of $M''$, which is impossible. So, if $M'' \setminus M' \neq \emptyset$ then $|M''| = 1$. We obtain symmetrically that $|M'| = 1$. Therefore, we have $u = a^p = b^p$, where $a, b \in Q$, $a \neq b$; furthermore, $u = c^p$ for every $c \in Q$.

In such a way we proved the following

Proposition 1.1. For every positive integer p and every $u \in V_p(Q)$ the set $[u;p]$ either contains least element M or every one element subset of Q is its minimal element. ⌘

The last statement suggests the following definition of a p-contents $c_p(u)$ of an element $u \in V_p(Q)$. First we put $c_p(u) = M$ if M is the least element of $[u;p]$, and $c_p(u) = \emptyset$ iff $|Q| \geq 2$ and all one element subsets of Q are minimal members in $[u;p]$.

2. Subgroupoids. We assume here that $(Q;f)$ is a given $(V;n,m)$-groupoid.

Proposition 2.1. If $\{P_i \mid i \in I\}$ is a family of subgroupoids of $(Q;f)$ and if $P = \cap \{P_i \mid i \in I\} \neq \emptyset$, then P is a subgroupoid of $(Q;f)$.

Proof. Let $a_1, \ldots, a_n \in P \subseteq P_i$, and let $f(a_1 \cdot \ldots \cdot a_n) = u \in V_m(Q)$. If $c_m(u) = \emptyset$, then we have $u = a^m$ for every $a \in P$, and thus it remains the case when $c_m(u) \neq \emptyset$. The fact that $P_i$ is a subgroupoid implies that there exist $b_{i_1}, \ldots, b_{i_m} \in P_i$ such that $u = b_{i_1} \cdot \ldots \cdot b_{i_m}$. If $M = c_m(u)$ then we have $u = c_1 \cdot \ldots \cdot c_m$, where $A = \{c_1, \ldots, c_m\} \subseteq \subseteq \{b_{i_1}, \ldots, b_{i_m}\}$ and therefore $M \subseteq P$. ⌘

Corollary 2.2. Every nonempty subset B of Q generates a uniquely determined subgroupoid $\langle B \rangle$ of $(Q;f)$. ⌘

Now we are going to give a suitable description of $\langle B \rangle$.

Proposition 2.3. Let B be a nonempty subset of Q and define a sequence $(B_\alpha \mid \alpha \geq 0)$ of subsets of Q as follows:

$$B_0 = B,\ B_{\alpha+1} = B_\alpha \cup (\bigcup \{c_m(f(u)) \mid u \in V_n(B_\alpha)\}).$$

Then

$$\langle B \rangle = \bigcup \{B_\alpha \mid \alpha \geq 0\}. ⌘$$

Consider now some connections between subgroupoids and strong subgroupoids.

<u>Proposition 2.4</u>. Let P be a subgroupoid of Q which is not a strong one. If R is a strong subgroupoid of (Q;f) such that $P \subseteq R \subseteq Q$, then R = Q.

<u>Proof</u>. The assumption that P is a subgroupoid but not a strong subgroupoid implies that there exists a $u \in V_n(Q)$ and $b_1, \dots, b_m, c_1, \dots, c_m \in Q$ such that

$$f(u) = b_1 \cdot \ldots \cdot b_m = c_1 \cdot \ldots \cdot c_m,$$

where $b_j \in P$, $c_k \in R$ and there is some i such that $c_i \in R \setminus P$. Let d be an arbitrary element of Q and define a sequence $d_1, \dots, d_m$ by

$$d_k = \begin{cases} c_k & \text{if } k \neq i \\ d & \text{if } k = i \end{cases}$$

Then we have $f(u) = d_1 \cdot \ldots \cdot d_m$, which implies that $d = d_i \in R$, i.e. Q = R. ✗

If B is a nonempty subset of Q then we denote by $\langle B \rangle_s$ the strong subgroupoid of (Q;f) generated by B. (The existence of $\langle B \rangle_s$ follows from the fact that a nonempty intersection of strong subgroupoids is a strong subgroupoid as well.)

<u>Corollary 2.5</u>. For every nonempty subset B of Q we have $\langle B \rangle_s = \langle B \rangle$ or $\langle B \rangle_s = Q$. ✗

Now we will describe the set of varieties $V$ of semigroups for which there are not any differences between subgroupoids and strong subgroupoids.

Let us say that $V$ is m-<u>regular</u> iff for every nonempty set Q and any element $u \in V_m(Q)$ $|[u;m]| = 1$. In other words, if $a_i, b_j \in Q$ are such that

$$u = a_1 \cdot \ldots \cdot a_m = b_1 \cdot \ldots \cdot b_m$$

then $(b_1, \dots, b_m)$ is a permutation of $(a_1, \dots, a_m)$.

<u>Proposition 2.6</u>. The following two conditions are equivalent

(a) $V$ is m-regular.

(b) For every $(V;n,m)$-groupoid (Q;f) each subgroupoid of (Q;f) is a strong subgroupoid of (Q;f) as well.

Proof. Let $V$ be m-regular and let P be a subgroupoid of a $(V;n,m)$-groupoid $(Q;f)$. Let $u \in V_n(P)$ and $f(u)=a_1 \cdot a_2 \cdot \ldots \cdot a_m$, where $a_i \in Q$. The fact that P is a subgroupoid of $(Q;f)$ implies that $f(u)=b_1 \cdot \ldots \cdot b_m$, for some $b_j \in P$. Thus, we have $a_1 \cdot \ldots \cdot a_m = b_1 \cdot \ldots \cdot b_m$, and from m-regularity of $V$ we obtain that $a_1,\ldots,a_m \in P$. Hence, P is a strong subgroupoid of $(Q;f)$.

Assume now that $V$ is not m-regular. Let Q be a set with at least m elements. The assumption that $V$ is not m-regular implies that there exist $a_i, b_j \in Q$ such that $A=\{a_1,\ldots,a_m\} \subsetneq \{b_1,\ldots,b_m\}$ and $a_1 \cdot \ldots \cdot a_m = b_1 \cdot \ldots \cdot b_m$ in $V(Q)$.

Define a $(V;n,m)$-groupoid $(Q;f)$ by $f(u)=a_1 \cdot \ldots \cdot a_m$ for every $u \in V_n(Q)$. Then A is a subgroupoid of $(Q;f)$, but it is not a strong one. ∎

Certainly, Pr. 2.6 does not mean that if $V$ is not m-regular then the set of strong subgroupoids of a $(V;n,m)$-groupoid $(Q;f)$ is a proper subset of the set of subgroupoids of $(Q;f)$. As an illustration, consider the following

Example 2.7. Let $m \geq 3$ and let $V$ = RB be the variety of rectangular bands. Let Q be an arbitrary set and $f:V_n(Q) \to V_m(Q)$ be defined by $f(u)=a^m$ (=a where a is a fixed element of Q). Then every subgroupoid of $(Q;f)$ is a strong subgroupoid as well. (Namely, P is a subgroupoid of $(Q;f)$ iff $a \in P$.)

3. Homomorphisms and congruences. First we note that in the case of $(V;n,m)$-groupoids the definition of a homomorphism can be restate as follows.

Proposition 3.1. If $(Q;f)$ and $(Q';f')$ are $(V;n,m)$-groupoids then a mapping $\phi:Q \to Q'$ is a homomorphism iff

$$\phi_m f = f' \phi_n \tag{3.1}$$

Proof. We have only to explain what are the meanings of $\phi_m$, $\phi_n$ in (3.1). First, the mapping $\phi:Q \to Q'$ induces a unique homomorphism $\tilde{\phi}:V(Q) \to V(Q')$ such that $\tilde{\phi}(V_\alpha(Q))=V_\alpha(Q')$, for every

$\alpha \geq 1$. Then we denote by $\phi_\alpha : V_\alpha(Q) \to V_\alpha(Q')$ the corresponding restriction of $\tilde{\phi}$. ✗

Proposition 3.2. If $\alpha : Q \to Q'$ is a bijective homomorphism then it is an isomorphism.

Proof. If $\phi : Q \to Q'$ is bijective, then $\tilde{\phi}$ is an isomorphism and $\phi_\alpha : V_\alpha(Q) \to V_\alpha(Q')$ is bijective as well, and $(\phi^{-1})_\alpha = (\phi_\alpha)^{-1}$. Then we have

$$(\phi^{-1})_m f' = (\phi^{-1})_m (f'\phi_n)(\phi_n)^{-1} = (\phi^{-1})_m \phi_m f(\phi_n)^{-1}$$
$$= f(\phi^{-1}).$$ ✗

We mentioned in Preliminaries that a homomorphic image of a subgroupoid is a subgroupoid, and that a complete inverse homomorphic image of a strong subgroupoid is a strong subgroupoid. The converse assertions are not true generally, as it show the following examples.

Example 3.3. Let $(Q;f)$ be a $(V;n,m)$-groupoid containing a subgroupoid P which is not a strong one, and let g be the restriction of f on P. Then P is a strong subgroupoid of $(P;g)$ and the embedding mapping from P into Q is a homomorphism such that P is a homomorphic image of a strong subgroupoid of $(P;g)$, but P is not strong in $(Q;f)$.

Example 3.4. Let $V$ be the variety of commutative semigroups which satisfies the identity $x^2=y^2$, where x,y are different variables. If $Q=\{a,b,c\}$ and $Q'=\{\alpha,\beta\}$ then

$$V_4(Q) = \{a^4, a^3b, a^3c, b^3c\}, \qquad V_2(Q) = \{a^2, ab, ac, bc\},$$
$$V_4(Q') = \{\alpha^4, \alpha^3\beta\}, \qquad V_2(Q') = \{\alpha^2, \alpha\beta\}.$$

Define $(V;4,2)$-groupoids $(Q;f)$ and $(Q';f')$ by

$$f(u) = bc, \qquad f'(u') = \alpha^2,$$

for every $u \in V_4(Q)$, $u' \in V_4(Q')$.

Then the mappings

$$\phi = \begin{pmatrix} a & b & c \\ \alpha & \beta & \beta \end{pmatrix}, \qquad \psi = \begin{pmatrix} a & b & c \\ \alpha & \alpha & \alpha \end{pmatrix}$$

are homomorphisms from $(Q;f)$ into $(Q';f')$. The set $A'=\{\alpha\}$ is a subgroupoid of $(Q';f')$, but $A=\{a\}=\phi^{-1}(A')$ is not a subgroupoid of $(Q;f)$. Furthermore, $A=\{a\}$ is a generating subset of $(Q;f)$, and $\phi$, $\psi$ are different homomorphisms which extend the mapping $a \mapsto \alpha$ from A into $Q'$.

It is natural to define congruences as follows. Let $(Q;f)$ be a $(V;n,m)$-groupoid and $\rho$ an equivalence on Q. We say that $\rho$ is a congruence on $(Q;f)$ iff there is a homomorphism $\phi:(Q;f) \to (Q';f')$, where $(Q';f')$ is a $(V;n,m)$-groupoid, such that $\rho=\ker\phi$, i.e. $a\rho b \iff \phi(a) = \phi(b)$.

Let $(Q;f)$, $(Q';f')$, $\phi,\rho$ be as above. Then $P'=\phi(Q)$ is a subgroupoid of $(Q';f')$ and $\phi$ induces a unique surjective homomorphism $\psi:(Q;f) \to (P';g')$, where $g'$ is the restriction of $f'$ on $P'$. Moreover, we have $\ker\psi=\rho=\ker\phi$. Thus, we can assume that $\phi$ is surjective. Then $\overline{\phi}:\overline{a} \mapsto \phi(a)$ is bijective mapping from $\overline{Q}=Q/\rho$ onto $Q'=\phi(Q)$, where

$$\overline{a} = \{b\in Q \mid a\rho b\} = \{b\in Q \mid \phi(a) = \phi(b)\}.$$

This implies that if we define $\overline{f}:V_n(\overline{Q}) \to V_m(\overline{Q})$ by

$$\overline{f}(\overline{a}_1\cdot\ldots\cdot\overline{a}_n)=\overline{b}_1\cdot\ldots\cdot\overline{b}_m \iff f'(\phi(a_1)\cdot\ldots\cdot\phi(a_n)) = \phi(b_1)\cdot\ldots\cdot\phi(b_m) \tag{3.2}$$

then we obtain a $(V;n,m)$-groupoid $(\overline{Q};\overline{f})$ such that $\overline{\phi}:\overline{a} \mapsto \phi(a)$ is an isomorphism from $(\overline{Q};\overline{f})$ onto $(Q';f')$.

Now we will give another characterization of congruences.

Proposition 3.5. Let $(Q;f)$ be a $(V;n,m)$-groupoid and $\rho$ an equivalence on Q such that

$$f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m,\ f(c_1\cdot\ldots\cdot c_n)=d_1\cdot\ldots\cdot d_m \text{ in } (Q;f) \tag{3.3}$$

and

$$\overline{a}_1\cdot\ldots\cdot\overline{a}_n = \overline{c}_1\cdot\ldots\cdot\overline{c}_n \text{ in } V(\overline{Q}) \tag{3.4}$$

implies

$$\overline{b}_1\cdot\ldots\cdot\overline{b}_m = \overline{d}_1\cdot\ldots\cdot\overline{d}_m \text{ in } V(\overline{Q}), \tag{3.5}$$

where $\overline{Q}=Q/\rho$, $\overline{a}=\{b\in Q \mid a\rho b\}$.

Then $\rho$ is a congruence on $(Q;f)$. Conversely, if $\rho$ is a congruence on $(Q;f)$ then every implication (3.3)&(3.4) $\Longrightarrow$ (3.5) holds.

Proof. Assume that $\phi:(Q;f) \to (Q';f')$ is a surjective homomorphism such that $\rho=\ker\phi$, and denote by $\overline{\phi}$ the corresponding isomorphism from $(\overline{Q};\overline{f})$ into $(Q';f')$.

If (3.3) holds in $(Q;f)$ then we have (in $(Q';f')$):

$$f'(\phi(a_1)\cdot\ldots\cdot\phi(a_n)) = \phi(b_1)\cdot\ldots\cdot\phi(b_m),$$
$$f'(\phi(c_1)\cdot\ldots\cdot\phi(c_n)) = \phi(d_1)\cdot\ldots\cdot\phi(d_m)$$

and therefore

$$\overline{f}(\overline{a}_1\cdot\ldots\cdot\overline{a}_n)=\overline{b}_1\cdot\ldots\cdot\overline{b}_m, \quad \overline{f}(\overline{c}_1\cdot\ldots\cdot\overline{c}_n)=\overline{d}_1\cdot\ldots\cdot\overline{d}_m$$

in $(\overline{Q};\overline{f})$. Assuming that (3.4) is satisfied, we obtain (3.5).

Conversely, assume that $\rho$ is an equivalence on $Q$ such that every implication (3.3)&(3.4) $\Longrightarrow$ (3.5) holds.

If $a_1,\ldots,a_n \in Q$ and $f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m$ in $(Q;f)$, then we define $\overline{f}(\overline{a}_1\cdot\ldots\cdot\overline{a}_n)$ by

$$\overline{f}(\overline{a}_1\cdot\ldots\cdot\overline{a}_n) = \overline{b}_1\cdot\ldots\cdot\overline{b}_m.$$

It follows from (3.3)&(3.4) $\Longrightarrow$ (3.5) that $\overline{f}$ is well defined, i.e. we obtain a $(\mathbb{V};n,m)$-groupoid $(\overline{Q};\overline{f})$. Clearly, $\overline{\phi}:a \to \overline{a}$ is a homomorphism from $(Q;f)$ onto $(\overline{Q};\overline{f})$ and $\rho=\ker\overline{\phi}$ , i.e. $\rho$ is a congruence. ∎

(We remark that the above definition and Pr. 3.5 imply that the well known isomorphism theorems ([1]) holds.)

4. Induced $(\mathbb{W};n,m)$-groupoids. We assume here that $\mathbb{W}$ is a nontrivial subvariety of a variety $\mathbb{V}$. Note that $\mathbb{W}(Q)\in\mathbb{V}$ for any nonempty set $Q$, which implies that there is a uniquely determined homomorphism $\pi:\mathbb{V}(Q) \to \mathbb{W}(Q)$ with the property $\pi(a)=a$ for all $a\in Q$. Moreover, for each positive integer $p$, $\pi(\mathbb{V}_p(Q))=\mathbb{W}_p(Q)$ and this implies that $\pi$ induces a surjective mapping $\pi_p:\mathbb{V}_p(Q) \to \mathbb{W}_p(Q)$.

We say that a $(\mathbb{W};n,m)$-groupoid $(Q;g)$ is <u>induced</u> by a $(\mathbb{V};n,m)$-groupoid $(Q;f)$ iff the following diagram commutes:

$$\begin{array}{ccc} \mathbb{V}_n(Q) & \xrightarrow{\ f\ } & \mathbb{V}_m(Q) \\ \big\downarrow{\scriptstyle \pi_n} & & \big\downarrow{\scriptstyle \pi_m} \\ \mathbb{W}_n(Q) & \xrightarrow{\ g\ } & \mathbb{W}_m(Q) \end{array}$$

An obvious consequence from this definition is

<u>Proposition 4.1</u>. If $(Q;f)$ is a $(\mathbb{V};n,m)$-groupoid then there exists at most one $(\mathbb{W};n,m)$-groupoid $(Q;g)$ which is induced by $(Q;f)$. Such a $(\mathbb{W};n,m)$-groupoid $(Q;g)$ do exist iff $(Q;f)$ satisfies the following condition:

$$(\forall u,v\in\mathbb{V}(Q))(\pi_n(u) = \pi_n(v) \implies \pi_m f(u) = \pi_m f(v)). \ \blacksquare \qquad (4.1)$$

If a $(\mathbb{V};n,m)$-groupoid $(Q;f)$ satisfies (4.1) then we say that it <u>admits</u> <u>weakly</u> $\mathbb{W}$. And $(Q;f)$ will be called a <u>$\mathbb{W}$-$(\mathbb{V};n,m)$-groupoid</u> iff the following statement holds:

$$(\forall u,v\in\mathbb{V}(Q))(\pi_n(u) = \pi_n(v) \implies f(u) = f(v)). \qquad (4.1')$$

<u>Proposition 4.2</u>. A $(\mathbb{W};n,m)$-groupoid $(Q;g)$ is induced by at least one $\mathbb{W}$-$(\mathbb{V};n,m)$-groupoid $(Q;f)$.

<u>Proof</u>. If $u\in\mathbb{W}_n(Q)$ then $\pi_n^{-1}(u) \in\mathbb{V}_n(Q)$, $\pi_m^{-1}(g(u))\in \mathbb{V}_m(Q)$.

If $f:\mathbb{V}_n(Q) \to \mathbb{V}_m(Q)$ is such that for every $x\in\mathbb{V}_n(Q)$ we have $f(x)\in\pi_m^{-1}(g\pi_n(x))$ then we obtain a $(\mathbb{V};n,m)$-groupoid $(Q;f)$ which induces $(Q;g)$. Certainly, we can define $f$ in such a way that it satisfies (4.1'). Namely, let $h:\mathbb{W}_n(Q) \to \mathbb{V}_m(Q)$ be such that $h(u)\in\pi_m^{-1}g(u)$ for every $u\in\mathbb{W}_n(Q)$. Now, if we define $f:\mathbb{V}_n(Q) \to \mathbb{V}_m(Q)$ by $f=h\pi_n$, then we will obtain a $\mathbb{W}$-$(\mathbb{V};n,m)$-groupoid $(Q;f)$ which induces $(Q;g)$. $\blacksquare$

The following statements are also clear.

<u>Proposition 4.3</u>. Let $(Q;g)$ be a $(\mathbb{W};n,m)$-groupoid which is induced by a $(\mathbb{V};n,m)$-groupoid $(Q;f)$. Then:

(a) If P is a subgroupoid of (Q;f) then P is a subgroupoid of (Q;g).

(b) If ρ is a congruence on (Q;f) then ρ is a congruence on (Q;g). ∎

Proposition 4.4. Let (Q;f) and (Q';f') be (V;n,m)-groupoids and let (Q;g), (Q';g') be (W;n,m)-groupoids such that (Q;g) is induced by (Q;f) and (Q';g') is induced by (Q';f'). If $\phi:Q \to Q'$ is a homomorphism from (Q;f) into (Q';f') then it is a homomorphism from (Q;g) into (Q';g') as well. ∎

The following example shows that Pr. 4.3 (a), in general, does not hold for strong subgroupoids.

Example 4.5. Let Q={a,b} and let $f:Q \to Q^3$ be defined by f(a)=f(b)=(a,a,a). Define a mapping $g:RB(Q)=Q \to RB_3(Q)$ by g(a)=g(b)=a. Then (Q;f) is a (Sem;1,3)-groupoid and (Q;g) is a (RB;1,3)-groupoid induced by (Q;f). A={a} is a strong subgroupoid of (Q;f), but A is not a strong subgroupoid of (Q;g).

R E F E R E N C E S

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ВЕКТОРСКО ВРЕДНОСНИ ГРУПОИДИ ИНДУЦИРАНИ ОД МНОГУОБРАЗИЈА ОД ПОЛУГРУПИ

Ѓ. Чупона и С. Марковски

Р е з и м е

Векторско вредносните групоиди индуцирани од полугрупи се разгледуваат во трудот [3]. Овде се разгледуваат истите прашања како и во претходно споменатиот труд со тоа што полугрупите се од дадено многуобразие од полугрупи. Се покажува дека некои резултати што не важат во општиот случај важат во вака извршената рестрикција.