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NONEXISTENCE OF CONTINUOUS (4,3)-GROUPS ON R

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Abstract. In this paper we show that continuous (4,3)-groups on R do not exist.

0. Introduction. Let $m, n, k=n-m$ be positive integers. A set $G\neq\emptyset$ together with a map $[\ ]: G^n \to G^m$ is called an (n,m)-group if:

(1) $[[x_1^n]x_{n+1}^{n+k}] = [x_1^i[x_{i+1}^{i+n}]x_{i+n+1}^{n+k}]$ for each $1 \leq i \leq k$; and

(2) For given $a_1^k, b_1^k \in G^k$, $c_1^m \in G^m$, there exist $x_1^m, y_1^m \in G^m$ such that $[a_1^k\ x_1^m]=c_1^m=[y_1^m\ b_1^k]$ (see [1], where this notion was introduced). Above, x_1^t denotes the vector $(x_1,\dots,x_t)\in G^t$, and $[x_1^n]$ denotes the image of x_1^n under the map $[\ ]$. We will denote by $\overset{t}{a}$ the vector $(a,a,\dots,a)\in G^t$.

We say that $(G;[\ ])$ is a continuous (n,m)-group, if: G is a topological space; $(G;[\ ])$ is an (n,m)-group; and the map $[\ ]$ is continuous, where G^n, G^m are equiped with the product topology.

If $(G;[\ ])$ is a $(2m,m)$-group, then $(G^m,\bullet)$ where $x_1^m\bullet y_1^m = [x_1^m\ y_1^m]$, is a group with identity element $\overset{m}{e}$, for some $e\in G$ (see [2]). We say that e is the identity of the $(2m,m)$-group $(G;[\ ])$. A $(2m,m)$-group $(G;[\ ])$ is called a topological $(2m,m)$-group, if G is a topological space and $(G^m,\bullet)$ is a topological group.

If $(G;[\ ])$ is an $(m+1,m)$-group, then $(G;[\ ]')$ where $[x_1^{2m}]'=[x_1^{2m}]$, is a $(2m,m)$-group, induced by $(G;[\ ])$. So, if $(G;[\ ])$ is an $(m+1,m)$-group, then there exists an element $e\in G$, such that $[x_1^m\ \overset{m}{e}]=[\overset{m}{e}\ x_1^m]=x_1^m$, and moreover $[x\ \overset{m}{e}] = [\overset{m}{e}\ x]$

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(see [3]). We say that an (m+1,m)-group is a <u>topological</u> <u>(m+1,m)-group</u> if its induced (2m,m)-group is a topological (2m,m)-group.

In [5] it was shown that continuous (3,2)-groups on R (where R is the set of the real numbers equiped with the usual topology) do not exist, but in [4] it was shown that topological (4,2)-groups on R do exist. The examples produced in [4] were obtained using Lie groups and Lie algebras.

In this paper we give an elementary proof that continuous (4,3)-groups on R do not exist. Also, we will give a sketch of a proof, that topological (4,3)-groups on R do not exist, using Lie groups and Lie algebras. Although the second result is a consequence of the first one, we include its proof, because of its method, which may be used for answering the existence question about topological (m+1,m)-groups for $m \geq 4$. Similar methods were used in [4].

<u>1. Elementary algebraic results</u>

We need several elementary results about (4,3)-groups, which are in fact, special cases of more general results about (m+1,m)-groups. Let $(G,[\ ])$ be a (4,3)-group, with identity element $e \in G$.

PROPOSITION 1. <u>The following conditions are equivalent</u>:

(1) $|G|=1$, <u>i.e.</u> G <u>has only one element</u>;

(2) $[x\ \overset{3}{e}] = \overset{3}{e}$, <u>for some</u> $x \in G$;

(3) $[x\ y\ \overset{3}{e}] = \overset{3}{e}$, <u>for some</u> $x,y \in G$; <u>and</u>

(4) $[x\ y\ \overset{3}{e}] = [z\ \overset{3}{e}]$, <u>for some</u> $x,y,z \in G$.

<u>Proof</u>. It is obvious that (1) $\Longrightarrow$ (2), (1) $\Longrightarrow$ (3) and (1) $\Longrightarrow$ (4). If $[x\ \overset{3}{e}] = \overset{3}{e}$ for some $x \in G$, then $xyz=[xyz\overset{3}{e}]= =[x\overset{3}{e}yz]=[\overset{3}{e}yz]=[yz\overset{3}{e}]=[yzx\overset{3}{e}]=yzx$, for each $y,z \in G$, implies that $|G|=1$; hence (2) $\Longrightarrow$ (1). If $[xy\overset{3}{e}]=\overset{3}{e}$ for some $x,y \in G$, then

$[xyxz]=[xy\overset{3}{e}xz]=[\overset{3}{e}xz]=[xz\overset{3}{e}]=[xzxy\overset{3}{e}]=[xzxy]$, for each $z\in G$, implies that $z=y$, i.e. $|G|=1$; hence (3) $\Longrightarrow$ (1). If $[xy\overset{3}{e}]=[z\overset{3}{e}]$ for some $x,y,z\in G$, then $xyz=[xyz\overset{3}{e}]=[xyxy\overset{3}{e}]=[xy\overset{3}{e}xy]=[z\overset{3}{e}xy]=zxy$, implies that $z=x$, and so, $[y\overset{3}{e}]=\overset{3}{e}$. Hence (4) $\Longrightarrow$ (2). ∎

For given $x,y\in G$, let $\alpha_x\beta_x\gamma_x$, $\alpha_{xy}\beta_{xy}\gamma_{xy}$ denote the vectors $[x\overset{3}{e}]$ and $[xy\overset{3}{e}]$ respectively.

PROPOSITION 2. _The following conditions are equivalent:_

(1) $|G|=1$;

(2) $\alpha_x = x$ _or_ $\gamma_x = x$ _for some_ $x\in G$;

(3) $\alpha_x = \gamma_y$ _for some_ $x,y\in G$;

(4) $\alpha_{xy} = x$ _or_ $\gamma_{xy} = y$ _for some_ $x,y\in G$;

(5) $\alpha_x\beta_x = \beta_y\gamma_y$ _for some_ $x,y\in G$;

(6) $\alpha_{xy}\beta_{xy} = \beta_{zt}\gamma_{zt}$ _for some_ $x,y,z,t\in G$;

(7) $\alpha_x\beta_x = \alpha_{xy}\beta_{xy}$ _or_ $\beta_x\gamma_x = \beta_{yx}\gamma_{yx}$ _for some_ $x,y\in G$.

Proof. It is obvious that (1) $\Longrightarrow$ (k) for each $k=2,\dots,7$. If $\alpha_x=x$ or $\gamma_x=x$, then $[\beta_x\ \gamma_x\ \overset{3}{e}]=\overset{3}{e}$ or $[\alpha_x\ \beta_x\ \overset{3}{e}]=\overset{3}{e}$, which implies that $|G|=1$, by P.1; hence (2) $\Longrightarrow$ (1). If $\alpha_x=\gamma_y$, then $\alpha_y\beta_y\ x = [\alpha_y\beta_y\ x\ \overset{3}{e}]=[\alpha_y\beta_y\alpha_x\beta_x\gamma_x]=[\alpha_y\beta_y\gamma_y\beta_x\gamma_x]=[y\ \overset{3}{e}\ \beta_x\gamma_x]=y\beta_x\gamma_x$, implies that $x=\gamma_x$; hence (3) $\Longrightarrow$ (2). If $\alpha_{xy}=x$ or $\gamma_{xy}=y$, then $[\beta_{xy}\gamma_{xy}\overset{3}{e}]=[y\overset{3}{e}]$ or $[\alpha_{xy}\beta_{xy}\overset{3}{e}]=[x\overset{3}{e}]$, which implies that $|G|=1$ by P.1; hence (4) $\Longrightarrow$ (1). If $\alpha_x\beta_x=\beta_y\gamma_y$, then $[\alpha_y\ x\ \overset{3}{e}]=[\alpha_y\alpha_x\beta_x\gamma_x]=$ $=[\alpha_y\beta_y\gamma_y\gamma_x]=[y\ \overset{3}{e}\ \gamma_x]=[y\ \gamma_x\ \overset{3}{e}]$ implies that $x=\gamma_x$; hence (5) $\Longrightarrow$ (2). If $\alpha_{xy}\beta_{xy}=\beta_{zt}\gamma_{zt}$, then $\alpha_{zt}\ x\ y=[\alpha_{zt}\ x\ y\ \overset{3}{e}]=$ $=[\alpha_{zt}\alpha_{xy}\beta_{xy}\gamma_{xy}]=[\alpha_{zt}\beta_{zt}\gamma_{zt}\gamma_{xy}]=[z\ t\ \overset{3}{e}\ \gamma_{xy}]=z\ t\ \gamma_{xy}$, implies that $y=\gamma_{xy}$; hence (6) $\Longrightarrow$ (4). If $\alpha_x\beta_x=\alpha_{xy}\beta_{xy}$, then $\alpha_{xy}\beta_{xy}\gamma_{xy}=$ $=[x\ y\ \overset{3}{e}]=[\overset{3}{e}\ x\ y]=[\alpha_x\beta_x\gamma_x\ y]=[\alpha_{xy}\beta_{xy}\gamma_x\ y]$ implies that $[\gamma_{xy}\ \overset{3}{e}]=[\gamma_x\ y\ \overset{3}{e}]$, which implies that $|G|=1$, by P.1; hence (7) $\Longrightarrow$ (1). ∎

PROPOSITION 3. <u>The element</u> $\alpha_e\beta_e\gamma_e$ <u>is in the centre of the group</u> $(G^3,\bullet)$ (<u>where</u> $xyz\bullet uvw=[xyzuvw]$), <u>if and only if</u> $|G|=1$.

<u>Proof</u>. If $\alpha_e\beta_e\gamma_e\bullet xyz = xyz\bullet\alpha_e\beta_e\gamma_e$ for each $xyz\in G^3$, then $[exyz]=[xyze]$ for each $x,y,z\in G$. For $x=e$, this implies $eyz=yze$ i.e. $y=z=e$; hence $|G|=1$. ∎

2. <u>Nonexistence of continuous</u> (4,3)-<u>groups on</u> R.

We start with the assumption that there is a continuous (4,3)-group on R, and denote it by $(R;[\ ])$. We denote by $[\ ]_1$, $[\ ]_2$, $[\ ]_3$ the components of $[\ ]$, i.e.

$$[xyzt] = [xyzt]_1[xyzt]_2[xyzt]_3.$$

Since $[\ ]$ is continuous, it follows that $[\ ]_i$, $i=1,2,3$, are also continuous. In the following several steps, the assumption that $(R;[\ ])$ is a continuous group will bring us to a contradiction.

<u>Step 1</u>. Let $\phi:R^3\to R$ be defined by $\phi(xyz)=[xyze]_1-x$. Since $[\ ]_1$ and $-$ are continuous, it follows that ϕ is also continuous.

<u>Fact 1</u>. $\phi^{-1}(0)\neq\emptyset$, i.e. there exists $xyz\in R^3$, such that $\phi(xyz)=0$, where 0 is the zero in R.

<u>Proof</u>. Consider $\phi(xee)=\alpha_x-x$, $\phi(\alpha_x\beta_x\gamma_x)=\alpha_{xe}-\alpha_x$ and $\phi(\alpha_{xe}\beta_{xe}\gamma_{xe})=x-\alpha_{xe}$. If $\phi(\alpha_x\beta_x\gamma_x)=0$, then $\phi^{-1}(0)\neq\emptyset$. So suppose that $\phi(\alpha_x\beta_x\gamma_x)\neq0$. Since $|R|>1$, P.2. implies that $\phi(xee)\neq0$ and $\phi(\alpha_{xe}\beta_{xe}\gamma_{xe})\neq0$. It is not possible all of the α_x-x, $\alpha_{xe}-\alpha_x$, $x-\alpha_{xe}$ to have the same sign, since their sum is equal to 0. So, two of them have different signs. This, together with the facts that ϕ is continuous and R is connected, implies that $\phi^{-1}(0)\neq\emptyset$. ∎

<u>Step 2</u>. Let $c,a,b\in R$ be such that $\phi(cab)=0$, i.e. $[cabe]=$ $=cuv$ for some $u,v\in R$. This implies that $[ab\overset{2}{e}]=uve$, $[ab\overset{3}{e}]=[uv\overset{2}{e}]$

and $abe=[uv\overset{3}{e}]$, i.e. $\alpha_{uv}=a$, $\beta_{uv}=b$, $\gamma_{uv}=e$. Now, let $\psi: R^2 \to R$ be defined by $\psi(xy)=[xy\overset{2}{e}]_1-x$. Again, since $[\ ]_1$ and $-$ are continuous, it follows that ψ is continuous.

Fact 2. $\psi^{-1}(0)\neq\emptyset$, i.e. there exists $xy\in R^2$ such that $\psi(xy)=0$.

Proof. Consider $\psi(uv)=\alpha_{ab}-u$ and $\psi(ab)=u-a$. Since the map $\eta: R^2 \to R$ defined by $\eta(xy)=[xy\overset{3}{e}]_1-x$ is continuous, $|R|>1$ and R is connected, P.2. implies that $\alpha_{xy}<x$ for each $x,y\in R$, or $\alpha_{xy}>x$ for each $x,y\in R$. If $u=a$, then $\psi^{-1}(0)\neq\emptyset$. If $u-a<0$, i.e. $u<a=\alpha_{uv}$, then $\alpha_{xy}>x$ for each $x,y\in R$. So $\alpha_{ab}-u>a-u>0$, i.e. $\psi(uv)>0$. This, together with $\psi(ab)<0$, implies that $\psi^{-1}(0)\neq\emptyset$. If $u-a>0$, i.e. $u>a=\alpha_{uv}$, then $\alpha_{xy}<x$ for each $x,y\in R$. So $\alpha_{ab}-u<a-u<0$, i.e. $\psi(uv)<0$. This, together with $\psi(ab)>0$, implies that $\psi^{-1}(0)\neq\emptyset$.

Step 3. Let $p,q\in R$ be such that $\psi(pq)=0$, i.e. $[pq\overset{2}{e}]=prs$ for some $r,s\in R$. This implies that $[rs\overset{3}{e}]=q\overset{2}{e}$, i.e. $\beta_{rs}\gamma_{rs}=\overset{2}{e}$.

Step 4. Symmetrically, there exist $z,w\in R$ such that $\alpha_{zw}\beta_{zw}=\overset{2}{e}$. (When we say symmetrically, we mean: change $[xyze]_1-x$ in Step 1, by $[exyz]_3-z$ and $[xy\overset{2}{e}]_1-x$ in Step 2, by $[\overset{2}{e}xy]_3-y$.)

Now, Step 3, Step 4 and P.2. imply that $|R|=1$, which is a contradiction.

3. The Lie groups and Lie algebras method

Now we will give a sketch of a proof, via Lie groups and Lie algebras, that topological (4,3)-groups on R do not exist. If $(R;[\ ])$ is a topological (4,3)-group, then $(R^3,\bullet)$ (where $xyz\bullet uvw=[xyzuvw]$) is a topological group. Using the positive answer to the Fifth Hilbert Problem [8] we obtain that $(R^3,\bullet)$ is a Lie group on R^3. The element $\alpha_e\beta_e\gamma_e\in R^3$ in this Lie group has the following properties: $(\alpha_e\beta_e\gamma_e)^3=\alpha_e\beta_e\gamma_e\bullet\alpha_e\beta_e\gamma_e\bullet\alpha_e\beta_e\gamma_e=\overset{3}{e}$; $\alpha_e\beta_e\gamma_e\neq\overset{3}{e}$; and $\alpha_e\beta_e\gamma_e$ is not in the centre of $(R^3,\bullet)$, by P.2 and P.3. We will prove that in arbitrary Lie group on R^3 there does not exist an element x satisfying these properties.

Suppose that $(R^3,*)$ is a Lie group, with the identity element e, and, $x \in R^3$ such that $x^3=e$, $x \neq e$ and x is not in the centre of $(R^3,*)$. The map $\psi: R^3 \to R^3$ defined by $\psi(y)=x*y*x^{-1}$ is an automorphism of R^3 of order 3, i.e. $\psi^3=id_{R^3}$ and $\psi \neq id_{R^3}$. Since R^3 is simply-connected manifold, there exists a bijection between the automotphisms of the Lie group and the automorphisms of its corresponding Lie algebra [6]. So, ψ corresponds to an automorphism, again denoted by ψ, of the corresponding Lie algebra on R^3, such that $\psi^3=id$ and $\psi \neq id$.

It is easy to check that if ψ is an autmorphism of a Lie algebra on R^3 of order 3, then there is a vector $X \in R^3$ such that X, $\psi(X)=Y$ and $\psi^2(X)=Z$ is a basis for the vector space R^3. Let the bracket on the Lie algebra be defined on X, Y by $[X,Y]=aX+bY+cZ$, for some $a,b,c \in R$. This implies that $[Y,Z]=cX+aY+bZ$, $[Z,X]=bX+cY+aZ$. The Jacobi identity implies that $(b-a)(a+b+c)=0$. So, the Lie algebras on R^3 with such an automorphism, can be classified into two classes:

Class 1, when $a=b$; and
Class 2, when $a+b+c=0$.

We will show that for each Lie algebra from these classes, which is a corresponding Lie algebra to a Lie group on R^3, in the Lie group there does not exist an element x such that $x \neq e$, $x^3=e$. This will complete the proof that topological (4,3)-groups on R do not exist, via this method.

If $a=b=c=0$, then the corresponding Lie group, up to isomorphism, is $(R^3,+)$ where + is the usual addition of vectors. So if $x^3=e$, then $x=e$.

Now, let $a^2+b^2+c^2 \neq 0$.

<u>Case 1</u>. Consider the Lie algebras from the class 1, i.e. when $a=b$. If $c \neq a$ and $c \neq -2a$, then each such Lie algebra is simple. It is known (see [7] p. 429) that there are only two

3-dimensional simple Lie groups up to a local isomorphism. So there are only two non-isomorphic simple Lie algebras on R^3. The following two Lie algebras:

1) a=b=1, c=0; and

2) a=b=0, c=1

on R are simple. They are not isomorphic, because the first one contains a 2-dimensional Lie subalgebra (generated by X and Y), and second does not, since in the second, the product-bracket is the usual vector-product on R^3. The second Lie algebra (i.e. a=b=0, c=1) is semisimple compact Lie algebra (see [7], p. 453). The Weyl's theorem ([7], p. 444), says that a connected Lie group with a semisimple compact Lie algebra must be compact. So there does not exist a Lie group on R^3 whose corresponding Lie algebra is isomorphic to the Lie algebra 2) i.e. a=b=0, c=1.

Next, we will give an example of a simple Lie group on a manifold homeomorphic to R^3, whose corresponding Lie algebra is isomorphic to the Lie algebra 1) i.e. a=b=1, c=0. Let $D=\{z\in C: |z|<1\}$ where C is the set of complex numbers, and $|\ |$ is the module. The map $z\to(1+z)/(1+\bar{z})$ from D into the set $\{z: z\in C, z\neq -1, |z|=1\}$ is continuous. Hence, for each $z\in D$, there is a unique $t\in(-\pi,\pi)$ such that $\exp(it)=(1+z)/(1+\bar{z})$, and we shall denote that number t by $(-i)\ln((1+z)/(1+\bar{z}))$. Define a binary operation on R×D by:

$$(x,u)\cdot(y,v)=\left(x+u+t,\frac{u+v(\exp(2iy))}{(\exp(2iy))+u\bar{v}}\right),$$

where $t=\frac{1}{2}(-i)\ln\frac{1+u\bar{v}(\exp(-2iy))}{1+\bar{u}v(\exp(2iy))}$. Then $(R\times D,\cdot)$ is a covering group for SL(2), and it is a semisimple Lie group (see [8], p.-417). In this group, $(x,u)^3=(0,0)$ implies $(x,u)=(0,0)$.

Now we consider the Lie algebras from Class 1, which are not simple. The Lie algebra for c=-2a, will be considered later as a Lie algebra from Class 2. So, we examine the Lie algebra for c=a, and hence a=b=c. The matrix group

$$G = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x,y,z \in R \right\}$$

can be considered as a Lie group on R^3. The corresponding Lie algebra is

$$\left\{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} : x,y,z \in R \right\}$$

where $[A,B]=AB-BA$, and the map

$$\phi(X) = \begin{bmatrix} 0 & 1 & 1/a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \phi(Y) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ \phi(Z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

is an isomorphism between this Lie algebra and the Lie algebra for $a=b=c$. In the matrix group G, if

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } x=y=z=0.$$

Case 2. Now consider the Lie algebras from Class 2, i.e. $a+b+c=0$. Let us suppose that the numbers $x,y,z,u,v,w \in R$, satisfy the conditions: $x+y+z=u+v+w=0$, $x^2+y^2+z^2 \neq 0 \neq u^2+v^2+w^2$ and $(x,y,z) \neq t(u,v,w)$. Then the vector product $(x,y,z) \times (u,v,w) = (t,t,t)$ for $t \neq 0$. It is easy to verify that the subspace U, of the Lie algebra for $a+b+c=0$, generated by the vector $xX+yY+zZ$ and $uX+vY+wZ$ is an invariant subalgebra. Moreover, it is a commutative Lie algebra. Suppose that Lie algebra for $a+b+c=0$, is a corresponding Lie algebra for a Lie group on R^3. Then, this Lie group contains a commutative connected 2-dimensional Lie subgroup H, whose corresponding Lie algebra is U (see [8]). Since R^3 is simply-connected, it follows that R^3/H is simply-connected Lie group (see [8], p. 255). Since $\dim(R^3/H)=1$, it follows that R^3/H is isomorphic to $(R,+)$. Let $x \in R^3$ such that $x^3=e$, and x is not in the centre of the Lie group on R^3. Then $(xH)^3=H$ and so $xH=H$, i.e. $x \in H$. Since H is a commutative subgroup, it follows that $\psi(z)=z$ for each $z \in H$, and hence $\psi(W)=W$ for each $W \in U$. Since $X+Y+Z \notin U$, and $\psi(X+Y+Z)=Y+Z+X=X+Y+Z$, it follows that $\psi=id$, hence $x=e$.

R E F E R E N C E S

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