INTRODUCTION TO COMBINATORIAL THEORY OF VECTOR VALUED SEMIGROUPS

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CONTENTS

§0. INTRODUCTION

The aim of this paper is to develop a combinatorial theory of vector valued (v.v.) semigroups via their presentations, and to give a satisfactory description of free objects in some varieties of v.v. semigroups. V.v. variants of Post and Cohn-Rebane theorems are obtained as applications of more general results.

In order this paper to be self-contained, we begin with a few necessary definitions, notations and results used in the main text, although they could be found in [2] (this volume). The introduction concludes with a short description of the paper.

Let Q be a nonempty set and r a positive integer. The r-th cartesian power of the set Q, denoted by Q^r, consists of the r-tuples $(a_1,\dots,a_r)$, where $a \in Q$. We will use the following notations: a_1^r, $a_1 \dots a_r$, $\underline{a}$ for $(a_1,\dots,a_r)$; a^r for a_1^r when $a_1=\dots=a_r=a$; x_i^j for $x_i \dots x_j$ if $i \le j$, and x_i^j for the empty sequence if $i > j$. The set of all nonempty finite sequences of elements from Q will be denoted by Q^+, and Q^+ together with the empty sequence (usually denoted by 1) will be denoted by Q^*. In fact, Q^+ is a free semigroup, and Q^* is a free monoid, with a basis Q, where the operation is the usual concatenation of sequences. (Sometimes, the elements of Q^* will be called words.)

By N we denote the set of all nonnegative integers, i.e. $N=\{0,1,2,\dots\}$, and by N_r we denote the set $\{1,2,\dots,r\}$. The mapping $d:Q^* \to N$ (called a dimension) is defined by: $d(1)=0$, $d(a_1^r)=r$, $a_\nu \in Q$.

From now on, $n,m,n-m=k$ will be positive integers. Let $Q\neq\emptyset$. A mapping $f:Q^n \to Q^m$ is called an (n,m)-operation (shortly a v.v. operation), and $\underline{Q}=(Q;f)$ is called an (n,m)-groupoid. If, in addition, the equation

$$f(f(a_1^n)a_{n+1}^{n+k}) = f(a_1^j f(a_{j+1}^{j+n})a_{j+n+1}^{n+k}) \tag{0.1}$$

is satisfied for each $a_\nu \in Q$, $j\in N_k$, then $\underline{Q}$ is an (n,m)-semigroup (shortly a v.v. semigroup).

Denote the set $\bigcup_{s\geq 1} Q^{m+sk}$ by $Q^{(n,m)}$[1]. A mapping $g:Q^{(n,m)} \to Q^m$ is said to be a poly-(n,m)-operation, and $\underline{Q}=(Q;g)$ a poly-(n,m)-groupoid. Moreover, if the equation

$$g(a_1^j g(b_1^{m+rk})a_{j+1}^{sk}) = g(a_1^j b_1^{m+rk} a_{j+1}^{sk}) \tag{0.2}$$

is satisfied for each $a_\nu, b_\lambda \in Q$, $r,s\geq 1$, $j\in N_{sk}\cup\{0\}$, then $\underline{Q}$ is said to be a poly-(n,m)-semigroup.

To each (n,m)-groupoid $\underline{Q}=(Q;f)$ one can associate a poly-(n,m)-groupoid $\underline{Q}^\# = (Q;f^\#)$ where $f^\#$ is defined by induction in the following manner:

$$f^\#(a_1^n) = f(a_1^n)\ ,\ f^\#(a_1^{m+(s+1)k}) = f(f^\#(a_1^{m+sk})a_{m+sk+1}^{m+(s+1)k}) \tag{0.3}$$

Conversely, to each poly-(n,m)-groupoid $\underline{Q}=(Q;g)$ we can associate an (n,m)-groupoid $\underline{Q}_\# = (Q;g_\#)$ by $g_\#(a_1^n)=g(a_1^n)$, i.e. $g_\#$ is the restriction of g on Q^n. It is obvious that $(\underline{Q}^\#)_\# = \underline{Q}$, but in general $(\underline{P}_\#)^\# \neq \underline{P}$.

In the case of (n,m)- and poly-(n,m)-semigroups we have the following:

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1) *If X is a nonepmty set, then by $X^{(n,m)}$ we denote the union* $\bigcup_{s\geq 1} X^{sk+m}$

Proposition 0.1. *(a) An (n,m)-groupoid $\underline{Q}$ is an (n,m)-semigroup iff $\underline{Q}^{\#}$ is a poly-(n,m)-semigroup.*

(b) A poly-(n,m)-groupoid $\underline{P}$ is a poly-(n,m)-semigroup iff $\underline{P}_{\#}$ is an (n,m)-semigroup.

(c) If $\underline{P}$ is a poly-(n,m)-semigroup then $(\underline{P}_{\#})^{\#} = \underline{P}$. □

(For the proof, see [2] p.p. 34-36.)

The notions of (n,m)- and poly-(n,m)-structures are easily thought of as algebras with m n-ary and poly-n-ary operations (called component or scalar operations). Namely, if (Q;f) is an (n,m)- or a poly-(n,m)-groupoid, the component operations $f_1,\dots,f_m: Q^n \to Q$ or $Q^{(n,m)} \to Q$ are defined by

$$f(a_1^{m+rk}) = b_1^m \text{ iff } f_i(a_1^{m+rk}) = b_i, \quad i \in N_m,$$

where r=1 in the first case, and r ≥ 1 in the second one.

It is easy to interpret the condition (0.1) and (0.2) via the corresponding component operations.

All of the notions such as: subalgebra, congruence, homomorphism, free object in the class of component (n,m)- or poly--(n,m)-algebras (i.e. algebras obtained from (n,m)- or poly-(n,m)-groupoids as above) are considered well known. Using the above notions the following ones can be obtained (without giving their explicit definitions): an (n,m)- and a poly-(n,m)-subgroupoid, a congruence on an (n,m)- and a poly-(n,m)-groupoid, and a homomorphism for (n,m)- and poly-(n,m)-groupoids.

Recall the construction of a free poly-(n,m)-groupoid given in ([2], P.6.3). Let $B \neq \emptyset$ and:

$$B_o = B, \quad B_{\alpha+1} = B_\alpha \cup N_m \times B_\alpha^{(n,m)}, \quad F(B) = \bigcup_\alpha B_\alpha.$$

Define a poly-(n,m)-operation g on F(B) by

$$g(u_1^{m+rk}) = v_1^m \iff (\forall i \in N_m),\ v_i = (i, u_1^{m+rk}) \qquad (0.4)$$

Proposition 0.2. *$\underline{F}(B) = (F(B);g)$ is a free poly-(n,m)-groupoid with a basis B.*

Proof. Let $\underline{Q}=(Q;f)$ be an (n,m)-groupoid and $\xi:B\to Q$ a mapping. Then there exists a unique extension $\bar{\xi}$ of ξ such that $\bar{\xi}$ is a homomorphism from $\underline{F}(B)$ into $\underline{Q}$. □

We have already defined an integer valued function, namely the dimension $d:Q^*\to N$. More such functions for $F(B)$ and $F(B)^+$ are defined and used in the main text. Naturally, in the definition of $F(B)$ there is a function denoted by χ and called hierarchy, where $\chi(u)=\min\{\alpha \mid u\in B_\alpha\}$ for $u\in F(B)$.

We define a norm on $F(B)$, i.e. a mapping $|\ |: F(B)^+\to N$ by induction on χ:

$$|b| = 0, \text{ for } b\in B, \tag{0.5}$$

$$|x| = \sum_{\nu=1}^{\alpha} |u_\nu| \text{ for } x=u_1^\alpha\in(F(B))^\alpha, \tag{0.6}$$

$$|(i,x)| = 1+|x| \text{ for } (i,x)\in F(B). \tag{0.7}$$

Sometimes in the text we will need an alternative definion of a norm. Namely, instead of (0.5) or (0.7) we can take

$$|b| = 1 \text{ for } b\in B, \tag{0.5'}$$

$$|(i,x)| = i+|x| \text{ for } (i,x)\in F(B), \tag{0.7'}$$

$$|(i,x)| = |x| \text{ for } (i,x)\in F(B), \tag{0.7"}$$

and instead of (0.6) we can take

$$|x| = \sum_{\lambda=1}^{\alpha} \prod_{\nu=1}^{\lambda} |u_\nu| \text{ for } x=u_1^\alpha\in(F(B))^\alpha. \tag{0.6'}$$

For the empty sequence 1, we always define $|1|=0$.

The norm used most often in the text is the one defined by (0.5), (0.6) and (0.7). (So, from now on when we say norm, we think of this one.)

A consize review of the results in this paper follows.

In §1 we define the notion of an (n,m)-semigroup determined by a presentation $\langle B;\Delta\rangle$ where $\Delta\subseteq F(B)^2$; it is the quotient structure $\underline{F}(B)/\bar{\Delta}$ where $\bar{\Delta}$ is the least congruence on $\underline{F}(B)$ such that $\Delta\subseteq\bar{\Delta}$ and $\underline{F}(B)/\bar{\Delta}$ is an (n,m)-semigroup. Two presentations $\langle B;\Delta\rangle$ and $\langle B';\Delta'\rangle$ are called equivalent if $F(B)/\bar{\Delta}\cong F(B')/\bar{\Delta}'$, and strongly equivalent if $B=B'$ and $\bar{\Delta}=\bar{\Delta}'$. A presentation $\langle B;\Delta\rangle$

is called a proper presentation if $(a,b)\in B^2\cap\bar{\Delta}$ implies a=b. It is shown that each presentation is equivalent to a proper one. A procedure (in general not sufficiently effective) for determining $\bar{\Delta}$, for given $\langle B;\Delta\rangle$, is described. At the end of §1 several simple examples are presented.

The question about a presentation $\langle B;\Delta\rangle$ which determines an (n,m)-semigroup $\underline{Q}=(Q;f)$ such that $Q\subseteq F(B)$, is investigated in §2. The answer to this question is via a retraction $\psi: F(B) \to F(B)$ (called a reduction for $\langle B;\Delta\rangle$) which satisfies several conditions. Next, we define reductions for the examples from §1 and consider two more examples of presentations together with reductions. One of these examples, the case of $\Delta=\emptyset$ gives a description of free v.v. semigroups.

In §3 we examine a special kind of presentations called vector (n,m)-presentations. It is shown that each (n,m)-presentation is equivalent to a vector one, but there are (n,m)-presentations which are not strongly equivalent with vector ones.

The notions of (n,m)-identities and vector - (n,m)-identidies in the class of poly-(n,m)-groupoids (and so in the class of (n,m)-semigroups as well) are introduced in §4. If θ is a set of (n,m)-identities, then by Varθ we denote the variety of (n,m)-semigroups which satisfy each identity from θ. Next we give the notion of a presentation $\langle B;\Delta;\theta\rangle$ in Varθ and prove some general results. In addition, it is shown that there are varieties of (n,m)-semigroups which could not be determined by vector (n,m)-identities.

In §5, §6 and §7 we consider only vector (n,m)-presentations. We use the fact that each vector (n,m)-presentation $\langle B;\Delta;\theta\rangle$ induces a corresponding presentation $\langle\overline{B;\Delta;\theta}\rangle$ of a semigroup, and investigate the question for a description of $\langle B;\Delta;\theta\rangle$ via a sequence of semigroups. This point of view is highly successful under the condition $(\bar{m})$ (see §5), since in this very case there is a general method for producing a reduction (in most cases - an effective reduction). As a consequence, the proofs of several results are made possible, such as the combinatorial descriptions of the free objects in several varieties of (n,m)-semigroups, and some v.v. variants of Post and Cohn-Rebane Theorems.

Section 8 ends with a commentary on the combinatorial theory of v.v. groups. It is worth mentioning that until now a good description of free v.v. groups is not known, and that in [6] (this volume) a satisfactory description of free (m+1,m)-groups is given.

§1. PRESENTATIONS OF VECTOR VALUED SEMIGROUPS

Let B be a nonempty set, and $\underline{F}(B)=F(B)^{(n,m)}$ be the free poly-(n,m)-groupoid with a basis B (see §0). If Δ is a subset of $F(B)\times F(B)$, i.e. a binary relation of F(B), let $\overline{\Delta}$ be the least congruence on $\underline{F}(B)$ such that $\Delta\subseteq\overline{\Delta}$ and $\underline{F}(B)/\overline{\Delta}$ is an (n,m)-semigroup. Then, we say that Δ is a set of (n,m)-defining (or: defining) relations on B, and $\langle B;\Delta\rangle$ is an (n,m)-presentation (or a presentation) of $\underline{F}(B)/\overline{\Delta}$. The notation $\langle B;\Delta\rangle$ will have the following three conotations: (i) an ordered pair of a set B and a set Δ of (n,m)-defining relations on B; (ii) an (n,m)-semigroup $\underline{F}(B)/\overline{\Delta}$; (iii) the carrier $F(B)/\overline{\Delta}$ of the (n,m)-semigroup $\underline{F}(B)/\overline{\Delta}$.

Let us give a more explicit description of the congruence $\overline{\Delta}$.

First we define a relation $\overset{o}{\vdash}$:

$u \overset{o}{\vdash} v$ iff $(u,v)\in\Delta$ or $u=(i,x'(1,y)\dots(m,y)x'')$, $v=(i,x'yx'')$ where $i\in N_m$, $y\in F(B)^{(n,m)}$, $x'x''\in F(B)^{sk}$ for some $s\geq 1$.

Assume that $\overset{\nu}{\vdash}$ is well defined, and define $\overset{\nu+1}{\vdash}$ as follows

$u,v\in F(B) \Longrightarrow (u \overset{\nu+1}{\vdash} v \Longleftrightarrow u=(i,xu'y),\ v=(i,xv'y),\ u' \overset{\nu}{\vdash} v')$.

Define a relation $\vdash$ on F(B) by:

$$u\vdash v \text{ iff } (\exists\lambda\geq 0)\ u \overset{\lambda}{\vdash} v.$$

Finally, let $\sim$ be the symmetric extension of $\vdash$ and $\approx$ be the reflexive and transitive extension of $\sim$. That is, $u\sim v$ iff $u\vdash v$ or $v\vdash u$, and $u\approx v$ iff there exist $t\geq 0$, $u_0,u_1,\dots,u_t\in F(B)$ such that $u=u_0$, $v=u_t$ and $u_{\lambda-1}\sim u_\lambda$ for any $\lambda\in N_t$.

Proposition 1.1. *If $u,v\in F(B)$, then: $(u,v)\in\overline{\Delta}$ iff $u \approx v$.* □

The (n,m)-semigroup $\underline{F}(B)/\overline{\Delta}=\langle B;\Delta\rangle$ can be more abstractly characterized by the notion of realizations of the pair (B,Δ) in (n,m)-semigroups.

Assume that $\underline{Q}=(Q;f)$ is an (n,m)-semigroup, and $\xi:b\mapsto\overline{b}$ a mapping from B into Q. By P.0.2, there is a unique homomorphism $\overline{\xi}:\underline{F}(B)\to Q$ which is an extension of ξ. We say that ξ is a realization of (B,Δ) in $\underline{Q}$ iff $\overline{\xi}(u)=\overline{\xi}(v)$ for every pair $(u,v)\in\Delta$. Moreover, if ξ is such that for every realization $\xi':B\to Q'$ of (B,Δ) in an (n,m)-semigroup $\underline{Q}'=(Q';f')$ there exists a unique homomorphism $\zeta:\underline{Q}\to Q'$ satisfying the equality $\xi'=\zeta\xi$, then we say that ξ is a universal realization of (B,Δ).

It is clear that the following two statements hold.

Proposition 1.2. *If ξ,η are universal realizations of (B,Δ) in $\underline{Q},\underline{P}$ - respectively, then there exists a unique isomorphism $\zeta:\underline{Q}\to\underline{P}$ such that $\eta=\zeta\xi$.* □

Proposition 1.3. *The natural mapping $nat\Delta:b\mapsto b^{\Delta}$ is a universal realization of (B,Δ) in $\underline{F}(B)/\approx$. (Here, if $u\in F(B)$, we denote by u^{Δ} the $\approx$ - equivalence class containing u, i.e. $u^{\Delta}=\{v\in F(B)\mid u\approx v\}$. Also, instead of $\overline{\Delta}$ we write $\approx$.)* □

Because of the last two properties, from now on, we will denote by $\langle B;\Delta\rangle$ any (n,m)-semigroup $\underline{Q}$ such that there exists a universal realization of (B,Δ) in $\underline{Q}$.

We say that a presentation $\langle B;\Delta\rangle$ is proper iff

$$(\forall a,b\in B)(a\approx b \Longrightarrow a=b).$$

In this case we may assume that B is a subset of $\langle B;\Delta\rangle$.

Proposition 1.4. *A presentation $\langle B;\Delta\rangle$ is proper iff there exists an injective realization of (B,Δ) in an (n,m)-semigroup.* □

Proposition 1.5. *If $\Delta\subseteq F(B)\times F(B)$ is such that $|u|\cdot|v|\geq 1$, for every pair $(u,v)\in\Delta$, then the presentation $\langle B;\Delta\rangle$ is proper.* □

Consider some trivial examples.

Example 1.6. If Δ is such that $\overline{\Delta}=F(B)\times F(B)$, then $<B;\Delta>$ is a one element (n,m)-semigroup.

Example 1.7. If $\Delta=\{(u,v)\in F(B)\times F(B) \mid |u|\cdot|v| \geq 1\}$, then $<B;\Delta>=B\cup\{o\}$, where $o\notin B$, and $f(c_1^n)=o^m$, for any $c_\nu\in B\cup\{o\}$.

If Δ_1 is a subset of Δ such that:

$$(u,v)\in\Delta_1 \iff (\exists i\in N_m)u=(i,x),\ v=(i,y),\ x,y\in F(B)^{(n,m)},$$

then:

$$<B;\Delta_1> = B\cup\{o_1,o_2,\ldots,o_m\}$$

is a constant (n,m)-semigroup, such that $f(c_1^n)=o_1^m$ for any $c_\nu\in B\cup\{o_1,\ldots,o_m\}$, where $B\cap\{o_1,\ldots,o_m\}=\emptyset$.

Example 1.8. If $\Delta=\{((i,b_1^n),b_i) \mid b_\nu\in B,\ i\in N_m\}$, then $<B;\Delta>=B$ is the left zero (n,m)-semigroup on B, i.e. $f(b_1^n)=b_1^m$ for any $b_\nu\in B$.

Example 1.9. Let $\underline{Q}=(Q;f)$ be an (n,m)-semigroup and $\Gamma(\underline{Q})\subseteq F(Q)\times F(Q)$ be defined as follows:

$$\Gamma(\underline{Q}) = \{((i,a_1^n),b_i) \mid f(a_1^n)=b_1^m \text{ in } \underline{Q},\ i\in N_m\}.$$

Then, the identity transformation $1:a\longmapsto a$ is a universal realization of $(Q,\Gamma(\underline{Q}))$ in $\underline{Q}$. Moreover, if $\underline{P}=(P;g)$ is an (n,m)-semigroup and $\xi:a\longmapsto\xi(a)$ is a mapping from Q in P, then ξ is a realization of $(Q,\Gamma(\underline{Q}))$ in $\underline{P}$ iff $\xi:\underline{Q}\to\underline{P}$ is a homomorphism. Thus $\underline{Q}=<Q;\Gamma(\underline{Q})>$. We say that $\Gamma(\underline{Q})$ is the graph of $\underline{Q}$ and that $<Q;\Gamma(\underline{Q})>$ is the graphical presentation of $\underline{Q}$.

Note that E.1.8 is a special case of E.1.9.

Proposition 1.10. *The presentations in E.1.7, E.1.8 and E.1.9 are proper, and if $|B|\geq 2$ then the presentation in E.1.6 is not proper.* □

We say that two presentations $<B;\Delta>$ and $<B';\Delta'>$ are equivalent iff the corresponding (n,m)-semigroups are isomorphic, i.e. if $\underline{F}(B)/\overline{\Delta}$ is isomorphic to $\underline{F}(B')/\overline{\Delta}'$. Then we write $<B;\Delta>\cong<B';\Delta'>$.

Proposition 1.11. *Every (n,m)-presentation is equivalent to a proper (n,m)-presentation.*

Proof. If $<B;\Delta>=\underline{Q}$, then $<B;\Delta>\cong<Q;\Gamma(\underline{Q})>$. □

Two presentations $\langle B;\Delta\rangle$ and $\langle B';\Delta'\rangle$ are called strongly equivalent iff $\overline{\Delta}=\overline{\Delta}'$. Thus:

Proposition 1.12. *The presentations* $\langle B;\Delta\rangle$ *and* $\langle B;\overline{\Delta}\rangle$ *are strongly equivalent.* □

Clearly, if two presentations are strongly equivalent then they are equivalent as well.

At the end of this section we note the following. If $\langle B;\Delta\rangle$ is not a proper (n,m)-presentation, then there exists a proper (n,m)-presentation $\langle B';\Delta'\rangle$ obtained as follows. Choose a unique element b' from $b^{\Delta}\cap B$, for each $b\in B$, and put $B'=\{b' \mid b\in B\}$. Construct $\Delta'\subseteq F(B')\times F(B')$ by replacing each appearence of b in $(u,v)\in\Delta$ with the unique corresponding element $b'\in B'$. Then $\langle B;\Delta\rangle$ is equivalent to $\langle B';\Delta'\rangle$, and $\langle B',\Delta'\rangle$ is a proper (n,m)-presentation.

§2. REDUCTIONS

The (n,m)-semigroup $\underline{F}(B)/\approx\ = \langle B;\Delta\rangle$ is a "quotient structure" and, it is usually desirable to find an (n,m)-semigroup isomorphic to $\underline{F}(B)/\approx$ whose carrier is a subset of $F(B)$. This can be achieved by a choice of one and only one element $\psi(u)$ from each $\approx$-equivalence class $u^{\Delta}=\{v \mid u\approx v\}$. Or, equivalently, by a mapping $\psi:F(B) \to F(B)$ with the following properties:

(i) $(u,v)\in\Delta \Longrightarrow \psi(u)=\psi(v)$;

(ii) $\psi(i,x'(1,y)(2,y)\dots(m,y)x") = \psi(i,x'yx")$;

(iii) $\psi(i,x'wx") = \psi(i,x'\psi(w)x")$;

(iv) $u \approx \psi(u)$;

(v) $\psi^2 = \psi$;

for every u,v,w, $(i,x'wx"),(i,x'(1,y)\dots(m,y)x")\in F(B)$.

A mapping $\psi:F(B) \to F(B)$ is said to be a reduction for $\langle B;\Delta\rangle$ if (i) to (v) are satisfied. We say that $u\in F(B)$ is reduced if $\psi(u)=u$, and reducible otherwise.

Proposition 2.1. *Let ψ be a reduction for $<B;\Delta>$ and let $Q=\psi(F(B))$. If an (n,m)-operation g is defined on Q by*

$$g(u_1^{m+sk}) = v_1^m \iff (\forall i\in N_m) v_i = \psi(i,u_1^{m+sk}),$$

then $\underline{Q}=(Q;g)$ is an (n,m)-semigroup and the restriction of ψ on B is a universal realization of (B,Δ) in $\underline{Q}$. Therefore, $\underline{Q}=<B;\Delta>$.

Proof. First, $Q=\psi(F(B))$ implies that g is a well defined (n,m)-operation on Q, and we may consider ψ as a surjective homomorphism from $\underline{F}(B)$ onto $\underline{Q}$ such that $\overline{\Delta}\subseteq\ker\psi$. If $(u,v)\in\ker\psi$, i.e. $\psi(u)=\psi(v)$, then $u \approx \psi(u)=\psi(v) \approx v$, whence $(u,v)\in\overline{\Delta}$. Thus, $\ker\psi=\overline{\Delta}$, and therefore $\underline{Q}$ and $\underline{F}(B)/\approx$ are isomorphic. □

Note that the condition (v) is a consequence of (i) - (iv).

A reduction ψ for $<B;\Delta>$ is called a proper one iff $\psi(b)=b$ for every $b\in B$.

Proposition 2.2. *A presentation $<B;\Delta>$ is proper iff it admits a proper reduction.* □

In general, there exist many reductions for a presentation $<B;\Delta>$. Usually, we look for a reduction which satisfies some conditions of "effectivness".

Let us consider E.1.6 to E.1.9.

First, if u_o is an arbitrary element of $F(B)$, and if we put $\psi(u)=u_o$ for all $u\in F(B)$, then we obtain a reduction for $<B;\Delta>$, where $\overline{\Delta}=F(B)\times F(B)$.

Let $<B;\Delta>$ and $<B;\Delta_1>$ be as in E.1.7, and let $u_o\in F(B)\setminus B$, $x_i\in F(B)^{(n,m)}$ for every $i\in N_m$ be fixed. Define two mappings $\psi,\psi_1:F(B) \to F(B)$ as follows:

$\psi(b) = \psi_1(b) = b$ for every $b\in B$,

$\psi(i,x) = u_o$, $\psi_1(i,x)=(i,x_i)$, for every $(i,x)\in F(B)\setminus B$.

Then, ψ is a reduction for $<B;\Delta>$, and ψ_1 is a reduction for $<B;\Delta_1>$.

If $<B;\Delta>$ is as in E.1.8, then we can define a reduction ψ by induction on the norm in the following manner: $\psi(b)=b$ for every $b\in B$, and $\psi(u)=\psi(u_i)$, for every $u=(i,u_1^{m+sk})\in F(B)\setminus B$. More generally, if $\underline{Q}=(Q;f)$ is an (n,m)-semigroup then a reduction for the graphical presentation $\langle Q;\Gamma(\underline{Q})\rangle$ of $\underline{Q}$ (E.1.9) can be defined as follows. First, $\psi(b)=b$, for every $b\in Q$. Assume that $u=(i,u_1^{m+sk})\in F(Q)\setminus Q$, and that $\psi(v)\in Q$ is well defined for every $v\in F(B)$ such that $|v|<|u|$. Then $\psi(u)$ is defined by:

$$\psi(u) = f_i(a_1^{m+sk}),$$

where $a_\nu=\psi(u_\nu)$.

Note that we do not have any particular use of the corresponding reductions in the above examples, as we know wery well their structure. However, in the next two examples the corresponding reductions are of substantial use.

Example 2.3. Let $B\neq\emptyset$, and $\Delta=\emptyset$. In this case $\langle B;\emptyset\rangle$ is the free (n,m)-semigroup with a basis B.

A reduction $\psi:F(B)\to F(B)$ will be defined by induction on the norm as follows.

(0) $(\forall b\in B)\psi(b) = b$.

Assume that $u=(i,x)\in F(B)\setminus B$ and that $\psi(v)\in F(B)$ is well defined for every $v\in F(B)$ such that $|v|<|u|$. Moreover, assume that the following condition is satisfied:

$$\psi(v) \neq v \Longrightarrow |\psi(v)| < |v|. \qquad (2.1)$$

If $x=u_1^{m+sk}$, $u_\nu\in F(B)$, then $v_\lambda=\psi(u_\lambda)$ is well-defined, and thus $v=(i,v_1^{m+sk})\in F(B)$. If there exists a λ such that $v_\lambda\neq u_\lambda$, then $|v|<|u|$. Consequently, we can define $\psi(u)$ by:

(1) $\psi(u) = \psi(v)$.

If $\psi(u_\lambda)=u_\lambda$ for every λ and if $x=x'(1,y)(2,y)\dots(m,y)x''$, where $x',x''\in F(B)$, $(\nu,y)\in F(B)$, and x' has the least possible dimension then we define $\psi(u)$ by:

(2) $\psi(u) = \psi(i,x'yx'')$.

And if $\psi(u)$ could not be defined by (1) or (2) then we put

(3) $\psi(u) = u$.

Clearly, if $\psi(u)$ is defined by (1) or (2) then we have $|\psi(u)| < |(u)|$ and this implies that $\psi: F(B) \to F(B)$ is a well-defined mapping. Moreover, (2.1) holds for every $v \in F(B)$.

By induction on the norm it can be checked that (i)-(v) are satisfied, i.e., that ψ is a reduction for $\langle B;\emptyset\rangle$. (see [5], [2]).

We also note that we have a good description of $S(B)=\psi(F(B))$. Namely, $u \in S(B)$, i.e. u is reduced, iff $u \in B$ or $u=(i,u_1^{m+sk})$ where $u_\nu \in S(B)$ for every ν and there is no $j \in N_{sk}$ such that $u_{j+\lambda}=(\lambda,y)$, for every $\lambda \in N_m$. Moreover, if u is a given element of F(B) then $\psi(u)$ is determined in a finite number of steps.

Further on we will always denote the above reduction ψ by ψ_o. Thus, $\underline{S}(B)=(S(B);f)$ is a free (n,m)-semigroup with a basis B, where f is defined by:

$$f(u_1^{m+sk}) = v_1^m \iff (\forall i \in N_m)\, v_i = \psi_o(i,u_1^{m+sk}).$$

In the case m=1 we have the following well known result.

Proposition 2.4. *The (n,1)-semigroup $\underline{S}(B)=(S(B);f)$ where*

$$S(B) = \{u \in B^+ \mid d(u) = 1 + sk,\ s \geq 1\}, \text{ and}$$

$$f(u_1^n) = u_1 u_2 \ldots u_n,$$

is a free (n,1)-semigroup, i.e. a free n-semigroup with a basis B. □

We say that a presentation $\langle B;\Delta\rangle$ is reduced iff $\Delta \subseteq S(B)\times S(B)$.

Proposition 2.5. *Let $\Delta \subseteq F(B)\times F(B)$ and*

$$\Delta_o = \{(\psi_o(u),\psi_o(v)) \mid (u,v)\in\Delta\}.$$

Then $\langle B;\Delta\rangle$ and $\langle B;\Delta_o\rangle$ are strongly equivalent and $\langle B;\Delta_o\rangle$ is a reduced presentation. □

From now on we will usually deal with reduced (n,m)-presentations.

Example 2.6. Let B be a nonempty set, $m \geq 3$ and let Δ be defined by:

$$\Delta = \{(u,v) \mid u=(1,x),\ v=(2,x)\in F(B)\}.$$

We will define a reduction ψ for <B;Δ> in the same way as in E.2.3. Namely, assume that (0), (1), (2) and (3) are as in E.2.3, and:

(1-) $\psi(2,x) = \psi(1,x)$.

In (2) it is assumed that $x=x'(1,y)(2,y)(3,y)\dots(m,y)x''$. The proof that (1-), (0), (1), (2) and (3) define a reduction for <B;Δ> is by an induction on a norm defined by (0.5), (0.6) and (0.7′) (see §0).

Note that in E.2.6 it is possible to take m=2, but then for the definition of a reduction one more step is needed, that is

$$\psi(i,x'(1,y)yx'') = \psi(i,x'y(1,y)x''),$$

and in the proof that ψ is indeed a reduction we need a norm defined by (0.5′), (0.6′) and (0.7′) (see §0).

Another remark about E.2.6 is that it is possible to take

$$\Delta = \{(u,v) \mid u = (i,x),\ v = (j,x),\ u,v\in F(B)\},$$

for $1 \le i < j \le m$. The cases $1 \le i < j < m$ and $1 < i < j \le m$ are the same as the case $i=1$, $j=2$, $m \ge 3$, and the case $i=1$, $j=m$, $m \ge 2$ is the same as the case $i=1$, $j=m=2$.

§3. VECTOR (n,m)-PRESENTATIONS

In the next part of the paper we will usually deal with a special kind of (n,m)-relations which will be called "vector (n,m)-relations".

Assume that B is a nonempty set and Λ a subset of $B^+\times B^+$ such that for every $(a_1^p,b_1^q)\in\Lambda$ we have $m \le q \le p$, and $q \equiv p \equiv m \pmod k$. Then we say that Λ is a set of vector (n,m)-relations on B. (Note that the assumption $q \le p$ is not essential.)

We can associate a set $\Lambda_{\#}$ of (n,m)-relations to a set Λ of vector (n,m)-relations in the following way.

Firstly, the preceeding notation is modified. Namely, below (i,u_1^m) will be another sign for u_i. Thus, $u\in F(B)$ iff $u=(i,u_1^p)$, where $i\in N_m$, $u_\nu\in F(B)$, $m \le p$, $p \equiv m \pmod k$.

Now, if $\Lambda \subseteq B^{+} \times B^{+}$ is a set of vector (n,m)-relations, then $\Lambda_{\#}$ is defined by:

$$\Lambda_{\#} = \{(u,v) \mid u=(i,a_1^p),\ v=(i,b_1^q),\ (a_1^p,b_1^q)\in\Lambda,\ i\in N_m\}.$$

If $\underline{Q}=(Q;f)$ is an (n,m)-semigroup and $\xi: B \to Q$ is a mapping, then we say that ξ is a (universal) realization of (B,Λ) in $\underline{Q}$ iff ξ is a (universal) realization of $(B,\Lambda_{\#})$ in $\underline{Q}$. Then $\langle B;\Lambda_{\#}\rangle$ is also denoted by $\langle B;\Lambda\rangle$. We say that $\langle B;\Lambda\rangle$ is a vector (n,m)-presentation or simply a vector presentation.

Proposition 3.1. *Let Λ be a set of vector (n,m)-relations on B, and $\underline{Q}=(Q;f)$ be an (n,m)-semigroup. Then:*

(i) A mapping $\xi: B \to Q$ is a realization of (B,Λ) in $\underline{Q}$ iff $f(\overline{a}_1^p)=f(\overline{b}_1^q)$ for every $(a_1^p,b_1^q)\in\Lambda$ where $\overline{c}=\xi(c)$.

(ii) $\langle B;\Lambda\rangle=\underline{Q}$ iff there is a universal realization ζ of (B,Λ) in $\underline{Q}$. □

Proposition 3.2. *Let $\Delta \subseteq F(B)\times F(B)$ be a set of (n,m)-relations on B with the following properties:*

If $(u,v)\in\Delta$ and $u\notin B$ (or $v\notin B$) then $u=(i,a_1^p)$ $(v=(i,b_1^q))$ where $i\in N_m$, $a_\nu,b_\lambda\in B$, $m\le q<p$, $q\equiv p\equiv m \pmod k$, and for every $j\in N_m$, $(u',v')\in\Delta$, where $u'=(j,a_1^p)$, $v'=(j,b_1^q)$.

Define a subset $\Delta^{\#}$ of $B^{+}\times B^{+}$ by:

$$\Delta = \{(\overset{m}{a},\overset{m}{b}) \mid (a,b)\in\Delta,\ a,b\in B\} \cup$$

$$\cup \{(a_1^p,b_1^q) \mid (\forall i\in N_m)((i,a_1^p),(i,b_1^q))\in\Delta,\ m\le q\le p,\ p>m\}.$$

Then $\Delta^{\#}$ is a set of vector (n,m)-relations such that $(\Delta^{\#})_{\#}=\Delta$. □

If $\Delta\subseteq F(B)\times F(B)$ is such that $\Delta=\Lambda_{\#}$ for a set Λ of vector (n,m)-relations on B, then we also say that $\langle B;\Delta\rangle$ is a vector (n,m)-presentation. Moreover, no distinction is being made between the two (n,m)-presentations, $\langle B;\Delta\rangle$, $\langle B;\Lambda\rangle$.

Proposition 3.3. *Every (n,m)-presentation is equivalent to a vector (n,m)-presentation.*

Proof. If $\langle B;\Delta\rangle=\underline{Q}=(Q;f)$, then $\langle B;\Delta\rangle$ and $\langle Q;\Gamma(Q)\rangle$ are equivalent, and, moreover, $\Gamma(\underline{Q})=\Lambda_{\#}$, where:

$$\Lambda = \{(a_1^n, b_1^m) \mid f(a_1^n) = b_1^m \text{ in } \underset{\sim}{Q}\}. \quad \square$$

It is natural to ask for an (n,m)-presentation which is not strongly equivalent to a vector (n,m)-presentation.

Proposition 3.4. *The presentation <B;Δ> given in E.2.6 is not strongly equivalent to a vector (n,m)-presentation.*

Proof. It is sufficient to show that there does not exist a vector (n,m)-presentation Λ on B such that $\overline{\Delta}=\overline{\Lambda}_{\not\#}$. Namely, if $(a_1^p, b_1^q) \in B^+ \times B^+$ is a vector (n,m)-relation on B such that $a_1^p \neq b_1^q$, then it can be easily seen that

$$(\exists i \in N_m)\ ((i, a_1^p), (i, b_1^q)) \notin \overline{\Delta}. \quad \square$$

As we have noticed in §2, if <B;Δ> is an (n,m)-presentation and if $(u,v) \in \Delta$, we can assume that u and v are reduced. And, if m=1 then $u \in F(B)$ is reduced iff $u \in B$ or $u=(1, a_1^{m+sk})$ where $s \geq 1$, $a_\nu \in B$. The last assertion implies the following:

Propisition 3.5. *Every (n,1)-presentation is strongly equivalent to a vector (n,1)-presentation.* □

Further on we will always assume that n,m and k are given positive integers such that n-m=k, $m \geq 2$. We will also assume that B is a nonempty set and Λ a set of vector (n,m)-relations on B. Then we can also consider Λ as a set of vector (2,1)-relations on B. This is the reason behind the use of different notations. Namely, we denote by <B;Λ> the corresponding (n,m)-presentation, and by $\overline{\langle B;\Lambda\rangle}$ the same presentation but now considered as a (2,1)-presentation. Thus, <B;Λ> is an (n,m)-semigroup, and $\overline{\langle B;\Lambda\rangle}$ is a semigroup.

Proposition 3.6. *Let <B;Λ> be a vector (n,m)-presentation, where $m \geq 2$, and let $\vdash$, $\sim$, $\overset{\Delta}{=}$ be relations in B^+ defined as follows.*

$u \vdash v$ iff $u=u_1u'u_2$, $v=u_1v'u_2$, $(u',v') \in \Lambda$, $u_\nu \in B^*$;

$u \sim v$ iff $u \vdash v$ or $v \vdash u$;

$u \overset{\Delta}{=} v$ iff there exist $t \geq 0$, $u_0, \ldots, u_t \in B^+$ such that $u=u_0$, $v=u_t$, $u_{\lambda-1} \sim u_\lambda$ for any $\lambda \in N_t$.

Then $\overset{\Lambda}{=}$ *is a congruence on* B^+ *such that* $B^+/\overset{\Lambda}{=} = \langle\overline{B;\Lambda}\rangle$.

Moreover:

(i) $u \in B^+,\ d(u) < m \Longrightarrow (u \overset{\Lambda}{=} v \Longleftrightarrow u = v)$

(ii) $u \overset{\Lambda}{=} v \Longrightarrow d(u) \equiv d(v) \pmod k$. □

It follows from (i) that we can assume that $B \cup B^2 \cup \dots \cup B^{m-1}$ is a subset of $B^+/\overset{\Lambda}{=} = \langle\overline{B;\Lambda}\rangle$. In general, two different elements u,v of B^m can define the same element in $\langle\overline{B;\Lambda}\rangle$, i.e.

$$u \overset{\Lambda}{=} v,\ u \neq v.$$

Let $\langle B;\Lambda\rangle$, n,m,k and $\overset{\Lambda}{=}$ be as above. If $u \in B^+$, then the set $\{d(v) \mid u \overset{\Lambda}{=} v\}$ is denoted by $\overline{d}(u)$. Clearly,

$$u \overset{\Lambda}{=} v \Longrightarrow \overline{d}(u) = \overline{d}(v)$$

and therefore if we put $\overline{d}(u^{\Lambda}) = \overline{d}(u)$ we get that for every $x \in \langle\overline{B;\Lambda}\rangle$ $\overline{d}(x)$ is a well defined set of positive integers. (Here $u^{\Lambda} = \{v \mid u \overset{\Lambda}{=} v\}$.)

Proposition 3.7. *If* $x \in \langle\overline{B;\Lambda}\rangle$ *and if* $\alpha \geq m$ *for some* $\alpha \in \overline{d}(x)$ *then* $\beta \geq m$ *for every* $\beta \in \overline{d}(x)$ *and moreover:* $\beta,\gamma \in \overline{d}(x) \Longrightarrow \beta \equiv \gamma \pmod k$. □

Proposition 3.8. *Let* $\langle B;\Lambda\rangle$ *be a vector (n,m)-presentation and* $\xi: c \longmapsto \overline{c}$ *be a realization of* (B,Λ) *in an (n,m)-semigroup* $\underline{P} = (P;g)$. *If* $a_\nu, b_\lambda \in B$, $r,s \geq 0$ *are such that* $a_1^{m+rk} \overset{\Lambda}{=} b_1^{m+sk}$, *then*

$$g(\overline{a}_1^{m+rk}) = g(\overline{b}_1^{m+sk}).$$

Proof. Let $u = a_1^{m+rk}$, $v = b_1^{m+sk}$, $\overline{u} = \overline{a}_1^{m+rk}$, $\overline{v} = \overline{b}_1^{m+sk}$.

If $u \sim v$ then $u = v_1u'v_2$, $v = v_1v'v_2$, where $v_1,v_2 \in B^*$ and $(u',v') \in \Lambda$ or $(v',u') \in \Lambda$; thus $g(\overline{u}') = g(\overline{v}')$. This implies that

$$\begin{aligned} g(\overline{u}) &= g(\overline{v}_1\overline{u}'\overline{v}_2) = g(\overline{v}_1 g(\overline{u}')\overline{v}_2) \\ &= g(\overline{v}_1 g(\overline{v}')\overline{v}_2) = g(\overline{v}_1\overline{v}'\overline{v}_2) \\ &= g(\overline{v}). \end{aligned}$$

If $u_0, u_1, \dots, u_t \in B^+$ are such that $t \geq 2$ and

$$u = u_0 \sim u_1 \sim \dots \sim u_t = v$$

then

$$g(\overline{u}) = g(\overline{u}_0) = g(\overline{u}_1) = \dots = g(\overline{u}_t) = g(\overline{v}).\quad □$$

§4. PRESENTATIONS IN VARIETIES OF VECTOR VALUED SEMIGROUPS

A class of (n,m)-semigroups is said to be a variety of (n,m)-semigroups if it has an axiom system which is a set of identities.

First we give a more precise definition of an identity in the class of poly-(n,m)-groupoids.

Denote by N the set of nonnegative integers and consider the free poly-(n,m)-groupoid F(N) with a basis N. The elements of F(N) will be denoted by $\rho, \omega, \tau, \ldots$.

Let $\underline{Q}=(Q;f)$ be a poly-(n,m)-groupoid. Every element

$$\rho \in F(N_t) \subset F(N)$$

define a t-ary operation $\rho^{\underline{Q}}$ on Q as follows.

(i) If $\rho = j \in N_t$ then $\rho^{\underline{Q}}(a_1^t) = a_j$;

(ii) If $\rho = (i, \rho_1^{m+sk})$ and $\rho_\nu^{\underline{Q}}(a_1^t) = b_\nu$

then

$$\rho^{\underline{Q}}(a_1^t) = f_i(b_1^{m+sk}).$$

We note that if $t < q$ and $\rho \in F(N_t)$ then $\rho \in F(N_q)$ and thus ρ defines a q-ary operation $\rho'^{\underline{Q}}$ on Q as well. Clearly, we have

$$\rho'^{\underline{Q}}(a_1^q) = \rho^{\underline{Q}}(a_1^t)$$

for any $a_\nu \in Q$. Further on, we ommit the upperscript, i.e. we write $\rho(a_1^t)$ instead of $\rho^{\underline{Q}}(a_1^t)$.

Let $\rho, \omega \in F(N_t)$. We say that a poly-(n,m)-groupoid $\underline{Q}$ satisfies the (n,m)-identity (ρ, ω), and write

$$\underline{Q} \models (\rho, \omega),$$

iff

$$\rho(a_1^t) = \omega(a_1^t) \text{ for any } a_1^t \in Q^t.$$

The reduction $\psi_o : F(N) \to F(N)$ is defined as in the general case. By induction on the norm it can be easily shown that:

Proposition 4.1. *If $\underline{Q}$ is an (n,m)-semigroup and $\rho \in F(N)$ then $\underline{Q} \models (\rho, \psi_o(\rho))$.* □

We say that an (n,m)-identity (ρ,ω) is reduced if $\psi_o(\omega)=\omega$ and $\psi_o(\rho)=\rho$. Since we are interested only in poly-(n,m)-groupoids which are (n,m)-semigroups, from now on we consider only reduced (n,m)-identities.

Proposition 4.2. *If (ρ,ω) is a reduced (n,m)-identity and $\underline{Q} \models (\rho,\omega)$ for every (n,m)-semigroup $\underline{Q}$, then $\rho=\omega$.*

Proof. If B is a nonempy set and $\rho\neq\omega$ then the free (n,m)-semigroup S(B) with a basis B does not satisfy the (n,m)-identity (ρ,ω). □

(We note that the conclusion of the proof holds in the case m=1 only if we assume that $|B|>2$. Namely the free (n,1)-semigroup with a free generator is commutative.)

If $\theta\subseteq F(N\setminus\{0\})\times F(N\setminus\{0\})$ then we say that θ is a set of (n,m)-identities. By $V=\mathrm{Var}\theta$ we denote the class of (n,m)-semigroups $\underline{Q}$ such that $\underline{Q}\models\theta$, where: $\underline{Q}\models\theta$ iff $\underline{Q}\models(\rho,\omega)$ for every $(\rho,\omega)\in\theta$. We say that $\mathrm{Var}\theta$ is the variety of (n,m)-semigroups generated by θ.

Let B be a nonempty set and let $\rho\in F(N_t)$, $u_1^t\in F(B)^t$. Then $\rho(u_1^t)\in F(B)$, because ρ induces a t-ary operation on F(B). If θ is a set of (n,m)-identities then we put:

$$\theta(F(B)) = \{(\rho(u_1^t),\ \omega(u_1^t)) \mid (\rho,\omega)\in\theta,\ \rho,\omega\in F(N_t),\ u_1^t\in F(B)^t,\ t\in N\}.$$

Proposition 4.3. *$\langle B;\theta(F(B))\rangle$ is a free object in $\mathrm{Var}\theta$ with a basis B.*

Proof. If $\Delta=\theta(F(B))$ then $\overline{\Delta}$ is the least congruence on $\underline{F}(B)$ such that $\underline{F}(B)/_{\overline{\Delta}}\in\mathrm{Var}\theta$. □

If Δ is a set of (n,m)-relations on B and θ is a set of (n,m)-identities then the (n,m)-semigroup $\langle B;\Delta\cup\theta(F(B))\rangle$ will be denoted by $\langle B;\Delta;\theta\rangle$. This (n,m)-semigroup can be characterized in the following way:

Proposition 4.4. *$\underline{Q}=(Q;f)=\langle B;\Delta;\theta\rangle$ iff the following conditions are satisfied:*

(i) $\underline{Q}\in\mathrm{Var}\theta$;

(ii) There is a realization ξ of (B,Δ) in $\underline{Q}$ such that for

any realization ξ' of (B,Δ) in an (n,m)-semigroup $\underline{Q}'\in Var\theta$ there is a unique homomorphism $\zeta:\underline{Q} \to \underline{Q}'$ such that $\zeta\xi=\xi'$. □

We say that $<B;\Delta;\theta>$ is an (n,m)-<u>presentation in</u> $Var\theta$.

Every set θ of (n,m)-identities induces an (n,m)-presentation $<N;\theta>$, and we say that θ is a set of <u>vector</u> (n,m)-<u>identities</u> iff $<N;\theta>$ is a vector (n,m)-presentation.

Vector (n,m)-identities can be defined directly, as well. Namely, let $p=m+sk$, $q=m+rk$, where $r,s \geq 0$, and let $(i_1^p, j_1^q)\in N^+\times N^+$. We say that an (n,m)-semigroup $\underline{Q}=(Q;f)$ satisfies the vector (n,m)-identity (i_1^p,j_1^q), and write $\underline{Q} \models (i_1^p,j_1^q)$, iff for every $a_1^t\in Q^t$ we have $f(b_1^p)=f(c_1^q)$, where $t=\max_{\nu,\lambda}\{i_\nu, j_\lambda\}$ and $b_\nu=a_{i_\nu}$, $c_\lambda=a_{j_\lambda}$, for any ν,λ. From now on we assume that $p\geq q$.

<u>Proposition 4.5</u>. *If $i_1^m \neq j_1^m$ and if $\underline{Q}\models(i_1^m,j_1^m)$ then $|Q| = 1$.* □

We say that a variety $\mathbb{V}$ of (n,m)-semigroups is a <u>vector variety</u> iff there exists a set of vector (n,m)-idnetities θ such that $\mathbb{V}=Var\theta$. Also, $<B;\Lambda;\theta>$ is a vector (n,m)-presentations iff $<B;\Lambda>$ and $<N;\theta>$ are vector (n,m)-presentations.

Let us consider some examples.

<u>Example 4.6</u>. If there exists an $(i_1^m,j_1^m)\in\theta$ such that $i_1^m\neq j_1^m$ then $Var\theta= \mathbb{O}(n,m)=\mathbb{O}$ is the least variety of (n,m)-semigroups, and $\underline{Q}=(Q;f)\in\mathbb{O}$ iff $|Q|=1$. Therefore for every B and Δ, $<B;\Delta;\theta>$ is a one element (n,m)-semigroup.

From now on we assume that $Var\theta\neq\mathbb{O}$. Therefore, we can also assume that in θ there are no identities of the form (i_1^m,j_1^m).

In the case $\theta=\{(i_1^p,j_1^q)\}$ we write $Var(i_1^p,j_1^q)$ and $<B;\Delta;(i_1^p,j_1^q)>$ instead of $Var\{(i_1^p,j_1^q)\}$ and $<B;\Delta;\{(i_1^p,j_1^q)\}>$, respectively.

<u>Example 4.7</u>. Let $i_\nu=\nu$ for each $\nu\in N_n$. Then $Var(i_1^n,i_1^m)= LZ$ is the variety of left zero (n,m)-semigroups, i.e., $(Q;f)\in LZ$ iff $f(a_1^{m+sk})=a_1^m$ for any $a_\nu\in Q$, $s\geq 0$. (See E.1.8.) Any (n,m)-semigroup $\underline{Q}=(Q;f)\in LZ$ is a free object in LZ with a basis Q. If $<B;\Delta;(i_1^n,i_1^m)>$ is a vector (n,m)-presentation in LZ, then it determines the left zero (n,m)-semigroup on $B/\approx$ where $\approx$ is the least equivalence on B such that

$$(a_1^p, b_1^q) \in \Delta \implies (\forall i \in N_m),\ a_i \approx b_i.$$

More generaly, let $\Delta \subseteq F(B)\times F(B)$ and let us define a mapping $\psi : F(B) \to B$ by induction on norm in the following way:

$\psi(b) = b$ for every $b \in B$

$\psi(i, u_1^{m+sk}) = \psi(u_i)$, for every $u_\nu \in F(B)$, $i \in N_m$, $s \geq 1$.

Then, if $\approx$ is the least equivalence in B such that

$$(u,v) \in \Delta \implies \psi(u) \approx \psi(v),$$

we obtain that $\langle B;\Delta;(i_1^n, i_1^m)\rangle$ is the left zero (n,m)-semigroup on $B/\approx$.

Example 4.8. Let $i_\nu = \nu$, $j_\nu = n+\nu$ for any $\nu \in N_n$. Then, $\mathrm{Var}(i_1^n, j_1^n) = Z(n,m)$ is the variety of constant (n,m)-semigroups (E.1.7.). We recall that an (n,m)-semigroup $\underline{Q} = (Q;f)$ is a constant one iff there exists a $o_1^m \in Q^m$ such that $f(a_1^n) = o_1^m$ for any $a_\nu \in Q$. Then we also have $f(a_1^{m+sk}) = o_1^m$ for any $s \geq 1$, $a_\nu \in Q$. As in the first two examples it is easy to give a description of an (n,m)-semigroup $\langle B;\Delta;(i_1^n, j_1^n)\rangle$, where $\Delta \subseteq F(B)\times F(B)$. Let $\{o_1, o_2, \ldots, o_m\}$ be a set disjoint with B and let $B_o = B \cup \{o_1, \ldots, o_m\}$, $o_i \neq o_j$ if $i \neq j$. Define a subset Δ_o of $B_o \times B_o$ by:

$$\Delta_o = \{(a,b) \mid (a,b) \in \Delta \cap B^2\}$$
$$\cup \{(o_i, a) \mid a \in B \text{ and } (a,u) \in \Delta \text{ or } (u,a) \in \Delta \text{ for some } u = (i,x) \in F(B)\}$$
$$\cup \{(o_i, o_j) \mid (u,v) \in \Delta \text{ for some } u = (i,x),\ v = (j,y) \in F(B)\}.$$

Let $\approx$ be the least equivalence in B_o containing Δ_o. If $c \in B_o$ then we denote by $\bar{c}$ the $\approx$-equivalence class containing c. Then $\langle B;\Delta;(i_1^n, j_1^n)\rangle$ is the constant (n,m)-semigroup $(B_o/\approx;f)$ defined by $f(\bar{c}_1^n) = o_1^m$.

We note that in E.4.6 we do not need any construction of F(B), and the same is true in the other two examples iff Δ is a set of vector (n,m)-relations on B.

Each one of the varieties considered above is an example of vector variety of (n,m)-semigroups, and certainly the class of all (n,m)-semigroups is such a variety. Below we give an example of a variety which is not a vector variety.

Example 4.9. Let $m \geq 2$ and let

$$\theta = \{(\rho,\omega) \mid \rho = (1,x),\ \omega = (2,x) \in F(N)\}.$$

If B is a nonempty set then $\theta(F(B))$ consists of all pairs (u,v) such that: $u=(1,u_1^{m+sk})$, $v=(2,u_1^{m+sk})$, $s \geq 1$, $u_\nu \in F(B)$. Thus, a free object in Varθ with a basis B is the (n,m)-semigroup $\underline{Q}=(Q;f)=$ $=\langle B;\Delta\rangle$ given in E.2.6.

Next, we show that if (i_1^p,j_1^q) is a nontrivial vector (n,m)-identity, then it does not hold in $\underline{Q}$.

Certainly, we can assume that $p \geq q \geq m$. If $p=q=m$ then the conclusion follows from the fact that Q is infinite. Let $b \in B$. If $p > q = m$ then we have $f(\overset{p}{b})=v_1^m \neq b$ where $v_2=(1,\overset{p}{b})$ and $v_\lambda=(\lambda,\overset{p}{b})$ for any $\lambda \neq 2$. Also, if $p > q > m$ we have $f(\overset{p}{b}) \neq f(\overset{q}{b})$, and therefore we can assume that $p = q > m$. Let $\lambda \in N_0$ be such that $i_\nu \neq j_\nu$. Let $a_{i_\lambda}=b$ for every λ, $a_{j_\lambda}=b$ for every λ, $\lambda \neq \nu$, and $a_{j_\nu}=(1,b)$. Then

$$f_1(\overset{p}{b}) = (1,\overset{p}{b}),\ f_1(\overset{\nu-1}{b}\,\overset{n}{(1,b)}\,\overset{p-\nu}{b}) = (1,\ \overset{\nu-1}{b}\,\overset{n}{(1,b)}\,\overset{p-\nu}{b}) \quad \text{and}$$

$$(1,\overset{p}{b}) \neq (1,\ \overset{\nu-1}{b}\,\overset{n}{(1,b)}\,\overset{p-\nu}{b}).$$

Thus we have proved that $\underline{Q}$ does not satisfy a nontrivial vector identity (i_1^p,j_1^q), which implies that Varθ is not a vector variety.

If $\theta \subseteq N^+ \times N^+$ is a set of vector (n,m)-identities, and C is a nonempty set then we define a subset $\theta(C)$ of $C^+ \times C^+$. Namely, $(b_1^p,c_1^q) \in \theta(C)$ iff there exists an $(i_1^p,j_1^q) \in \theta$ and a sequence $a_1,a_2,\ldots$ of elements of C, such that $b_\nu=a_{i_\nu}$, $c_\lambda=a_{j_\lambda}$, for every $\nu \in N_p$, $\lambda \in N_q$.

Proposition 4.10. *Let $\Lambda \subseteq B^+ \times B^+$ be a set of vector (n,m)-relations on a set B, and let $B \subseteq C$. If θ is a set of vector (n,m)-identities, then the embedding mapping $b \mapsto b$ from B into C induces an injective homomorphism $\xi:\underline{S} \to \underline{T}$, where $\underline{S}=\langle B;\Lambda \cup \theta(B)\rangle$, $\underline{T}=\langle C;\Lambda \cup \theta(C)\rangle$.* (In other words, $\underline{S}$ can be considered as a subsemigroup of $\underline{T}$.)

Proof. Denote the relation $\overset{\Lambda\cup\theta(C)}{\sim}$ by $\overset{T}{\sim}$. In the same manner we will use the notations $\overset{T}{\equiv}$, $\overset{S}{\sim}$, $\overset{S}{\equiv}$.

We have to show that if $u,v\in B^+$ are such that $u \overset{T}{\equiv} v$, then $u \overset{S}{\equiv} v$. Namely, $u \overset{T}{\equiv} v$ implies that there exist $u_o,u_1,\dots,u_t\in C^+$ such that $u=u_o$, $v=u_t$, and $u_{\lambda-1} \overset{T}{-} u_\lambda$ for every $\lambda\in N_t$. If in each u_ν we substitute each element of $C\backslash B$ by a fixed element $b\in B$ we obtain a sequence $u'_o=u_o,u'_1,\dots,u'_{t-1}, u'_t=u_k$ such that $u'_{\lambda-1} \overset{S}{-} u'_\lambda$ for each $\lambda\in N_t$. This implies that $u \overset{S}{\equiv} v$. □

Proposition 4.11. *Let $<B;\Lambda;\theta>$ be a vector (n,m)-presentation and let $\xi:c \mapsto \bar{c}$ be a realization of (B,Λ) in an (n,m)-semigroup $\underline{P}=(P;g)\in Var\theta$. If $p,q \geq m$ and $p \equiv q \equiv m \pmod k$, $a_1^p\in B^p$, $b_1^q\in B^q$, and $a_1^p=b_1^q$ in $\overline{<B;\Lambda\cup\theta(B)>}$, then $g(\bar{a}_1^p)=g(\bar{b}_1^q)$ in $\underline{P}$.*

Proof. This is a corollary of P.3.8, since $\bar{\xi}$ is a realization of $(B,\Lambda\cup\theta(B))$ in $\underline{P}$. □

From the well known Birkhoff Theorem, a class of (n,m)-semigroups is a variety of (n,m)-semigroups iff it is non-empty and is closed under the operations of taking (n,m)-subsemigroups, homomorphic images and dirrect products. (See, for example, [1] p. 169). As it has been shown above, the set of vector varieties of (n,m)-semigroups (in the case $m \geq 2$) is a proper subset of the set of varieties of (n,m)-semigroup. It is certainly desirable to find a corresponding "Birkhoff Theorem" for vector varieties of (n,m)-semigroups.

§5. VECTOR (n,m)-PRESENTATIONS WHICH SATISFY THE CONDITION ($\bar{m}$)

The general method for describing an (n,m)-semigroup determined by an (n,m)-presentation can be certainly applied in the case of vector (n,m)-presentations. Sometimes, however, if $\underline{Q}=<B;\Lambda;\theta>$ is a vector (n,m)-presentation then $\underline{Q}$ can be more easily obtained by corresponding presentations of semigroups.

We always assume below that $<B;\Lambda;\theta>$ is a vector (n,m)-presentation with $m \geq 2$.

Let: $G_1=B$, $G_{\alpha+1}=G_\alpha \cup N_m\times E_\alpha$, $\underline{S}_\alpha=\overline{<G_\alpha;\Lambda\cup\theta(G_\alpha)>}$

$$G = \bigcup_\alpha G_\alpha,\ \underline{S} = \overline{<G;\Lambda \cup \theta(G)>}$$

where

$E_1 = \{x \in S_1 \mid (\forall r \in \overline{d}(x))\ (r \equiv m \pmod k \ \&\ r > m)\}$

$E_{\alpha+1} = \{x \in S_{\alpha+1} \setminus S_\alpha \mid (\forall r \in \overline{d}(x))\ (r \equiv m \pmod k \ \&\ r > m)\}$.

By P.4.10 $\underline{S}_\alpha$ is a subsemigroup of $\underline{S}_{\alpha+1}$ and of $\underline{S}$ as well. Moreover, $S = \bigcup_\alpha S_\alpha$.

Here we also assume that the following condition $(\overline{m})$ is satisfied:

$(\overline{m})$ If $u_1^m, v_1^m \in G^m$ are such that $u_1^m = v_1^m$ in $\underline{S}$, then $u_1^m = v_1^m$ in G^m, i.e. $u_\lambda = v_\lambda$ for every $\lambda \in N_m$.

From the definition of G it follows that if $u_1^{m+sk} \in G^{m+sk}$ and if there does not exist a v_1^m in G^m such that $v_1^m = u_1^{m+sk}$ in $\underline{S}$, then $(i, u_1^{m+sk}) \in G$, for any $i \in N_m$. And

$(i,x) = (j,y)$ iff $i=j$ and $x=y$ in $\underline{S}$.

If we have $u_1^{m+sk} = v_1^m$ in $\underline{S}$ then by the condition $(\overline{m})$ v_1^m is uniquely determined. In this case $(i, u_1^{m+sk}) = (i, v_1^m)$ will be only another notation for v_i.

Thus, we have a poly-(n,m)-groupoid $\underline{G} = (G;f)$, where f is defined by

$$f_i(u_1^{m+sk}) = (i, u_1^{m+sk}).$$

(We should emphasize that u_1^{m+sk} in $f_i(u_1^{m+sk})$ is an element of G^{m+sk}, and in (i, u_1^{m+sk}) an element of S.)

If $(a_1^p, b_1^q) \in \Lambda$ then we have $a_1^p = b_1^q$ in $\underline{S}$, and thus $f(a_1^p) = f(b_1^q)$. Let $(i_1^p, j_1^q) \in \theta$ and $u_1, u_2, \ldots \in G$. Then

$$u_{i_1} u_{i_2} \ldots u_{i_p} = u_{j_1} u_{j_2} \ldots u_{j_q}$$

in $\underline{S}$, and thus $f(v_1^p) = f(w_1^q)$, where $v_\lambda = u_{i_\lambda}$, $w_\nu = u_{j_\nu}$.

Proposition 5.1. *Let $(P;g) = \underline{P} \in Var\theta$ and let $\xi : b \mapsto \overline{b}$ be a realization of (B,Λ) in $\underline{P}$. Then there exists a unique homomorphism $\overline{\xi} : \underline{G} \to \underline{P}$ which is an extension of ξ.*

Proof. We define $\overline{\xi}$ by induction on the norm. Let $u = (i,x) \in G_{\alpha+1} \setminus G_\alpha$. Then $x = u_1^{m+sk}$, $s \geq 1$, $u_\nu \in G_\alpha$, and thus $\overline{u}_\nu = \overline{\xi}(u_\nu) \in P$ is well defined. Now, we can define $\xi(u)$ by

$$\overline{\xi}(u) = g_i(\overline{u_1^{m+sk}}).$$

If $x=v_1^{m+rk}$, where $v_\lambda \in G_\alpha$, $r \geq 1$, then we have

$$g_i(\overline{v_1^{m+rk}}) = g_i(\overline{u_1^{m+sk}}),$$

and therefore $\overline{\xi}$ is well defined. Clearly, $\overline{\xi}$ is a homomorphism from the poly-(n,m)-groupoid $\underline{G}$ into the (poly) (n,m)-semigroup $\underline{P}$. □

Proposition 5.2. *If $\approx$ is the least congruence on $\underline{G}$ such that $\underline{G}/\approx$ is an (n,m)-semigroup then $\underline{G}/\approx = \langle B;\Lambda;\theta\rangle$.*

Proof. Clearly, $\underline{G}/\approx \in \mathrm{Var}\theta$ and the mapping $\pi : b \mapsto b^{\approx}$ is a realization of (B,Λ) in $\underline{G}/\approx$. If $\xi : b \mapsto \overline{b}$ is a realization of (B,Λ) in an (n,m)-semigroup $\underline{P}=(P,g) \in \mathrm{Var}\theta$ then by P.5.1 there exists a unique homomorphism $\overline{\xi} : \underline{G} \to \underline{P}$ which is an extension of ξ. We also have $\approx \subseteq \ker\overline{\xi}$, and thus there exists a unique homomorphism $\zeta : \underline{G}/\approx \to \underline{P}$ such that $\zeta(b^{\approx})=\xi(b)$ for every $b\in B$. □

Let us state some properties concerning the condition $(\overline{m})$.

Proposition 5.3. *A vector (n,m)-presentation $\langle B;\Lambda;\theta\rangle$ satisfies the condition $(\overline{m})$ iff for every $\alpha \geq 1$ the following condition $(\overline{m\cdot\alpha})$ holds:*

$$(\overline{m\cdot\alpha}) \quad \left[\begin{array}{l} \textit{If } u_i, v_i \in G_\alpha \textit{ and } u_1^m = v_1^m \textit{ in } S_\alpha \textit{ then} \\ (\forall i \in N_m) \quad u_i = v_i. \end{array}\right.$$ □

Proposition 5.4. *Assume that $\langle B;\Lambda;\theta\rangle$ satisfies $(\overline{m\cdot 1})$ and that one of the following conditions is satisfied:*

a) $|B| \geq 2$;

b) $p \geq q > m$ for every $(i_1^p, j_1^q) \in \theta$;

c) $E_1 = \emptyset$.

Then the condition $(\overline{m})$ holds.

Proof. Let $u_i, v_i \in G$, $u_i \neq v_i$ for some $i\in N_m$ and $u_1^m = v_1^m$ in $\underline{S}$. If $u_1^m, v_1^m \in B^m$, then by $(\overline{m\cdot 1})$ we have $u_i = v_i$. Thus, we suppose $u_1^m \notin B^m$ or $v_1^m \notin B^m$.

Assume that a) holds. There exist $t \geq 0$, $x_1, \ldots, x_t \in G^+$ such that

$$u_1^m \sim x_1 \sim \dots \sim x_t \sim v^m. \tag{5.1}$$

If $u_i \in B$, take an $a \in B$, $a \neq u_i$, and replace any occurence of $v \in G\setminus B$ in (5.1) by a. If $v_i \in B$, the proof is by symmetry. If $u_i \in B$ and $v_i \notin B$, then we take $a,b \in B$, $a \neq b$, and replace any occurence of u_i in (5.1) by a, and any occurence of $v \in (G\setminus B)\setminus\{u_i\}$ in (5.1) by b. In such a way we obtain $a_1^m = b_1^m$ in S_1, $a_\nu, b_\nu \in B$, and $a_i \neq b_i$, which contradicts $(\overline{m \cdot 1})$.

Assume that b) is satisfied, and that $u_1^m \in B^m$. Then neither an (n,m)-relation nor an (n,m)-identity could be applied on u_1^m, which contradicts the fact that $u_1^m = v_1^m$ in $\underline{S}$.

The conclusion is clear in the case $E_1 = \emptyset$, for this implies that $G = B$, $\underline{S} = \underline{S}_1$. □

Proposition 5.5. *For every vector (n,m)-presentation* $\langle B;\Lambda;\theta\rangle$ *the conditions* $(\overline{m})$ *and* $(\overline{m.2})$ *are equivalent.*

Proof. By P.5.3 we have to show that $(\overline{m \cdot 2})$ implies $(\overline{m})$. Clearly, $(\overline{m \cdot 2})$ implies $(\overline{m \cdot 1})$, and therefore we can assume that $E_1 \neq \emptyset$, whence it follows $|G_2| \geq 2$. Now, we can show the implication $(\overline{m \cdot 2})$ & $|G_2| \geq 2 \Longrightarrow (\overline{m})$ in the same way as we have proved the implication $(\overline{m \cdot 1})$ & $|B| \geq 2 \Longrightarrow (\overline{m})$. □

Proposition 5.6. *A vector (n,m)-presentation* $\langle B;\Lambda;\theta\rangle$ *satisfies the conditions* $E_1 = \emptyset$ *and* $(\overline{m.1})$ *iff there is an (n,m)-semigroup* $\underline{B} = (B;f)$ *such that*

$$(\forall a_\nu, b_\lambda \in B,\ r,s \geq 0)(a_1^{m+rk} = b_1^{m+sk} \text{ in } \underline{S}_1 \Longleftrightarrow f(a_1^{m+rk}) = f(b_1^{m+sk})) \tag{5.2}$$

Moreover, we have $\langle B;\Lambda;\theta\rangle = \underline{B}$.

Proof. If $E_1 = \emptyset$ and $(\overline{m \cdot 1})$ is satisfied then for every $r \geq 0$ and $a_1^{m+rk} \in B^{m+rk}$ there exists a unique $b_1^m \in B^m$ such that $a_1^{m+rk} = b_1^m$ in $\underline{S}_1$. Thus, we can define a poly-(n,m)-operation f on B by:

$$(\forall a_\nu, b_\lambda \in B,\ r \geq 1)(f(a_1^{m+rk}) = b_1^m \Longleftrightarrow a_1^{m+rk} = b_1^m \text{ in } \underline{S}_1),$$

and it can be easily seen that then $\underline{B} = (B;f) \in \mathrm{Var}\,\theta$ and (5.2) holds as well. Moreover, then the identity mapping $b \mapsto b$ is a universal realization of (B,Λ) in $\underline{B}$.

The converse conclusion is clear also. □

§6. REDUCTIONS FOR VECTOR (n,m)-PRESENTATIONS WITH THE CONDITION ($\overline{m}$)

Below we assume that $\langle B;\Lambda;\theta\rangle$ is a given vector (n,m)-presentation which satisfies the condition ($\overline{m}$), where $m\geq 2$.

The description of the (n,m)-semigroup $\langle B;\Lambda;\theta\rangle$ given by P.5.2 can not be considered as good enough, for we have not a convenient description of the congruence $\approx$. Nevertheless in some cases it is possible to obtain a convenient description for $\langle B;\Lambda;\theta\rangle$.

Let $E=G^m\cup(\bigcup_{\alpha\geq 1} E_\alpha)$. Namely, E consists of all elements $x\in S$ such that $r\equiv m(\mathrm{mod} k)$ for all $r\in\overline{d}(x)$, and the above union is disjoint. Assume also that $L\subseteq G$ and $R\subseteq E$ have the following properties:

a) $B\subseteq L$, $B^m\subseteq R$;

b) $(i,x)\in L \iff x\in R$;

c) $R\subseteq T$, where $T=E\cap H$ and $\underline{H}=\langle L;\Lambda\cup\theta(L)\rangle$;

d) $\psi:T\to T$ is a transformation of T with the following properties:

d.1) $\psi(x)=x \iff x\in R$;

d.2) $\psi(x'(1,y)(2,y)\dots(m,y)x'')=\psi(x'yx'')$, for every $x'(1,y)\dots(m,y)x''\in T$, $(\nu,y)\in L$;

d.3) $\psi(x'xx'')=\psi(x'\psi(x)x'')$, for every $x,x'xx''\in T$.

By d.1) and d.3) we obtain:

<u>Proposition 6.1</u>. *$\psi^2=\psi$ and $\psi(T)=R$.* □

Define a poly-(n,m)-groupoid (L;g) by

$$g_i(u_1^{m+sk})=(i,\psi(u_1^{m+sk})).$$

The operation g is well defined, for $\psi(u_1^{m+sk})\in R$ and this, by b), implies $(i,\psi(u_1^{m+sk}))\in L$. (Here, if $\psi(u_1^{m+sk})=v_1^m$ then by definition $(i,v_1^m)=v_i$.)

<u>Proposition 6.2</u>. *$(L;g)=\underline{L}\in Var\theta$ and the inclusion $b\mapsto b$ from B into L is a realization of (B,Λ) in $\underline{L}$.* □

Theorem 6.3. *Assume that L,R and ψ satisfy the conditions a),b),c),d.1),d.2),d.3) and the following condition:*

d.4) If $\xi: b \mapsto \overline{b}$ is a realization of (B,Λ) in an (n,m)-semigroup $\underline{Q} \in Var\theta$ and if $\overline{\xi}: \underline{G} \to \underline{Q}$ is the (n,m)-homomorphism defined in P.5.1, then the restriction η of $\overline{\xi}$ on L is an (n,m)-homomorphism from $\underline{L}$ into $\underline{Q}$.

Then $\underline{L}$=<B;Λ;θ>.

Proof. First, it can be easily checked that B is a generating subset of $\underline{L}$, and then the conclusion follows from P.4.4 and d.4). □

We do not have a general method to answer the question whether such sets L, R and a mapping ψ:T → T do exist. But, the assumed condition suggest the following procedure.

Let $L_1=B$, $R_0=B^m=T_0$, $R_1=E_1=T_1$, $L_2=B\cup N_m\times R_1$, $\underline{H}_\alpha=\overline{<L_\alpha;\Lambda\cup\theta(L_\alpha)>}$ (i.e. $\underline{H}_\alpha$ is the subsemigroup of $\underline{S}_\alpha$ generated by L_α) and

$$T_{\alpha+1} = \{x\in H_{\alpha+1}\setminus H_\alpha \mid (\forall r\in\overline{d}(x)) \quad r\equiv m \pmod k\}.$$

An element $x\in T_{\alpha+1}$, $(\alpha\geq 1)$ is said to be reducible iff $x=x'(1,y)\dots(m,y)x''$, where $(\nu,y)\in L_{\alpha+1}$ and $x\neq x'yx''$. Denote by $R_{\alpha+1}$ the set of elements of $T_{\alpha+1}$ which are not reducible, i.e. which are reduced. Then

$$L_{\alpha+2} = L_{\alpha+1}\cup N_m\times R_{\alpha+1}.$$

Finally, let

$$L = \bigcup_{\alpha\geq 1} L_\alpha,\ R = \bigcup_{\beta\geq 0} R_\beta,\ T = \bigcup_{\beta\geq 0} T_\beta,\ H = \bigcup_{\alpha\geq 1} H_\alpha.$$

(Note that $L_\alpha\subseteq L_{\alpha+1}$, $H_\alpha\subseteq H_{\alpha+1}$ and if $\gamma\neq\delta$ then $T_\gamma\cap T_\delta=\emptyset$, $R_\gamma\cap R_\delta=\emptyset$.) Then we say that L,R and T are defined by the "standard construction".

It remains to define a corresponding mapping ψ:T → T. For that purpose it is useful to generalize the notion of a "norm" |x| of an element x∈S. Namely, if x∈S, then we define a set |x| of nonnegative integers as follows.

$x\in S_1 \Longrightarrow |x| = \{0\}$,
$(i,x)\in G\setminus B \Longrightarrow |(i,x)| = |x|+1 = \{\alpha+1 \mid \alpha\in|x|\}$,

$x \in S \implies |x| = \{\alpha_1+\ldots+\alpha_t \mid x=u_1^t,\ \alpha_\nu \in |u_\nu|,\ u_\nu \in G\}$.

Denote by $\|x\|$ the least element of $|x|$.

Let $\psi' : T\backslash R \to T$ be a mapping such that:

$$x \in T\backslash R \implies \|\psi'(x)\| < \|x\|.$$

Now we can define a mapping $\psi : T \to T$ as follows:

(i) $\psi(x) = x$ for every $x \in R$.

Assume that $x \in T\backslash R$ and that if $y \in T$ is such that $\|y\| < \|x\|$ then $\psi(y) \in R$ is well defined. Moreover, we assume that the following condition is satisfied:

$$\psi(y) \neq y \implies \|\psi(y)\| < \|y\|. \qquad (6.1)$$

Then $\psi(\psi'(x)) \in R$ is well defined and thus we put

(ii) $\psi(x) = \psi(\psi'(x))$.

So we have $\psi(x) \in R$ for all $x \in T$, and furthermore

$$\|\psi(x)\| = \|\psi(\psi'(x))\| \leq \|\psi'(x)\| < \|x\|,$$

i.e. (6.1) holds if y is replaced by x. This implies that $\psi : T \to T$ is a well defined mapping such that $\psi(T)=R$, and moreover d.1) and (6.1) hold. In any particular case it has to be checked whether d.2)-d.4) hold. In fact, we should choose ψ' such that d.2)-d.4) hold.

Assume that the following condition is satisfied.

(ass) $\left[\begin{array}{l} \text{Let } x \in T\backslash R,\ y,z \in R\backslash R_o,\ x=x'(1,y)\ldots(m,y)x''=t'(1,z)\ldots(m,z)t'' \\ \text{in } \underline{H}, \text{ where } x' \text{ has the least possible norm and} \\ \|x'\| = \|t'\|. \text{ Then } x'yx''=t'zt'' \text{ in } \underline{H}, \text{ and } \|x'yx''\| < \|x\|. \end{array}\right.$

(The above condition will be referred as (ass).)

Then, if $x \in T\backslash R$, x', x'', y are as above, we define $\psi'(x)$ by: $\psi'(x)=x'yx''$, and we say that this is the "standard definition of ψ'".

Consider the case when $\theta=\emptyset$, i.e. $\langle B;\Lambda;\theta\rangle=\langle B;\Lambda\rangle$. By P.5.4, the condition $(\overline{m})$ is equivalent to $(\overline{m.1})$, i.e. to the assertion that "B^m is a subset of $\overline{\langle B;\Lambda\rangle}$". If $E_1=\emptyset$ then L=B and $\langle B;\Lambda\rangle=(B;f)$, where

$$f(a_1^{m+sk}) = b_1^m \text{ iff } a_1^{m+sk} = \cdot b_1^m \text{ in } \langle \overline{B;\Lambda} \rangle.$$

Next, suppose that $E_1 \neq \emptyset$. Then every element $x \in S$ has a unique form

$$x = a_0 u_1 a_1 u_2 \ldots a_{\alpha-1} u_\alpha a_\alpha,$$

where $u_\lambda \in G \setminus B$, $a_\nu \in \langle \overline{B;\Lambda} \rangle^1$. This implies that the norm of x is uniquely determined, $|x| = \sum_{\nu=1} |u_\nu|$ and $\|x\| = |x|$.

Let L,R,T be obtained by the standard construction.

If $x \in T \setminus R$, $y \in R \setminus R_o$ and $x = x'(1,y)\ldots(m,y)x''$, where x' has the least possible norm, then x', y, x'' are uniquely determined, and thus ψ' can be defined by the standard definition.

By induction on the norm it can be easily checked that the corresponding mapping $\psi : T \to T$ satisfies the conditions d.1)-d.4), and thus

$$(L;f) = \underline{L} = \langle B;\Lambda \rangle.$$

(For $\Lambda = \emptyset$ the above discussion gives another description of the free (n,m)-semigroup with a basis B.)

We note that if "we have a good description of $\langle \overline{B;\Lambda} \rangle$" and if (m.1) holds, then "we have a good description of $\langle B;\Lambda \rangle$", as well.

Next we will state and prove corresponding vector valued variants of Post and Cohn-Rebane Theorems, using the above results.

An (m+rk,m)-semigroup $\underline{Q} = (Q;g)$ is said to be an r-subsemigroup of an (m+k,m)-semigroup $\underline{P} = (P;f)$ iff $Q \subseteq P$ and $f(a_1^{m+rk}) = g(a_1^{m+rk})$ for any $a_\nu \in Q$.

Theorem 6.4. (Post Theorem) *Every (m+rk,m)-semigroup is an r-subsemigroup of an (m+k,m)-semigroup.*

Proof. In the case m=1 this result is well known and that is why we assume that $m \geq 2$. We also assume $r \geq 2$, for in the case r=1 the conclusion is trivial.

Let $\underline{Q}=(Q;g)$ be an $(m+rk,m)$-semigroup, and consider the $(m+k,m)$-presentation $\underline{P}=\langle Q;\Gamma(\underline{Q})\rangle$. (Note that if $\langle Q;\Gamma(\underline{Q})\rangle$ is considered as an $(m+rk,m)$-presentation then $\langle Q;\Gamma(\underline{Q})\rangle=\underline{Q}$!)

Let $a_1^m \in Q^m$ and $a_1^m=b_1^{m+sk}$ in $\underline{P}$. Then it can be checked that r is a divisor of s and that $g(b_1^{m+sk})=a_1^m$. This implies that the condition $(\overline{m.I})$ is satisfied. Hence, $\underline{Q}$ is an r-subsemigroup of $\langle Q;\Gamma(\underline{Q})\rangle$. □

A vector valued variant of Cohn-Rebane Theorem (see [1], [3]) is stated and proved below. First we give a few definitions.

If A is a nonempty set, $D\subseteq A^{n(\omega)}$, and ω a mapping from D in $A^{m(\omega)}$, then we say that ω is a <u>partial</u> $(n(\omega),m(\omega))$-<u>operation</u> (i.e. a <u>partial vector valued operation</u>) on A, with a domain D. (Here, the condition $n(\omega) > m(\omega)$ is not assumed.) If Ω is a set of partial vector valued operations on A then we say that $\underline{A}=(A;\Omega)$ is a <u>partial vector valued algebra</u>.

<u>Proposition 6.5</u>. (Cohn-Rebane Theorem). *Let $\underline{A}=(A;\Omega)$ be a partial vector valued algebra, n,m positive integers such that $n-m=k\geq 1$ and let*

$$(\forall\omega\in\Omega)(\exists\alpha(\omega)\geq 1,\ \beta(\omega)\geq 0)(n(\omega)+1=m+\alpha(\omega)k, m(\omega)=m+\beta(\omega)k).$$

Then, there is an (n,m)-semigroup $\underline{Q}=(Q;f)$ and a mapping $\omega\mapsto\overline{\omega}$ from Ω in Q such that $A\subseteq Q$ and

$$(\forall\omega\in\Omega,\ a_1^{n(\omega)}, b_1^{m(\omega)}\in A^+)(\omega(a_1^{n(\omega)})=b_1^{m(\omega)} \iff f(a_1^{n(\omega)}) = f(b_1^{n(\omega)})).$$

<u>Proof</u>. We assume that $m\geq 2$, because in the case m=1 this Theorem is shown in [1]. Let $\overline{\Omega}=\{\overline{\omega}\mid\omega\in\Omega\}$ be such that $\omega\mapsto\overline{\omega}$ is a bijection and $\overline{\Omega}\cap A=\emptyset$. Denote by B the set $A\cup\overline{\Omega}$, and let Λ be a set of vector (n,m)-relations defined by

$$\Lambda = \{(\overline{\omega}a_1^{n(\omega)},\ b_1^{m(\omega)}) \mid \omega(a_1^{n(\omega)}) = b_1^{m(\omega)} \text{ in } \underline{A}\}.$$

We will show that the (n,m)-presentation $\langle B;\Lambda\rangle$ satisfies the condition $(\overline{m.I})$, and this will imply that $\underline{Q}=\langle B;\Lambda\rangle$ is an (n,m)-semigroup with the desired property.

Thus we have only to show that $(\overline{m.1})$ holds. For that purpose we will give a complete description of the semigroup $\langle\overline{B;\Lambda}\rangle$.

The dimension d in B^+ is defined in the usual way. Let dg(=degree) in B^+ be defined by

$dg(b)=0$, $dg(\bar{\omega})=1$ for every $b\in A$, $\omega\in\Omega$,

$dg(xy)=dg(x)+dg(y)$ for all $x,y\in B^+$.

Let $d(1)=dg(1)=0$.

An element $x\in B^+$ is said to be reducible iff it has a form $x=x'\bar{\omega}a_1^{n(\omega)}x''$, where $x',x''\in B^*$, $\omega\in\Omega$, $a_1^{n(\omega)}\in D$. Otherwise x is called reduced. Denote by P the set of all the reduced elements of B^+, and define a mapping $\eta:B^+\to P$ as follows.

1) $\eta(x) = x \iff x\in P$.

Let $x\in B^+\setminus P$ and let for every $y\in B^+$ such that $dg(y)<dg(x)$, $\eta(y)\in P$ is well defined, and moreover

$$\eta(y) \neq y \Longrightarrow dg(\eta(y)) < dg(y). \tag{6.2}$$

If x has a form $x=x'\bar{\omega}a_1^{n(\omega)}x''$, where $\omega(a_1^{n(\omega)})=b_1^{m(\omega)}$ in $\underline{A}$ and $dg(x')$ is the least possible and if $y=x'b_1^{m(\omega)}x''$, then we have $dg(y)=dg(x)-1$. Then we define $\eta(x)$ by

2) $\eta(x) = \eta(y)$.

By the assumption (6.2), $\eta(x)=\eta(y)\in P$ and $dg(\eta(x))=dg(\eta(y)) \leq \leq dg(y)<dg(x)$. Thus, x also satisfies (6.2) and this implies that $\eta:B^+\to P$ is well defined and that (6.2) holds for every $x\in B^+$. Also, it can be easily checked that

$$\eta(x\,y) = \eta(\eta(x)y) = \eta(x\eta(y)) \tag{6.3}$$

for every $x,y\in B^+$. This implies that if we define an operation $\bullet$ on P by

$$x\bullet y = \eta(xy)$$

then we obtain a semigroup $\underline{P}=(P;\bullet)$. Moreover, the embedding $b\mapsto b$ of B in P is a universal realization of (B,Λ) in $\underline{P}$, i.e. $\underline{P}=\langle B;\Lambda\rangle$.

If $b_\nu \in B$ then

$$b_1 \bullet b_2 \bullet \ldots \bullet b_m = \eta(b_1^m) = b_1^m$$

and this implies that $(\overline{m.1})$ holds. □

It is natural to ask for a "partial variant" of the Post Theorem. For that purpose we introduce the notion of a "partial poly-(n,m)-groupoid". Namely, if $D = D_g \subseteq Q^{(n,m)}$ and g is a mapping from D into Q^m, then we say that $\underline{Q}=(Q;g)$ is a partial poly-(n,m)-groupoid with a domain D. If $\underline{Q}=(Q;g)$ is a partial poly-(m+rk,m)-groupoid with domain D and if $\underline{P}=(P;f)$ is an (n,m)-semigroup such that $Q \subseteq P$ and

$$a_1^{m+srk} \in D \implies g(a_1^{m+srk}) = f(a_1^{m+srk}),$$

then we say that $\underline{Q}$ is a partial r-subsemigroup of P.

Proposition 6.6. *Let $\underline{Q}=(Q;g)$ be a partial poly-(m+rk,m)-groupoid and let Λ be a set of (n,m)-relations on Q defined by*

$$\Lambda = \{(a_1^{m+srk}, b_1^m) \mid a_1^{m+srk} \in D,\ g(a_1^{m+srk}) = b_1^m\}.$$

Then $\underline{Q}$ is a partial r-subsemigroup of an (n,m)-semigroup $\underline{P}=(P;f)$ iff the presentation $\langle Q;\Lambda\rangle$ satisfies the condition $(\overline{m.1})$. □

§7. PRESENTATIONS IN SOME VARIETIES OF VECTOR VALUED SEMIGROUPS

Here we consider some vector varieties $\mathcal{V} = \mathrm{Var}\,\theta$ of (n,m)-semigroups such that $p \geq q > m$ for every $(i_1^p, j_1^q) \in \theta$, and thus P.5.4. is applicable.

First we give some definitions.

An (n,m)-semigroup $\underline{Q}=(Q;f)$ is said to be γ-nilpotent iff

$$f(a_1^{m+\gamma k}) = f(b_1^{m+\gamma k}) \tag{7.1}$$

for any $a_\nu, b_\lambda \in Q$.

If $\underline{Q}=(Q;f)$ is γ-nilpotent, there exist $0_1,\ldots,0_m \in Q$ such that

$$f(a_1^{m+\alpha k}) = 0_1^m \tag{7.1'}$$

for any $\alpha \geq \gamma$ and $a_\nu \in Q$. And conversely, if $0_1^m \in Q^m$ is such that (7.1′) holds for any $\alpha \geq \gamma$, $a_\nu \in Q$, then Q is γ-nilpotent. The class of γ-nilpotent (n,m)-semigroups will be denoted by $Z(\gamma;n,m)$. (We assume that $\gamma \geq 1$, for $Z(0;n,m)=0(n,m)$.)

If $\theta=\{(i_1^{m+\alpha k}, j_1^{m+\beta k}) \mid i_\nu, j_\lambda \in N\setminus\{0\}, \alpha,\beta \geq \gamma\}$, then $\mathrm{Var}\theta=Z(\gamma;n,m)$.

An (n,m)-semigroup $Q=(Q;f)$ is called an α-left zero (n,m)-semigroup iff

$$f(a_1^\alpha b_1^\beta) = f(a_1^\alpha c_1^\gamma) \tag{7.2}$$

for any $a_\nu, b_\lambda, c_\varepsilon \in Q$, where α is a given positive integer and $\beta,\gamma \in N$ are such that $\alpha+\beta \equiv \alpha+\gamma \equiv m(\mathrm{mod}k)$, $\alpha+\beta > m$, $\alpha+\gamma > m$. The class of α-left zero (n,m)-semigroups will be denoted by $LZ(\alpha;n,m)$. Thus,

$$LZ(\alpha;n,m) = \mathrm{Var}\theta$$

where θ consists of all the vector (n,m)-idnetities $(i_1^{\alpha+\beta}, j_1^{\alpha+\gamma})$ such that $i_\nu=j_\nu=\nu$ for every $\nu \in N_\alpha$, and $i_{\alpha+\lambda}$, $j_{\lambda+\varepsilon}$ are arbitrary; here, as above, we assume that $\alpha+\beta \equiv \alpha+\gamma \equiv m(\mathrm{mod}k)$, $\alpha+\beta > m$, $\alpha+\gamma > m$.

Assume now that $\alpha,\beta \geq 1$ are given positive integers and denote by θ the set of all vector (n,m)-identities $(i_1^{\alpha+\gamma+\beta}, j_1^{\alpha+\delta+\beta})$, where

$$\alpha+\delta+\beta \equiv \alpha+\gamma+\beta \equiv m(\mathrm{mod}k),\ \alpha+\delta+\beta > m,\ \alpha+\gamma+\beta > m \tag{7.3}$$

and $i_\nu=j_\nu=\nu$ for every $\nu \in N_\alpha$, $i_{\alpha+\gamma+\lambda}=j_{\alpha+\delta+\lambda}$ for every $\lambda \in N_\beta$. Then we denote by $RB(\alpha,\beta;n,m)$ the variety $\mathrm{Var}\theta$. Thus, if $Q=(Q;f)$ is an (n,m)-semigroup then $Q \in RB(\alpha,\beta;n,m)$ iff

$$f(a_1^\alpha c_1^\gamma b_1^\beta) = f(a_1^\alpha d_1^\delta b_1^\beta)$$

for any $a_i, b_j, c_s, d_t \in Q$, where (7.3) holds. The (n,m)-semigroups of $RB(\alpha,\beta;n,m)$ are called $(\alpha,\beta;n,m)$-rectangular bands.

Proposition 7.1. *Assume that:*

(i) $\mathrm{Var}\theta \in \{Z(\gamma;n,m),\ LZ(\alpha;n,m),\ RB(\alpha,\beta;n,m)\}$,

(ii) Λ *is a set of vector (n,m)-relations on a set* $B \neq \emptyset$,

(iii) $\langle B;\Lambda;\theta\rangle$ *satisfies* $(\overline{m.1})$ *and* $E_1 \neq \emptyset$,

(iv) L,R *and* T *are defined by the standard construction.*

Then the condition (ass) of section 6 is satisfied, and if ψ' *is defined in a standard way then the corresponding mapping* $\psi: T \to T$ *satisfies the conditions d.1)-d.4).* □

Denote by Absem(n,m) the variety of commutative (n,m)-semigroups, i.e. Absem(n,m)=Varθ, where

$$\theta = \{(i_1^p, j_1^p) \mid j_1^p \text{ is a permutation of } i_1^p,\ p=m+sk,\ s \geq 1\}.$$

<u>Proposition 7.2</u>. *Let* $\langle B;\Lambda;\theta\rangle$ *be a vector presentation such that*

(i') $Var\theta = Absem(n,m)$

and, moreover, let the conditions (ii)-(iv) of P.7.1 be satisfied. Define $\psi': T\backslash R \to T$ *in the following way:* $\psi'(x)=x'y_1y_2\ldots y_t$, *where* $x=x'(1,y_1)\ldots(m,y_1)\ldots(1,y_t)\ldots(m,y_t)$, $(\nu,y_\lambda)\in L$ *and* $x'\in S^1$ *has the least possible dimension (i.e.* x' *is not equal to* $x''(1,z)\ldots(m,z)$*). Then* ψ' *is well defined and satisfies the condition*

$$|\psi'(x)| < |x| \text{ for every } x\in T\backslash R,$$

and the corresponding mapping $\psi: T \to T$ *satisfies the conditions d.1)-d.4).* □

If $\mathcal{C}$ is a class of (n,m)-semigroups and r is a positive integer then we denote by $\mathcal{C}(r)$ the class of (m+rk,m)-semigroups which are r-subsemigroups of (n,m)-semigroups belonging to $\mathcal{C}$. By a "Post Theorem" for the class $\mathcal{C}$ we mean a statement which gives a description for the set of classes $\{\mathcal{C}(r) \mid r \geq 1\}$. Thus T.6.4 is a "Post Theorem" for the variety Sem (n,m) of (n,m)-semigroups. The theorem actually states that

$$\text{Sem}(n,m)(r) = \text{Sem}(m+rk,m).$$

Assume that $\mathcal{V}$=Varθ is a variety of (n,m)-semigroups and $r \geq 1$. If $\underline{Q}=(Q;g)$ is an (m+rk,m)-semigroup then the graph

$$\Gamma(\underline{Q}) = \{(a_1^{m+rk}, b_1^m) \mid g(a_1^{m+rk}) = b_1^m \text{ in } \underline{Q}\}$$

is a set of (m+rk,m)-relations and (n,m)-relations as well. Further on we will always consider $\langle Q;\Gamma(\underline{Q});\theta\rangle$ as an (n,m)-presentation.

Proposition 7.3. *Let* $\mathcal{V}=Var\theta$. *Then* $\underline{Q}\in\mathcal{V}(r)$ *iff the (n,m)-presentation* $\langle Q;\Gamma(\underline{Q});\theta\rangle$ *is proper.*

Proof. If $\underline{P}=(P;f)$ is an (n,m)-semigroup and $Q\subseteq P$, then $\underline{Q}$ is an r-subsemigroup of $\underline{P}$ iff the embedding mapping of Q into P is a realization of $(Q,\Gamma(\underline{Q}))$ in $\underline{P}$. The conclusion follows by P.1.4. □

Proposition 7.4. *If* δ *is the least positive integer such that* $\delta r\geq\gamma$, *then*

$$Z(\gamma;n,m)(r) = Z(\delta;m+rk,m). \quad \square$$

Proposition 7.5. $LZ(\alpha;n,m)(r) = LZ(\alpha;m+rk,m)$. □

Proposition 7.6. $RB(\alpha,\beta;n,m)(r) = RB(\alpha,\beta;m+rk,m)$. □

Proposition 7.7. $Absem(n,m)(r) = Absem(m+rk,m)$. □

The proofs of all the four propositions are similar. First, it is clear that the varieties on the left sides are subvarieties of the corresponding varieties on the right sides. Further, if $\underline{Q}$ is an (m+rk,m)-semigroup belonging to the corresponding variety on the right, then one can easily check that the corresponding (n,m)-presentation $\langle Q;\Gamma(\underline{Q});\theta\rangle$ satisfies the condition $(\overline{m.I})$. Then P.7.1, P.7.2, and P.7.3 imply that the corresponding equalities hold.

An (n,m)-semigroup $\underline{Q}=(Q;f)$ is called an (n,m)-semilattice iff it satisfies each (n,m)-identity (i_1^p,j_1^q), where $p\geq q\geq n$ and

$$\{i_1,i_2,\ldots,i_p\} = \{j_1,j_2,\ldots,j_q\}.$$

(Here $\{i_1,i_2,\ldots,i_p\}$ has the usual meaning, i.e. $i\in\{i_1,\ldots,i_p\}$ iff $i=i_\nu$ for some $\nu\in N_p$.) The class of (n,m)-semilattices will be denoted by $S\mathfrak{L}(n,m)$. Clearly, $SL(n,m)$ is a subclass of $Absem(n,m)$.

Proposition 7.8. $SL(n,m)(r) = SL(m+rk,m)$.

Proof. It is clear that $SL(n,m)(r) \subseteq SL(m+rk,m)$. Let $(Q;g)\in SL(m+rk,m)$ and let an (n,m)-operation f on Q be defined by

$$f(x_1^n) = g(x_1^n \overset{(r-1)k}{x_1}).$$

It can be easily checked that $(Q;f)\in SL(n,m)$ and that

$$f(x_1^{m+rk}) = g(x_1^{m+rk})$$

i.e. $(Q;g)$ is an r-subsemigroup of $(Q;f)$. □

P.7.1 and P.7.2 imply that if $\langle B;\Lambda;\theta\rangle$ is a vector (n,m)-presentation such that $(\overline{m.1})$ is satisfied, where $Var\theta\in\{Z(\gamma;n,m)$, $LZ(\alpha;n,m)$, $RB(\alpha,\beta;n,m)$, $Absem(n,m)\}$, and if we have an effective description of the semigroup $\underline{S}_1=\overline{\langle B;\Lambda\cup\theta(B)\rangle}$, then we have an effective description of the (n,m)-semigroup $\langle B;\Lambda;\theta\rangle$ as well. (Until now, we do not have a corresponding result in the case $Var\theta=SL(n,m)$.)

Next, we will give more detailed description of free objects in the mentioned varieties of (n,m)-semigroups, except the variety $SL(n,m)$, by giving explicit definitions of the sequences of sets $(R_\lambda \mid \lambda \geq 1)$ and then $L_1=B$, $L_{\lambda+1}=L_\lambda\cup N_m\times R_\lambda$, $L=\bigcup_{\lambda\geq 1} L_\lambda$. In the following we assume that B is a nonempty set.

Example 7.9. The free object in $Z(1;n,m)=Z(n,m)$ with a basis B is the constant (n,m)-semigroup on the set $L=B\cup\{0_1,0_2,\dots,0_m\}$, where 0_1^m is the constant, i.e. $f(x_1^n)=0_1^m$ for any $x_1^n\in L^n$, and $0_\nu\notin B$, $0_\nu\neq 0_\lambda$ for $\nu\neq\lambda$.

Assume that $\gamma \geq 2$.

$R_1 = \{0\}\cup\{x\in B^+ \mid d(x) = m+\delta k,\ 1 \leq \delta < \gamma\}$. (We note that 0 is a notation for any element $x\in B^+$ such that $d(x)=m+\delta k$, where $\delta \geq \gamma$.) $R_{\lambda+1}=\{x\in L_{\lambda+1}^+\setminus L_\lambda^+ \mid m<d(x) \equiv m(\mathrm{mod}\ k),\ x \text{ is reduced}\}$. Then L is (the carrier of) a free object in $Z(\gamma;n,m)$.

Example 7.10. Consider $LZ(\alpha;n,m)$.

$R_\lambda = R'_\lambda\cup R''_\lambda$, where:

$R'_\lambda=\{x\in L_\lambda^+\setminus L_{\lambda-1}^+ \mid m < d(x) < \alpha,\ d(x) \equiv m(\mathrm{mod}\ k),\ x \text{ is reduced}\}$,

$R_\lambda'' = \{x \in L_\lambda^\alpha \setminus L_{\lambda-1}^\alpha \mid x \text{ is reduced}\}$.

Here, for $x \in L_\lambda^\alpha$ "x is reduced" means x has not a form $x = x'(1,y)\ldots(m,y)x''$ neither a form $x = x'(1,y)\ldots(\beta,y)$ for some $1 \le \beta < m$. We note that here u_1^α is in fact a notation for any $u_1^\alpha v_1^\gamma$, where $\alpha+\gamma = m+\delta k$, $\delta \ge 1$. This implies that if $x = x'(1,y)\ldots(m,y)x'' \in T\setminus R$, then we put

$$\psi'(x) = \begin{cases} x'yx'' & \text{if } d(x'yx'') \le \alpha \\ u_1^\alpha & \text{if } x'yx'' = u_1^\alpha v_1^\gamma. \end{cases}$$

And, if $x = x'(1,y)\ldots(\beta,y) \in L^\alpha$, then

$$\psi'(x) = x'v_1^\beta$$

where $y = v_1^\beta w_1^\delta (u_\nu, v_\lambda, w_\varepsilon \in L)$. Since ψ' is defined, we proceed as in Section 6.

We treat separately the case $\alpha=1$. Note that the objects of $LZ(1;n,m)$ can be considered as algebras $(Q;h_1,h_2,\ldots,h_m)$ with m unary operations, i.e. transformations of Q, satisfying the following condition:

$$h_i h_1 = h_i \tag{7.4}$$

for every $i \in N_m$. Namely, if $(Q;h_1,\ldots,h_m)$ is such an algebra and if an (n,m)-groupoid $\underline{Q} = (Q;g)$ is defined by

$$g(a_1^n) = b_1^m \iff b_i = h_i(a_1) \tag{7.5}$$

then $\underline{Q} \in LZ(1;n,m)$; conversely, if $\underline{Q} = (Q;g) \in LZ(1;n,m)$ and if $h_1,h_2,\ldots,h_m$ are defined by (7.5), then (7.4) is satisfied for any $i \in N_m$.

Now, we put

$$L = B \cup \{(i_1^p;b) \mid b \in B,\ i_\nu \in N_m,\ i \notin \{i_2,\ldots,i_p\}\},$$

and the operation g on L is defined by

$$g_i(u_1^{m+sk}) = \begin{cases} (i\ i_1^p;b) & \text{if } u_1 = (i_1^p;b), i_1 \ne 1 \\ (i\ i_2^p;b) & \text{if } u_1 = (1 i_2^p;b) \end{cases}$$

(More generally, if $1 \le \alpha < n$, then the variety $LZ(\alpha;n,m)$ is equivalent to a variety of (α,m)-groupoids.)

Example 7.11. Consider the variety RB$(\alpha,\beta;n,m)$.

$L_1 = L$, $R_{\lambda+1} = R'_\lambda \cup R''_\lambda$, where

$R'_\lambda = \{x\in L^+_\lambda \setminus L^+_{\lambda-1} \mid m < d(x) = m+\delta k < \alpha+\beta$, x is reduced$\}$,

$R''_\lambda = \{x\in L^{\alpha+\beta}_\lambda \setminus L^{\alpha+\beta}_{\lambda-1} \mid x$ is reduced$\}$.

Then, $x=u^\alpha_1 v^\beta_1$ is reducible iff one of the following is satisfied:

$u^\alpha_1 = x'(1,y)\dots(j,y)$, $1\le j\le m$,

$v^\beta_1 = (t,z)\dots(m,z)x''$, $i\le t\le m$,

or x is reducible in the usual sense. (Here $u^\alpha_1 v^\beta_1$ is a notation for $u^\alpha_1 w^\delta_1 v^\beta_1$, where $\alpha+\delta+\beta=m+\varepsilon k$. This implies corresponding modifications in the definition of ψ' from the previous example.)

Example 7.12. If C is a nonempty set, and if we define a relation $\approx$ on C^+ by

$$u \approx v \iff v \text{ is a permutation of } u,$$

then we obtain a congruence $\approx$ on C^+ such that the quotient semigroup $C^+/\approx$ is a free abelian (i.e. commutative) semigroup with a basis C. Denote $C^+/\approx$ by $C^\oplus$. Thus, we can consider $C^\oplus$ as the set of nonempty finite sequence on C, where two sequences are considered to be equal iff one of them is a permutation of the other one. In the same sense, we denote by $C^{ⓡ}$ the set of all elements of $C^\oplus$ with a dimension r.

A free object in Absem(n,m) with a basis B can be described as wollows.

$L_1 = B$, $E_1=R_1=\{x\in B^\oplus \mid m < d(x) \equiv m \pmod k\}$,

$L_2 = L_1 \cup N_m\times R_1$.

An element $x\in \bigcup_{r\ge 1} L_\lambda^{ⓜ+rk}$ is said to be reducible iff

$x = x'(1,y)(2,y)\dots(m,y)$

where $(\nu,y)\in L_\lambda$.

Then:

$R_\lambda = \{x\in L^\oplus_\lambda \setminus L^\oplus_{\lambda-1} \mid m < d(x) \equiv m \pmod k$, x is reduced$\}$

$L_{\lambda+1} = L_\lambda \cup N_m\times R_\lambda$, $L = \bigcup_{\lambda\ge 1} L_\lambda$, $R = \bigcup_{\lambda\ge 1} R_\lambda$,

$T = \{x \in L^{\oplus} \mid m < d(x) \equiv m (\mathrm{mod}\ k)\}$.

If $x \in T \setminus R$ and

$x = x'(1,y_1)\ldots(m,y_1)\ldots(1,y_t)\ldots(m,y_t)$,

where $x'=1$ or $x' \in L^{\oplus}$ but x' has the least possible dimension, then:

$$\psi'(x) = x'y_1y_2\ldots y_t.$$

Although we have considered several vector (n,m)-varieties, we do not have a general method for producing an effective reduction for free objects in a given vector (n,m)-variety. It is interesting to search for such effective reductions for example in:

the mentioned variety SL(n,m);

the variety $PAb(n,m)=Var\theta$, where $\theta=\{(i_1^n,j_1^n)\}$, $i_\nu=\nu$, $j_1=1$, $j_n=n$, and j_2^{n-1} is a permutation of i_2^{n-1}; and

the variety $Var\theta= U(n,m)$, where $(i_1^p,j_1^q)\in\theta$ implies that $p=q>m$, $\{i_1,\ldots,i_p\}=\{j_1,\ldots,j_p\}$ as sets, and the number of occurences of i_ν in i_1^p is equal to the number of occurences of i_ν in j_1^p.

§8. PRESENTATIONS OF VECTOR VALUED GROUPS

It is natural to ask the question about developing a "combinatorial theory of vector valued groups". The reason for this comes from the facts that the combinatorial group theory is well developed and that the connection between vector valued groups and ordinary groups is very strong.

Since the notion of vector valued groups was not mentioned until now, we give here several characteristic properties of vector valued groups. See [2] for a definition.

<u>Proposition 8.1</u>. *Let Q be a nonempty set, $n,m,k=n-m$ positive integers, and $f, {}^-f, f^-$ three poly-(n,m)-operations on Q satisfying the following:*

(i) $(Q;f)$ is an (n,m)-semigroup,
(ii) For each $s \geq 1$, $x_1^{sk} \in Q^{sk}$, $y_1^m \in Q^m$,

$$f(x_1^{sk}\ {}^{-}f(x_1^{sk}y_1^m)) = y_1^m = f(f^{-}(y_1^m x_1^{sk})x_1^{sk}). \tag{8.1}$$

Then (Q;f) is an (n,m)-group. Conversely, if (Q;f) is an (n,m)-group and if we define $^{-}f, f^{-}$ *by*

$$f^{-}(y_1^m x_1^{sk}) = z_1^m \iff f(z_1^m x_1^{sk}) = y_1^m,$$
$$^{-}f(x_1^{sk}y_1^m) = z_1^m \iff f(x_1^{sk}z_1^m) = y_1^m, \tag{8.2}$$

then (i) and (ii) are satisfied for $Q, f, {}^{-}f, f^{-}$. □

Similar to the constructions of free poly-(n,m)-groupoids, is the construction of free structures with three poly-(n,m)-operations. (Of course, all the corresponding definitions for structures with three poly-(n,m)-operations should be given. But, since our opinion is that they are clear enough or could be reduced to the corresponding notions from universal algebras by using component algebras, we do not give them explicitely here.)

Theorem 8.2. *Let B be a nonempty set, and the sequence of sets* $(G_\alpha \mid \alpha \geq 1)$ *be constructed by:*

$$G_1 = B,\quad G_{\alpha+1} = G_\alpha \cup \{-1,0,1\}\times N_m \times G_\alpha^{(n,m)}.$$

Let $G(B) = \bigcup_{\alpha\geq 1} G_\alpha$ *and let* $f, {}^{-}f, f^{-}$ *be three poly-(n,m)-operations on G(B) defined via their components by*

$$f_i(u_1^{m+sk}) = (0,i,u_1^{m+sk}),\quad {}^{-}f_i(u_1^{m+sk}) = (-1,i,u_1^{m+sk}),$$
$$f_i^{-}(u_1^{m+sk}) = (1,i,u_1^{m+sk}).$$

Then the structure $(G(B);f,{}^{-}f,f^{-})$ *is a free structure with three poly-(n,m)-operations with basis B.* □

Now suppose that $\Delta \subseteq G(B)\times G(B)$ is a set of defining relations. Denote by $\overline{\Delta}$ the least congruence on $(G(B);f,{}^{-}f,f^{-})$ such that:

(iii) $\Delta \subseteq \overline{\Delta}$

(iv) the factor structure $(G(B)/\overline{\Delta};f,{}^{-}f,f^{-})$ satisfies the conditions (i) and (ii). Then (G(B);f) is an (n,m)-group.

We say that (G(B);f) is <u>determined by</u> (or <u>has</u>) <u>the group (n,m)-presentation</u> $\langle B;\Delta\rangle$.

It is not hard to translate the notions of: a realization of (B; Δ) in an (n,m)-semigroup $\underline{Q}$, a universal realization, an (n,m)-identity, a vector (n,m)-presentation, etc., from the class of vector valued semigroups to the class of vector valued groups, and using them, to give the corresponding definitions for group (n,m)-presentations and vector group (n,m)-presentations <B;Λ> and <B;Λ;θ>, via realizations. But we do not give their explicit definitions, because they are clear enough, and because at this moment we are not able to apply them in a fruitfull way. Thus, we are not able to answer the "reduction question" even in some special cases, such as n=m-1, which is connected to the question about the following vector valued variant of Post Theorem: "Every (m+rk,m)-group is an (m+rk,m)-subgroup of an (m+k,m)-group". One of the possible ways for answering the Post Theorem question is to show that the "graphical (m+k,m)-presentation" of an (m+rk,m)-group is proper. Another way is to show that every (m+rk,m)-group is an (m+rk,m)-subgroup of an (m+1,m)-group. There are reasons to believe that the second way would give us sooner a positive answer, since there exists a good combinatorial description of free (m+1,m)-groups.

In the rest of this section we will consider only (m+1,m)-groups, and for such vector valued groups we will give a combinatorial theory different from the previous one, i.e. a theory analogous to the theory of ordinary groups, while using the combinatorial theory of vector valued semigroups developed in the previous sections. First we state the following characterization of (m+1,m)-groups. See [6].

<u>Proposition 8.3</u>. *Let (G;f) be an (m+1,m)-semigroup. Then (G;f) is an (m+1,m)-group iff there are a nullary operation* $e\theta G$ *and a unary operation* $h:G \to G$ *such that:*

(a) $f(x\overset{m}{e}) = f(\overset{m}{e}x)$

(b) $f(x_1^m\overset{m}{e}) = x_1^m$

(c) $f(xh(x)h^2(x)\dots h^m(x)) = \overset{m}{e}$

for every $x\theta G$, $x_1^m\theta G^m$.

Moreover, if $|G|>1$, *then the unary operation* h *is of order* $m+1$, *i.e.* $h^{m+1}=1_G$, *and* $h^k \neq 1_G$ *for* $1 \leq k \leq m$. □

Now, let B be a nonempty set. Let $Y=\bigcup_{i=1}^{m+1} C_i$, where $C_i = B \times \{i\}$, and let $e \notin Y$, $X=\{e\} \cup Y$. Let F(X) be the free poly-(n,m)-groupoid and let Δ be the minimal subset of F(X)×F(X) such that (1) to (4) hold.

(1) $((i,x\overset{m}{e}),(i,\overset{m}{e}x)) \in \Delta$.

(2) $((i,x_1^m\overset{m}{e}),x_i) \in \Delta$.

(3) $((i,(b,1)(b,2)\dots(b,m)(b,m+1)),e) \in \Delta$, $b \in B$.

(4) $((i,x_1^{m+s}),e) \in \Delta$ for each $i \in N_m \Longrightarrow ((i,x_2^{m+s}x_1),e) \in \Delta$ for each $i \in N_m$.

The proof of the following theorem is given in [6].

<u>Theorem 8.4</u>. *The (m+1,m)-semigroup* $\langle X;\Delta\rangle$ *is a free (m+1,m)-group with a basis B. Moreover, there exists an explicit reduction for* $\langle X;\Delta\rangle$. □

Because T.8.4 is the only example, that we know of, for presentations of (m+1,m)-groups where we do have an explicit reduction, at the end of this section we will only briefly mention the general case for presentations of (m+1,m)-groups, without giving the explicit definitions for the used notions and terms.

<u>Proposition 8.5</u>. *Let B,X and Δ be as above. If* $\Lambda \subseteq F(B)\times F(B)$ *is a set of (m+1,m)-defining relations on B, and θ is a set of (m+1,m)-identities, then the semigroup* $\langle X;\Delta\cup\Lambda;\theta\rangle$ *is an (m+1,m)-group in* Varθ. □

R E F E R E N C E S

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