VECTOR VALUED GROUPOIDS, SEMIGROUPS AND GROUPS

G.Čupona, N.Celakoski, S.Markovski, D.Dimovski

CONTENTS

§0. INTRODUCTION

The main aim of this work is an investigation of structures with one vector valued operation, i.e. vector valued groupoids, with a special attention on vector valued semigroups and groups. Almost all the results of this kind of structures known up to now are given here and many new results are obtained too (they are noted in §10).

This work is divided into ten section. In §1 we define the notion of "(n,m)-groupoid" as an ordered pair $\underline{Q}=(Q;f)$, where Q is a nonempty set, n and m are positive integers and $f:Q^n \to Q^m$ is a mapping. In §2 some classes of v.v. groupoids are considered (here and further on, "v.v." is an abbreviation for "vector valued"). Here we define the classes of commutative v.v. groupoids, v.v. quasigroups, v.v. semigroups and v.v. groups, and we investigate some elementary relations between these classes.

Although well-known, the necessary definitions and properties for presentations of semigroups are given in §3, the reason being clearness and completeness of the subsequent investigation. The problem of embedding of a v.v. groupoid into a semigroup, given with a corresponding presentation, is often placed in this work. To any v.v. groupoid $\underline{Q}$ one associates a semigroup $\underline{Q}^\wedge$ which is generated by the set Q and is defined by a set of defining relations. ($\underline{Q}^\wedge$ is called the universal semigroup for $\underline{Q}$.)

In §4 we consider the question for embedding of a v.v. groupoid into a semigroup and we give a complete answer to this question. We consider also the notion of "pure embedding" of (n,m)-groupoids into semigroups and we give a complete answer for $n \geq m$. For the case $n < m$, we show that every free (n,m)-groupoid has the above property; however, we have not a satisfactory description for the class of (n,m)-groupoids which are pure (n,m)-subgroupoids of semigroups when $1 < n < m$.

In §5 the general associative law for v.v. semigroups is proved and a number of characteristic properties of v.v. semigroups and v.v. groups are obtained. The notions "poly-(n,m)-groupoid" and "poly-(n,m)-semigroups" are introduced too, and it is shown that there is no essential difference between the *class* of (n,m)-semigroups and the class of poly-(n,m)-semigroups.

In §6 the free v.v. semigroups are described and it is shown that every free v.v. semigroup is cancellative. Also a description of the universal semigroups of the free v.v. semigroups is given. Further investigations of the universal semigroups for v.v. semigroups, called "universal coverings", are done in §7. Explicit descriptions of universal coverings of v.v. semigroups are obtained and it is shown that every cancellative v.v. semigroup is embedable into a cancellative semigroup.

In §8, v.v. groups are investigated by means of their universal coverings and corresponding v.v. variants of Post and Hosszu-Gluskin Theorems are proved. The main goal of §9 is the investigation of the (n,m)-groups in the cases $n=2m$ and $n=m+1$. It is shown that the theory of (2m,m)-groups in its great part is analogous to the theory of the (usual) groups and that every set is a carrier of a (2m,m)-group. The (m+1,m)-groups are also closely connected to the groups, but the situation is essentially different when existence is in question. Namely, if G is a finite set such that $|G| > 1$, then there is no (m+1,m)-group with a carrier G. By the results of [14] it follows that every infinite set G is a carrier of an (m+1,m)-group. However, we know only one kind of nontrivial (m+1,m)-groups, namely the free ones. Examples of (2m+1,m)-groups are given, and it is shown that there is no finite nontrivial (5,3)-group with an odd order.

Further discussions - the notes and the comments made in §10, are related to the preceding nine section and some results of other papers.

The work ends with an index, a list of notations, and also, according to our knowledge, a complete bibliography on vector valued algebraic structures.

§1. VECTOR VALUED GROUPOIDS

If m and n are positive integers, then an (n,m)-operation on a nonempty set Q is any mapping f from Q^n into Q^m, where Q^s denotes the s-th Cartesian power of Q: $Q^2 = Q\times Q$, $Q^3 = Q\times Q\times Q, \dots$, and $Q^1 = Q$.

For example, a (3,2)-operation on Q is a mapping f: $Q^3 \longrightarrow Q^2$, a (3,3)-operation on Q is a mapping g: $Q^3 \longrightarrow Q^3$, and (1,3)-operation on Q is a mapping h: $Q \longrightarrow Q^3$. In this sense, a (2,1)-operation means a binary operation and an (n,1)-operation means an n-ary operation.

In some cases, when it will not be necessary to emphasize the integers n and m, we will say vector valued operation (v.v.o.) instead of (n,m)-operation.

Let f be an (n,m)-operation on a set Q. We can associate to f a sequence of n-ary operations $f_1, f_2, \dots, f_m$ by putting.

$$((\forall i\in\{1,2,\dots,m\})\ f_i(a_1,\dots,a_n) = b_i) \iff f(a_1,\dots,a_n) = (b_1,\dots,b_m). \quad (1.1)$$

Then we call f_i the i-th component operation of f and we write

$$f = (f_1, f_2, \dots, f_m). \quad (1.2)$$

Conversely, if $f_1, f_2, \dots, f_m$ is a sequence of n-ary operations on the set Q, then there exists a unique (n,m)-operation f on Q such that (1.2) is true.

Thus, every (n,m)-operation f: $Q^n \longrightarrow Q^m$ induces a sequence $f_1, f_2, \dots, f_m$ of n-ary operations on the set Q, and the converse is also true.

If f is an (n,m)-operation on a set Q, then by the analogy with the case $n \geq 2$, $m = 1$, we call the pair $\underline{Q} = (Q;f)$ an (n,m)-groupoid. In that case, if the equality (1.2) is true, we say that $cp\underline{Q} = (Q; f_1, f_2, \dots, f_m)$ is the component algebra of $\underline{Q}$ and $(Q;f_i)$ is the i-th component n-groupoid of $\underline{Q}$.

We will call an (n,m)-groupoid also a vector valued groupoid (v.v.g.).

Here, we will introduce some short notations which will be used frequently further on.

1) The elements of Q^s, i.e. the sequences $(a_1,a_2,\dots,a_s)$ $(a_i\in Q)$ will be denoted by $a_1a_2\cdots a_s$ or a_1^s; in some cases when there will be no risk of misunderstanding the sequence $(a_1,a_2,\dots,a_s)$ will be denoted by one letter, often underlined: $\underline{a}$. Thus the symbols

$$(a_1,a_2,\dots,a_s);\ a_1a_2\cdots a_s;\ a_1^s;\ \underline{a}$$

will denote the very same object, namely an element of Q^s.

2) The symbol x_i^j will denote the sequence $x_ix_{i+1}\cdots x_j$ when $i\le j$, and the empty sequence when $i>j$.

3) If $x_1=x_2=\dots=x_p=x$ $(x_i\in Q)$, then the sequence x_1^p is denoted by the symbol $\overset{p}{x}$.

4) The set $\{1,2,\dots,s\}$ will be denoted by N_s and by N_o sometimes will be denoted the empty set. The set of positive integers will be denoted by N.

Let us return to the v.v. groupoids. The fact that we can associate the algebra cp$\underline{Q}$ to a given v.v.groupoid $\underline{Q}$, allows us to carry over all the notions which make sense for universal algebras to v.v. groupoids without giving their explicit definitions. Such notions are: a subgroupoid, a direct product, a homomorphism, a congruence, a factor groupoid etc.

For example, if $(Q;f)$ is an (n,m)-groupoid, then a nonempty subset P of Q is an (n,m)-*subgroupoid* of $(Q;f)$ iff:

$$a_1^n\in P^n\ \&\ f(a_1^n) = b_1^m \Longrightarrow b_1^m\in P^m. \tag{1.3}$$

Another example: if $\underline{Q} = (Q;f)$ and $\underline{Q}' = (Q';f')$ are (n,m)-groupoids, then a mapping ϕ: $x\mapsto \bar{x}$ from Q into Q' is a *homomorphism* from $\underline{Q}$ into $\underline{Q}'$ iff:

$$f(a_1^n) = b_1^m \Longrightarrow f'(\bar{a}_1^n) = \bar{b}_1^m. \tag{1.4}$$

One can define the other mentioned notions analogously.

Next we give a description of a free (n,m)-groupoid with a given basis B $(B\neq\emptyset)$, which ilustrates the use of homomorphisms as well. The description is simple enough, since it is the (absolute) free algebra with the basis B and signature $F=\{f_1,f_2,\ldots,f_m\}$, where f_i is an n-ary operator and $f_i \neq f_j$ for $i \neq j$. This algebra can be described in the following way.

Put $H_o=B$ and suppose that the sequence $H_o,H_1,\ldots,H_p$ of disjoint sets is built up. Let C_p be the set defined by:

$$C_p = \{u_1^n \in (H_o \cup H_1 \cup \ldots \cup H_p)^n \mid (\exists i \in N_n) u_i \in H_p\}.$$

Then the set H_{p+1} is defined by:

$$H_{p+1} = N_m \times C_p.$$

Without loss of generality, we suppose that $H_{p+1} \cap H_i = \emptyset$ for every $i \in N$. Further on we will make such suppositions without an explicite explanation.

If $u \in H_p$, then we say that the element u has the <u>hierarchy</u> p and we write $\chi(u) = p$.

Now, put $\overline{B} = \bigcup_{p \geq 0} H_p$ and define n-ary operations $f_1,f_2,\ldots,f_m$ in $\overline{B}$ by:

$$(\forall i \in N_m)\ f_i(u_1^n) = (i,u_1^n). \qquad (1.5)$$

So we obtain an (n,m)-groupoid $(\overline{B};f)$, where $f=(f_1,f_2,\ldots,f_m)$.

The following theorem shows that $(\overline{B};f)$ is a <u>free</u> (n,m)-<u>groupoid</u> with a basis B.

<u>Theorem 1.1</u>. *If $\underline{Q} = (Q;g)$ is an (n,m)-groupoid, and ξ is a mapping from a nonempty set B into Q, then there exists a unique homomorphism ζ of the (n,m)-groupoid $(\overline{B};f)$ into the groupoid $(Q;g)$ which is an extension of ξ.*

<u>Proof</u>. Let $\underline{D}$ be an (n,m)-subgroupoid of $(\overline{B};f)$ such that $B=H_o \subseteq D$. Suppose that every element which has a hierarchy less than or equal to p is in D. If $\chi(u)=p+1$, then

$$u = (i,u_1^n) = f_i(u_1^n) \in D.$$

Thus, $H_{p+1} \subseteq D$ and so $\overline{B} = D$, which means that B is a generating set of $(\overline{B};f)$.

Let $(Q;g)$ be an (n,m)-groupoid and let $\xi: B \longrightarrow Q$ be a mapping. So we have a mapping ζ defined on H_o. Further on we use induction on hierarchy. If $\chi(u)=p+1$, then $u=(i,u_1^n)$, where $\chi(u_\nu) \le p$, and so $\zeta(u_\nu)=\overline{u}_\nu \in Q$. Setting

$$\zeta(u) = g_i(\overline{u}_1^n),$$

we obtain

$$\zeta(f_i(u_1^n)) = g_i^*(\overline{u}_1^n).$$

Thus, ζ is a homomorphism which is an extension of ξ. The fact that B is a generating set implies the conclusion. □

<u>Example 1.2</u>. Let $A=\bigcup_{\alpha\ge 0} N_m^\alpha$ and define unary operations g_i $(i\in N_m)$ on A by:

$$u\in N_m^\alpha \Longrightarrow g_i(u) = iu\in N_m^{\alpha+1}$$

for $i\in N_m$. Let $g=(g_1,g_2,\ldots,g_m)$. Then it is easily seen that $(A;g)$ is isomorphic to the free $(1,m)$-groupoid with a basis consisting of m elements. Thus, if $m=1$, then $(A;g)$ is isomorphic to $(N;s)$, where $s(x)=x+1$ for all $x\in N$.

§2. SOME TYPES OF VECTOR VALUED GROUPOIDS

Here we will consider the problem of classification of v.v. groupoids in general. A corresponding classification can be made according to assumed useful properties or by following the classifications of the binary groupoids. Here we will use the second possibility.

Suppose that $\mathcal{G}$ is a class of groupoids and consider the problem of defining a set $\{\mathcal{G}(n,m) \mid n,m\ge 1\}$ such that, for every $n,m\ge 1$, $\mathcal{G}(n,m)$ is a class of (n,m)-groupoids, where $\mathcal{G}(2,1)=\mathcal{G}$. Of course, this problem is not uniquely solvable because the only demand $\mathcal{G}(2,1)=\mathcal{G}$ has not some essential limitation.

One of the possibly „acceptable" solutions is the component definition. In that case, it is necessary first to define the set $\{\mathcal{G}(n) \mid n\ge 1\}$ such that for every $n\ge 1$, $\mathcal{G}(n)$ is a class of n-groupoids which satisfies the condition $\mathcal{G}(2)=\mathcal{G}$.

Then, $\mathcal{G}(n,m)$ will be defined by:

„$(Q;f)\in\mathcal{G}(n,m) \iff (\forall i\in N_m)(Q;f_i)\in\mathcal{G}(n)$",

where f_i is the i-th component operation of f, i.e. $f=(f_1,f_2,\ldots,f_m)$.

For example, if $\mathcal{G}$ is some of the classes

(a) commutative groupoids, (b) cancellative groupoids,

(c) quasigroups, (d) semigroups, (e) groups,

then the definition of $\mathcal{G}(n)$ for every $n\geq 1$[1] is well-known in every of the above five cases. Therefore, if we use the component method, we will obtain five classes of (n,m)-groupoids and it is not necessary to define them explicitly. However, these definitions, except in the case of commutativity, are not equivalent with the corresponding "direct" v.v. definitions.

(a) An (n,m)-groupoid (Q;f) is said to be commutative iff for every permutation $\sigma\in S_n$ the following identity holds

$$f(x_1^n) = f(\sigma(x_1^n)), \tag{2.1}$$

where $\sigma(x_1^n) = x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$.

This definition makes sense for every $n,m\geq 1$. On the other hand, it is clear that an (n,m)-groupoid is commutative iff each one of its component n-groupoid is commutative. Therefore the given direct definition (2.1) of the concept of commutative (n,m)-groupoid coincides with the corresponding component definition.

Here we will define a congruence in the (n,m)-groupoid $(\bar{B};f)$ (see T.1.1) which will bring us to a free c o m m u t a t i v e (n,m)-groupoid.

Define a relation $\approx$ on the free (n,m)-groupoid $(\bar{B};f)$ with a basis B by induction on hierarchy, i.e. if $u,v\in\bar{B}$, then:

1) $\chi(u) = 0 \implies (u \approx v \text{ iff } u = v)$,

2) $\chi(u) = p+1 \implies (u \approx v \text{ iff } u = (i,u_1^n),\ v = (i,v_1^n)$ for some

[1] *1-groupoid is in fact a unar and so the cancellation is equivalent with injectivity, the quasigroupness with bijectivity, while the commutativity and associativity are always satisfied.*

$i\in N_m$ and there exists a permutation $(i_1,i_2,\dots,i_n)$ of $(1,2,\dots,n)$ such that $v_\nu = u_{i_\nu}$ for every ν).

Then the following statement is true:

Proposition 2.1. *The relation $\approx$ is a congruence on $(\overline{B};f)$ and the factor (n,m)-groupoid $(\overline{B}/\approx;f)$ is a free commutative (n,m)-groupoid with a basis B.* □

The direct definitions of the corresponding generalizations of the classes (b)-(e) differ essentially from the component ones.

Two different non-component definitions for cancellative v.v. groupoids and v.v. quasigroups can be immediate realized: one for arbitrary n,m and the other for $n-m\geq 1$.

Namely, an (n,m)-groupoid $(Q;f)$ is said to be a partial (n,m)-quasigroup iff the following condition is satisfied:

(b.1) If $x_\nu, y_\nu \in Q$ are such that

$$f(x_1^n) = x_{n+1}^{n+m}, \quad f(y_1^n) = y_{n+1}^{n+m}$$

and if there exists a sequence of positive integers $i_1,i_2,\dots,i_n$ such that $1\leq i_1 < i_2 < \dots < i_n \leq n+m$, $x_{i_\nu} = y_{i_\nu}$, for every $\nu\in N_n$, then

$$x_1^{n+m} = y_1^{n+m}.$$

Examples 2.2. 1) Let + denote the usual addition on N, and define an n-ary operation f on N by $f(x_1^n)=x_1+x_2+\dots+x_n$. Then one obtains a partial $(n,1)$-quasigroup.

2) Define a $(2,2)$-operation f on Z by $f(x_1^2)=(x_1+x_2,\ x_1-x_2)$. Then one obtains a partial $(2,2)$-quasigroup.

3) A partial (n,m)-quasigroup $(Z;f)$ for every $n,m\geq 1$ one obtains if f is defined by:

$$f(x_1^n) = y_1^m \iff y_i = \sum_{j=1}^{n} j^{i-1} x_j, \quad i\in N_m.$$

An (n,m)-groupoid $(Q;f)$ is called an (n,m)-quasigroup iff the following condition is satisfied:

(c.1) For every element $y_1^n\in Q^n$ and for every sequence of positive integers i_1^n such that $i_\nu < i_{\nu+1}$ there exists a uniquelly determined element $x_1^{n+m} \in Q^{n+m}$ such that $f(x_1^n) = x_{n+1}^{n+m}$ and $y_\nu = x_{i_\nu}$ for every $\nu \in N_n$.

(Clearly, every quasigroup is a (2,1)-quasigroup.)

Obviously, the following statement is true:

Proposition 2.3. *Every (n,m)-quasigroup is a partial (n,m)-quasigroup.* □

E.2.2. shows that the converse of P.2.3. is not true, i.e. there are partial (n,m)-quasigroups which are not (n,m)-quasigroups.

Example 2.4. Let $Q=Z_7$ and define a (3,3)-operation f on Q by:

$$f(x_1^3) = y_1^3 \iff y_1 = x_1 + x_2 + x_3,\ y_2 = x_1 + 2x_2 + 3x_3,\ y_3 = x_1 + 4x_2 + 2x_3$$

Then it is easy to check that (Q;f) is a (3,3)-quasigroup.

Now suppose that $n-m=k \geq 1$. An (n,m)-groupoid (Q;f) is said to be cancellative iff every implication of the following form is true in (Q;f):

(b.2) $f(a_1^i x_1^m a_{i+1}^k) = f(a_1^i y_1^m a_{i+1}^k) \implies x_1^m = y_1^m$,

for every $i \in \{0,1,\ldots,k\}$.

Using a shortened notation, we can write this implication in the following way:

(b.2′) $f(\underline{a}\underline{x}\underline{b}) = f(\underline{a}\underline{y}\underline{b}) \implies \underline{x} = \underline{y}$,

where $\underline{a}\underline{b} \in Q^k$ and $\underline{x}, \underline{y} \in Q^m$.

An (n,m)-groupoid (Q;f) is called a weak (n,m)-quasigroup ($n-m=k \geq 1$), iff the following condition is satisfied:

(c.2) For every $\underline{a}\underline{b} \in Q^k$ and $\underline{c} \in Q^m$, there exists a unique element $\underline{x} \in Q^m$ such that

$$f(\underline{a}\underline{x}\underline{b}) = \underline{c}.$$

The following proposition follows directly from the preceding definitions:

Proposition 2.5. *If $n-m=k\geq 1$, then:*

(i) Every (n,m)-quasigroup is a weak (n,m)-quasigroup.

(ii) Every weak (n,m)-quasigroup is a cancellative (n,m)-groupoid.

(iii) Every partial (n,m)-quasigroup is a cancellative (n,m)-groupoid. □

Let $n-m=k\geq 1$. An (n,m)-groupoid $\underline{Q}=(Q;f)$ is called an (n,m)-semigroup iff the following equality

(d.1) $f(f(x_1^n)x_{n+1}^{n+k}) = f(x_1^j f(x_{j+1}^{j+n})x_{j+n+1}^{n+k})$

is an identity in $(Q;f)$, for every $j\in N_k$.

Examples 2.6. Here we suppose that Q is a nonempty set and that $n-m=k\geq 1$.

1) Fix an element $a_1^m\in Q^m$ and put $f(x_1^n)=a_1^m$ for every $x_1^n\in Q^n$. Then $(Q;f)$ is an (n,m)-semigroup, called a constant (n,m)-semigroup on Q.

2) Define an (n,m)-operation f on Q by $f(x_1^n)=x_1^m$ for every $x_1^n\in Q^n$. Then $(Q;f)$ is an (n,m)-semigroup, called a left zero (n,m)-semigroup. Dually, a right zero (n,m)-semigroup $(Q;g)$ can be defined by $g(x_1^n)=x_{k+1}^n$.

3) Let $Q=A\times B=\{(a,b)\mid a\in A,\ b\in B\}$ and define an (n,m)-operation f on Q by

$$f(c_1^n) = d_1^m \iff (c_i=(a_i,b_i),\ d_j=(a_j,b_{j+k}),\ i\in N_n,\ j\in N_m).$$

Then $(Q;f)$ is an (n,m)-semigroup, and it is a direct product of a left-zero (n,m)-semigroup on A and a right zero (n,m)-semigroup on B. We say that $(Q;f)$ is an (n,m)-rectangular band.

4) Let $(Q;g)$ be a $t+1$-semigroup, $t\geq 1$, $m\geq 1$ and $n=(t+1)m$. Define an (n,m)-operation f on Q, by:

$$f(x_1^n) = y_1^n \iff y_i = g(x_i x_{i+m}\cdots x_{i+tm}),\quad i\in N_m.$$

Then $(Q;f)$ is an (n,m)-semigroup.

Note that a given semigroup $(Q;\cdot)$ induces a $t+1$-semigroup $(Q;g)$ by $g(x_1^{t+1})=x_1\cdot x_2\cdots x_{t+1}$. Together with the above, this gives new examples of $((t+1)m,m)$-semigroups.

5) Given a lattice $(Q;\wedge,\vee)$ one can define a (3,2)-semigroup $(Q;f)$ by

$$f(x_1^3) = (x_1\wedge x_2\wedge x_3,\ x_1\vee x_2\vee x_3).$$

Similarly, if $(Q;\wedge)$ is a semilattice and if f is defined by

$$f(x_1^n) = y_1^m \iff y_i = x_1\wedge x_2\wedge\ldots\wedge x_n,\ i\in N_m,$$

then one obtains an (n,m)-semigroup $(Q;f)$.

6) Let Q=N and define f by

$$f(x_1^n) = (1,\ldots,1,\ x_1\cdot x_2\cdots x_n).$$

Then $(Q;f)$ is an (n,m)-semigroup.

An (n,m)-groupoid $\underline{Q}=(Q;f)$ is called an (n,m)-<u>group</u> iff:

(e.1) $\underline{Q}$ is an (n,m)-semigroup; and

(e.2) $(\forall \underline{a}\in Q^k)(\forall \underline{b}\in Q^m)(\exists \underline{x},\underline{y}\in Q^m)(f(\underline{a}\underline{x})=\underline{b}=f(\underline{y}\underline{a}))$.

<u>Examples 2.7</u>. 1) Let $(Q;g)$ be a t+1-group and k=tm, $t\geq 1$. Then $(Q;f)$, where f is defined in E.2.6. 4), is an (n,m)-group, n=(t+1)m.

2) If $(G;+)$ is an abelian group, then $(G;f)$, where $f_i(a_1^m b c_1^m) = a_i-b+c_i$, is a (2m+1,m)-group.

3) Let $G=Z_4=\{0,1,2,3\}$ and $\overline{0}=\overline{2}=0$, $\overline{1}=\overline{3}=1$. Then $(G;f)$, where

$$f(xyzt) = (x+z-\overline{y}-\overline{t}+\overline{y+t},\ y+t-\overline{x}-\overline{z}+\overline{x+z})$$

is a (4,2)-group.

4) Let $\phi: R^2 \longrightarrow R^2$ be defined by

$$\phi(s,u) = (s+\tfrac{1}{2}\sin u,\ u+\tfrac{1}{2}\sin s),$$

where R is the set of real numbers. Then ϕ is a bijection and $(R;f)$, where

$$f(stuv) = \phi^{-1}(s+u+\tfrac{1}{2}(\sin t+\sin v),\ t+v+\tfrac{1}{2}(\sin s+\sin u)),$$

is a (4,2)-group.

5) If $f: C^5 \longrightarrow C^2$, where C is the set of complex numbers, is defined by:

$$f(z_1^5) = (z_1+z_4 - \frac{1+i\sqrt{3}}{2} z_3,\ z_2+z_5 - \frac{1-i\sqrt{3}}{2} z_3),$$

then (C;f) is a (5,2)-group.

The proof of the following proposition will be given in § 8.

Proposition 2.8. *If $\underline{Q}=(Q;f)$ is an (n,m)-semigroup, then the following statements are equivalent:*

(i) $\underline{Q}$ is an (n,m)-group;

(ii) $(\forall \underline{a}\in Q^k)(\forall \underline{b}\in Q^m)(\exists !\underline{x},\underline{y}\in Q^m)(f(\underline{a}\underline{x})=\underline{b}=f(\underline{y}\underline{a}))$;

(iii) $\underline{Q}$ is a weak (n,m)-quasigroup. □

As consequences of the above proposition one can obtain the following two descriptions of v.v. groups.

Corollary 2.9. *An (n,m)-semigroup $(Q;f)$ is an (n,m)-group iff there exist (n,m)-groupoids $(Q;{}^{-}f)$ and $(Q;f^{-})$ such that for every $\underline{a}\in Q^k$, $\underline{b}\in Q^m$*

$$f(\underline{a}\,{}^{-}f(\underline{a}\underline{b})) = \underline{b},\quad f(f^{-}(\underline{b}\underline{a})\underline{a}) = \underline{b}.\ \square$$

Corollary 2.10. *Let $n-m=k \geq 2$. An (n,m)-semigroup $(Q;f)$ is an (n,m)-group iff there exist a positive integer $i\in N_{k-1}$ and an (n,m)-groupoid $(Q;f^{-i})$ such that for every $\underline{a}\in Q^i$, $\underline{b}\in Q^{k-i}$, $\underline{c}\in Q^m$*

$$f(\underline{a}f^{-i}(\underline{a}\underline{c}\underline{b})\underline{b}) = \underline{c}.\ \square$$

The following proposition is obviously true:

Proposition 2.11. *Let $n \geq 2$. If $\underline{Q}=(Q;f)$ is an $(n,1)$-groupoid, then:*

(i) $\underline{Q}$ is an n-quasigroup <=> $\underline{Q}$ is an $(n,1)$-quasigroup <=> $\underline{Q}$ is a weak $(n,1)$-quasigroup.

(ii) $\underline{Q}$ is an n-semigroup <=> $\underline{Q}$ is an $(n,1)$-semigroup.

(iii) $\underline{Q}$ is an n-group <=> $\underline{Q}$ is an $(n,1)$-group. □

Further on we will say also: v.v. semigroup, v.v. group,... instead of (n,m)-semigroup, (n,m)-group,... .

If $\mathcal{C}$ is a property of v.v. groupoids, then we say that a v.v. semigroup has the property $\mathcal{C}$ iff it has that property as a v.v. groupoid. Therefore, the meanings of the following expessions are clear: "a commutative v.v. semigroup", "a cancellative v.v. semigroup", etc.

In this sense, it could be said that an (n,m)-group is commutative iff it is commutative as an (n,m)-groupoid. However, if $n,m \geq 2$, there are no nontrivial commutative (n,m)-groups as we can see by the following

Proposition 2.12. *Let $n,m \geq 2$. If $\underline{Q}=(Q;f)$ is a commutative (n,m)-groupoid, then the following statements are equivalent:*

(i) $|Q|=1$, i.e. Q is a one-element set.

(ii) $\underline{Q}$ is a cancellative (n,m)-groupoid.

(iii) $\underline{Q}$ is an (n,m)-group.

Proof (ii) ⟹ (i). Let (ii) be true and $a,b \in Q$. Since $\underline{Q}$ is commutative, we have

$$f(\overset{k+1}{a}\ \overset{m-1}{b}) = f(\overset{k}{a}\, b\, a\ \overset{m-2}{b})$$

and because of (ii), i.e. (b.2), it follows that

$$a\overset{m-1}{b} = ba\overset{m-2}{b} \quad \text{i.e. } a = b.$$

(iii) ⟹ (ii). Follows from P.2.5. and P.2.8. □

Each of the defined classes of v.v. groupoids can be described by the component operations too. So, if $cp\underline{Q}=(Q;f_1,f_2,\dots,f_m)$ is the component algebra which corresponds to the (n,m)-groupoid $\underline{Q}=(Q;f)$, then $\underline{Q}$ is an (n,m)-semigroup iff the following equality

$$\begin{aligned} &f_i(f_1(x_1^n)f_2(x_1^n)\cdots f_m(x_1^n)x_{n+1}^{n+k}) = \\ &= f_i(x_1^j f_1(x_{j+1}^{j+n})\cdots f_m(x_{j+1}^{j+n})x_{j+n+1}^{n+k}) \end{aligned} \tag{ij}$$

is an identity in $cp\underline{Q}$, for every $j \in N_k$ and $i \in N_m$.

Note that the identity (ij) can be written in the following „component-vector" form:

$$f_i(f(x_1^n)x_{n+1}^{n+k}) = f_i(x_1^j f(x_{j+1}^{j+n})x_{j+n+1}^{n+k}) \tag{ij*}$$

for every $j \in N_k$, $i \in N_m$.

If $\mathscr{C}(n,m)$ is a class of (n,m)-groupoids, then the question for a suitable description of the free objects in the class $\mathscr{C}(n,m)$ (if such objects do exist) occurs naturally.

By the construction of the free (n,m)-groupoid in §1 and the definition of partial (n,m)-quasigroup, it is clear that:

Proposition 2.13. *Every free (n,m)-groupoid ($\overline{B}$;f), with a basis B, is a partial (n,m)-quasigroup.*

Proof. Let the conditions of (b.1) be satisfied. If there exists a positive integer ν such that $i_\nu > n$, i.e. $x_{n+i}=y_{n+i}$ for some $i \in N_m$, then

$$f_i(x_1^n) = f_i(y_1^n), \text{ i.e. } (i,x_1^n) = (i,y_1^n), \text{ i.e. } x_1^n = y_1^n,$$

which implies $x_1^{n+m} = y_1^{n+m}$.

If all $i_\nu \le n$, then we have again $x_1^n = y_1^n$, which implies the same conclusion. □

§3. PRESENTATIONS OF SEMIGROUPS

If $\underline{Q}$=(Q;f) is a given v.v. groupoid, then we can associate to $\underline{Q}$ a semigroup $\underline{Q}^\wedge$, obtained by a corresponding presentation induced by $\underline{Q}$. Therefore we will make a brief discussion of presentations in the class of semigroups.

Let B be a nonempty set and denote by B^+ the set of all non empty finite sequences on B, i.e.

$$B^+ = \bigcup_{p\ge 1} B^p.$$

Sometimes the elements of B^+ will be called words in B. Thus,

$$B^+ = \{b_1 b_2 \cdots b_t \mid b_\nu \in B,\ t \ge 1\},$$

and if $a_\nu, b_\lambda \in B$, $p,q \ge 1$, then

$$a_1 a_2 \cdots a_p = b_1 b_2 \cdots b_q \iff p = q \text{ and } (\forall \nu \in N_p)\ a_\nu = b_\nu.$$

We denote by B^* the set $B^+ \cup \{1\}$, where 1 denotes the empty word and $1 \notin B^+$.

If $u \in B^p \subseteq B^*$, then p is called the dimension of u and we write p=d(u). (Thus, d(1)=0).

The set B^+ with the operation of concatenation of sequences is a semigroup, and moreover it is a free semigroup with a basis B.

In other words, if $\underline{S}=(S;\cdot)$ is a semigroup and $\xi: B \longrightarrow S$ is a mapping, then there exists a unique homomorphism ξ^+ from B^+ into $\underline{S}$ which is an extension of ξ. So, if $u=b_1b_2\cdots b_t \in B^+$, then

$$\xi^+(u) = \xi(b_1)\cdot\xi(b_2)\cdot\ldots\cdot\xi(b_t).$$

The set B^* with the operation of concatenation is a free monoid with a basis B. So, if $\underline{S}=(S;\cdot)$ is a monoid with the identity e and $\xi: B \longrightarrow S$ is a mapping, then ξ can be extended, in a unique way, to a homomorphism ξ^* from B^* into $\underline{S}$, where

$$\xi^*(1)=e \quad \& \quad (\forall u\in B^+)\,\xi^*(u)=\xi^+(u).$$

Further on we will often write $\xi(u)$ instead of $\xi^+(u)$ and $\xi^*(u)$.

Any subset Λ of the set $B^+\times B^+$[1] is called a set of defining relations on B, and the pair $\langle B;\Lambda\rangle$ is called a (semigroup) presentation.

Let $\underline{S}=(S;\cdot)$ be a semigroup and $\xi: B \longrightarrow S$ be a mapping such that $\xi(u)=\xi(v)$[2], for every pair $(u,v)\in\Lambda$. Then we say that ξ is a realization of the pair (B,Λ) in $\underline{S}$. If, moreover, for any realization ξ' of (B,Λ) in a semigroup $\underline{S}'=(S';\cdot)$ there exists a unique homomorphism $\zeta: \underline{S} \longrightarrow S'$ such that $\xi'=\zeta\xi$, then we say that the semigroup $\underline{S}$ is determined by the presentation $\langle B;\Lambda\rangle$.

Proposition 3.1. *If $\underline{S}$ and $\underline{T}$ are two semigroups determined by a presentation $\langle B;\Lambda\rangle$, then $\underline{S}$ and $\underline{T}$ are isomorphic.* □

Further on we will usually write „$\underline{S} = \langle B;\Lambda\rangle$" instead of „$\underline{S} = (S;\cdot)$ is a semigroup determined by the presentation $\langle B;\Lambda\rangle$". We will also write $S=\langle B;\Lambda\rangle$. Thus, $\langle B;\Lambda\rangle$ will have three meanings.

Proposition 3.2. *Let $\overset{\Lambda}{=}$ be the least congruence on the semigroup B^+ containing Λ. Then $B^+/\overset{\Lambda}{=} \; = \langle B;\Lambda\rangle$.* □

Here we give a more explicit description of the congruence $\overset{\Lambda}{=}$.

---

1) *To avoid any confusion, the elements of $B^+\times B^+$ will be denoted by (u,v), where $u,v\in B^+$.*

2) *We write ξ instead of ξ^+ (as we said above).*

First, if $u,v\in B^+$ are such that $u=u_1u'u_2$, $v=u_1v'u_2$, where $(u',v')\in\Lambda$, and $u_1,u_2\in B^*$, then we write $u \overset{\Lambda}{\longmapsto} v$. Let $\overset{\Lambda}{\sim}$ be the symmetric extension of $\overset{\Lambda}{\longmapsto}$, i.e.

$$u \overset{\Lambda}{\sim} v \iff u \overset{\Lambda}{\longmapsto} v \text{ or } v \overset{\Lambda}{\longmapsto} u.$$

Finally, let $\overset{\Lambda}{=}$ be the reflexive and transitive extension of $\overset{\Lambda}{\sim}$, i.e. $u \overset{\Lambda}{=} v$ iff there exist $u_0,u_1,\dots,u_t\in B^+$, such that $t\geq 0$, $u=u_0$, $v=u_t$, and $u_{i-1} \overset{\Lambda}{\sim} u_i$ for any $i\in N_t$.

If $u\in B^+$, then we denote by u^Λ the element of $B^+/\overset{\Lambda}{=}$ containing u, i.e.

$$u^\Lambda = \{v\in B^+ \mid u \overset{\Lambda}{=} v\}. \tag{3.1}$$

The <u>dimension</u> $\bar{d}(u^\Lambda)$ of u^Λ is defined by:

$$\bar{d}(u^\Lambda) = \{d(v) \mid v\in u^\Lambda\}. \tag{3.2}$$

A presentation $\langle B;\Lambda\rangle$ is said to be <u>proper</u> iff

$$(\forall a,b\in B)(a\overset{\Lambda}{=}b \implies a=b). \tag{3.3}$$

In this case we may assume that B is a subset of $B^+/\overset{\Lambda}{=}$. (We note that the assertion „$\langle B;\Lambda\rangle$ is proper", does not mean „$(\forall b\in B)\, b^\Lambda=\{b\}$".)

We will denote by $\langle B;\Lambda\rangle$ the semigroup $B^+/\overset{\Lambda}{=}$. In order to simplify the notations, we will write u instead of u^Λ, i.e. we will use the elements of B^+ as „names" for the elements of $\langle B;\Lambda\rangle$. Thus, „$u=v$ in $\langle B;\Lambda\rangle$" means „$u \overset{\Lambda}{=} v$", and $\bar{d}(u)=\{d(v) \mid u \overset{\Lambda}{=} v\}$. Of course, if $\langle B;\Lambda\rangle$ is not proper, it may happen $a=b$ in $\langle B;\Lambda\rangle$, but $a\neq b$ in B.

<u>Proposition 3.3</u>. *Let m and k be two positive integers such that for every $(u,v)\in\Lambda$, $d(u)\geq m$, $d(v)\geq m$ and $d(u)\equiv d(v)\equiv m \pmod k$.*

a) If $w_1\in B^+$ and $d(w_1)<m$, then for all $w_2\in B^+$, $w_1\overset{\Lambda}{=}w_2$ iff $w_1=w_2$.

b) If $m>1$, then the presentation $\langle B;\Lambda\rangle$ is proper and moreover we may assume that

$$B\cup B^2\cup\dots\cup B^{m-1}\subset \langle B;\Lambda\rangle.$$

c) If $u,v\in B^+$ and $u \overset{\Lambda}{=} v$, then $d(u)\equiv d(v) \pmod k$. □

It is usually desirable to have a procedure for answering the question whether or not two given words $u,v\in B^+$ are equal in $\langle B;\Lambda\rangle$. For this aim one uses very often a special mapping ψ of B^+ into B^+ which associate, to every equivalent class u^Λ ($u\in B^+$), a uniquely determined element $\psi(u)\in u^\Lambda$.

Namely, a mapping $\psi\colon B^+ \longrightarrow B^+$ is called a reduction for $\langle B;\Lambda\rangle$ iff the following conditions are satisfied:

(i) $\psi(uvw) = \psi(u\psi(v)w)$ for every $u,w\in B^*$, $v\in B^+$.

(ii) $(u,v)\in\Lambda \implies \psi(u)=\psi(v)$.

(iii) $\psi(u) \overset{\Lambda}{=} u$ for every $u\in B^+$.

Proposition 3.4. *If ψ is a reduction for $\langle B;\Lambda\rangle$, then $\psi(u)=\psi(v)$ iff $u \overset{\Lambda}{=} v$, for all $u,v\in B^+$, i.e. $\ker\psi = \overset{\Lambda}{=}$.* □

Proposition 3.5. *Let $\psi\colon B^+ \longrightarrow B^+$ be a reduction for $\langle B;\Lambda\rangle$ and let $S=\psi(B^+)$. If the operation $\bullet$ is defined on S by:*

$$(\forall u,v\in S)\ u\bullet v = \psi(uv), \tag{3.4}$$

then $(S;\bullet) = \langle B;\Lambda\rangle$.

Proof. First, (i) implies that $\psi^2=\psi$, i.e. ψ is a retract and, moreover, ψ is a surjective homomorphism from B^+ onto $(S;\bullet)$. The conditions (ii) and (iii) imply that $\overset{\Lambda}{=}$ is a kernel of ψ, and therefore $B^+/\overset{\Lambda}{=}$ is isomorphic to $(S;\bullet)$. □

Proposition 3.6. *A presentation $\langle B;\Lambda\rangle$ is proper iff there exists a reduction $\psi\colon B^+ \longrightarrow B^+$ for $\langle B;\Lambda\rangle$ such that $(\forall b\in B)\ \psi(b)=b$.* □

We note that any presentation admits a reduction. Namely, let S be a subset of B^+ such that for every $u\in B^+$ there exists exactly one element $v\in S$ which satisfies the relation $u \overset{\Lambda}{=} v$, i.e. S contains one and only one element from each class of the equivalence $\overset{\Lambda}{=}$. Then a reduction ψ can be defined by:

$$(\forall u\in B^+)\big(\psi(u)\in S \ \ \&\ \ u \overset{\Lambda}{=} \psi(u)\big).$$

Now we will apply the notion of reduction for a presentation in order to prove a v.v. variant of Cohn-Rebane's Theorem. First we will introduce two more notions.

A partial (n,m)-operation (p.v.v.o.) on a set A is a mapping f: $\mathcal{D} \to A^m$, where $\mathcal{D} = \mathcal{D}_f$ („the domain of f") is a subset of A^n. If F is a set of p.v.v.o. on the set A, then $\underline{A}=(A;F)$ is called a partial vector valued algebra (p.v.v.a.).

Let $\underline{A}=(A;F)$ be a p.v.v.a., and let F´ be a set such that $F' \cap (A \cup F) = \emptyset$ and $f \mapsto f'$ is a bijection from F to F´. Denote by B the set $A \cup F'$, and consider the presentation $\langle B;\Lambda\rangle$, where

$$\Lambda = \{(f'a_1^n, b_1^m) \mid f(a_1^n) = b_1^m \text{ in } \underline{A}\}.$$

If $u \in B^+$, then we denote by dg(u) (= degree of u) the number of occurences of elements of F´ in the word u. (Namely, dg(a)=0, dg(f´)=1, dg(uv)=dg(u)+dg(v), for any $a \in A$, $f \in F$, $u,v \in B^+$.)

If $u \in B^+$ has a form $u=u'f'a_1^n u''$, where $u',u'' \in B^*$, $f \in F$, $a_1^n \in \mathcal{D}_f$ (= the domain of f), then u is said to be reducible. We say that $u \in B^+$ is reduced if it is not reducible. We define a reduction ψ: $B^+ \to B^+$ by induction on degree. Denote by S the set of reduced words. Then

(0) $\quad (\forall u \in S)\ \psi(u) = u.$

Let $u=u'f'a_1^n u''$ be a reducible word, where $f \in F$, $a_1^n \in \mathcal{D}_f$ and u´ has the least possible dimension. If $b_1^m = f(a_1^n)$ in $\underline{A}$, and $v=u'b_1^m u''$, then dg(v)=dg(u)-1. Therefore, by the inductive hypothesis, $\psi(v) \in S$ is well defined. Now we define $\psi(u)$ by:

(1) $\quad \psi(u) = \psi(v)$

So, we have defined a mapping ψ: $B^+ \to S$, and by induction on dimensions and degrees it can be shown that the conditions (i), (ii), (iii) are satisfied, i.e. ψ is a reduction for the presentation $\langle B;\Lambda\rangle$. Therefore $(S;\bullet)=\langle B;\Lambda\rangle$, where „•" is defined by (3.4).

Clearly, $A \cup F' \subseteq S$, i.e. ψ is a proper reduction.

If $f(a_1^n)=b_1^m$ in $\underline{A}$, then we have:

$$f' \bullet a_1 \bullet \ldots \bullet a_n = \psi(f'a_1 \ldots a_n) = \psi(b_1^m) = b_1 \bullet b_2 \bullet \ldots \bullet b_m.$$

Conversely, if $f \in F$ is an (n,m)-operation and $a_1^n \in A^n$, $b_1^m \in A^m$ are such that

$$f'\bullet a_1\bullet\ldots\bullet a_n = b_1\bullet b_2\bullet\ldots\bullet b_m$$

in $(S;\bullet)$, then we have

$$\psi(f'a_1^n)=f'\bullet a_1\bullet\ldots\bullet a_n=b_1\bullet\ldots\bullet b_m=\psi(b_1^m)=b_1^m,$$

and therefore $f(a_1^n)=b_1^m$ in $\underline{A}$.

This proves the following v.v. variant of Cohn-Rebane's Theorem:

Theorem 3.7. *If $\underline{A}=(A;F)$ is a p.v.v.a., then there exists a semigroup $\underline{S}=(S;\bullet)$ and a mapping $f\mapsto f'$ from F into S such that $A\subseteq S$ and*

$$f(a_1^n) = b_1^m \text{ in } \underline{A} \iff f'\bullet a_1\bullet a_2\bullet\ldots\bullet a_n = b_1\bullet b_2\bullet\ldots\bullet b_m \text{ in } \underline{S},$$

for any $a_\nu, b_\lambda\in A$ and any (n,m)-operation $f\in F$. □

Now we will consider the question of associating a semigroup to a given v.v. groupoid by using presentations of semigroups.

Let $\underline{Q}=(Q;f)$ be an (n,m)-groupoid and let $\Lambda=\Lambda_{\underline{Q}}$ where $\Lambda_{\underline{Q}}$ is the following set of defining relations:

$$\Lambda_{\underline{Q}} = \{(a_1^n,b_1^m)\mid f(a_1^n) = b_1^m \text{ in } \underline{Q}\}. \tag{3.5}$$

Then $\underline{Q}^\wedge=\langle Q;\Lambda\rangle$ is called a universal semigroup for the given (n,m)-groupoid $\underline{Q}$. (Note that $\Lambda_{\underline{Q}}$ is the graph of f.)

Having in mind P.3.2., we have the following:

Proposition 3.8. *Let $\underline{Q}$ be an (n,m)-groupoid and $\Lambda=\Lambda_{\underline{Q}}$. Then $\underline{Q}^\wedge=\underline{Q}^+/\overset{\Lambda}{=}$ and the natural mapping*

$$nat^\wedge\colon a\mapsto a^\Lambda$$

is a realization of (Q,Λ) in $\underline{Q}^\wedge$.

For every realization $\eta\colon Q\to S$ of (Q,Λ) in a semigroup $\underline{S}=(S;\cdot)$, there is a unique homomorphism $\zeta\colon \underline{Q}^\wedge\to\underline{S}$, such that

$$(\forall b\in Q)\ \eta(b) = \zeta(b^\Lambda).\ \square$$

Next we will give a description of the universal semigroup for the free (n,m)-groupoid $(\overline{B};f)$ with a basis B, defined in §1.

We say that a word $x \in \bar{B}^+$ is reducible iff it has a form

$$x = x'(1,y)(2,y)\cdots(m,y)x'',$$

where $x', x'' \in \bar{B}^*$ and $y = u_1^n \in \bar{B}^n$. And, if this is not satisfied, then we say that x is reduced. The set of all reduced elements of $\bar{B}^+$ will be denoted by R.

We note that in this case $\Lambda = \Lambda_{\bar{B}}$ has the following form:

$$\Lambda = \{(u_1^n, (1,u_1^n)(2,u_1^n)\cdots(m,u_1^n)) \mid u_\nu \in \bar{B}\}.$$

The length $|x|$ of an element x of $\bar{B}^+$ is defined in the usual manner. Namely

$$|b| = 1, \quad |(i,u^n)| = \sum_\nu |u_\nu|, \quad |yz| = |y| + |z|,$$

for every $b \in B$, $u_1^n \in \bar{B}^n$, $y,z \in \bar{B}^+$.

By induction on lengths we define a reduction ψ in $\langle \bar{B};\Lambda \rangle$ as follows:

(0) $\psi(x) = x$, for every $x \in R$.

Let x be a reducible element of $\bar{B}^+$ and assume that $\psi(y)$ is a well defined element of R for every $y \in \bar{B}^+$ such that $|y| < |x|$.

Consider, first, the case $m \geq 2$.

If $x = x'(1,u^n)(2,u_1^n)\cdots(m,u_1^n)x''$, where x' has the least possible dimension, then we put $y = x'u_1^n x''$. Clearly, $|y| < |x|$, and thus, $\psi(y) \in R$ is well defined. Then $\psi(x)$ is defined by:

(1) $\psi(x) = \psi(y)$,

and therefore $\psi\colon \bar{B}^+ \to R$ is a well defined mapping in the case $m \geq 2$.

In the case when $m = 1$, a reduction $\psi\colon \bar{B}^+ \to \bar{B}^+$ can be defined by induction on hierarchy. If $u = (1,u_1^n) \in \bar{B}$ then $\psi(u)$ is defined by:

(1′) $\psi(u) = \psi(u_1)\psi(u_2)\cdots\psi(u_n)$.

And, if $x = u_1^p \in \bar{B}^p$, then $\psi(x)$ is defined by

(1″) $\psi(x) = \psi(u_1)\cdots\psi(u_p)$.

By induction on lengths (for $m \geq 2$) or on lengths and hierarchies (for $m=1$), it can be shown that ψ is a reduction for $\langle\overline{B};\Lambda\rangle$. Therefore, by P.3.5., if we define an operation $\bullet$ on R by

$$x \bullet y = \psi(xy),$$

we obtain that $\underline{R}=(R;\bullet)$ is the universal semigroup for $(\overline{B};f)$.

We note that in the case m=1, $\underline{R}=B^+$, i.e. B^+ is the universal semigroup for $(\overline{B};f)$.

§4. VECTOR VALUED SUBGROUPOIDS OF SEMIGROUPS

An (n,m)-groupoid $\underline{Q}=(Q;f)$ is called an (n,m)-subgroupoid of a semigroup $(S;\cdot)$ iff $Q \subseteq S$ and for every $a_1^n \in Q^n$, $b_1^m \in Q^m$

$$f(a_1^n) = b_1^m \text{ in } \underline{Q} \implies a_1 \cdot a_2 \cdot \ldots \cdot a_n = b_1 \cdot b_2 \cdot \ldots \cdot b_m \text{ in } \underline{S}. \quad (4.1)$$

If, in addition, for every $a_1^m, b_1^m \in Q^m$ the following implication is true

$$a_1 \cdot a_2 \cdot \ldots \cdot a_m = b_1 \cdot b_2 \cdot \ldots \cdot b_m \text{ in } \underline{S} \implies a_1^m = b_1^m, \quad (4.2)$$

then we say that $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{S}$.

It is clear that:

Proposition 4.1. *An (n,1)-groupoid $\underline{Q}$ is an (n,1)-subgroupoid of a semigroup $\underline{S}$ iff $\underline{Q}$ is a pure (n,1)-subgroupoid of $\underline{S}$.* □

Proposition 4.2. *An (n,n)-groupoid $\underline{Q}=(Q;f)$ is a pure (n,n)-subgroupoid of a semigroup iff*

$$(\forall a_1^n \in Q^n) \quad f(a_1^n) = a_1^n,$$

i.e. f is the identity transformation on Q^n.

Proof. If f is the identity transformation on Q^n, then $(Q;f)$ is a pure (n,n)-subgroupoid of Q^+. □

Let $\underline{Q}=(Q;f)$ be an (n,m)-groupoid, $\Lambda=\Lambda_{\underline{Q}}$ be the set of defining relations as in (3.5) and $\underline{Q}^\wedge$ be the corresponding semigroup $\langle Q;\Lambda\rangle$. The congruence $\stackrel{\Lambda}{=}$ is also defined in §3. Here we will write $\vdash$, $\sim$ instead of $\stackrel{\Lambda}{\vdash}$, $\stackrel{\Lambda}{\sim}$, respectively.

The definition of (n,m)-subgroupoids of semigroups can be restated by the following:

Proposition 4.3. *An (n,m)-groupoid $\underline{Q}=(Q;f)$ is an (n,m)-subgroupoid of a semigroup $\underline{S}=(S;\cdot)$ iff $Q\subseteq S$, and the inclusion $a\mapsto a$ from Q into S is a realization of (Q,Λ) in $\underline{S}$.* □

We recall that the presentation $\langle Q;\Lambda\rangle$ is proper iff

$$(\forall a,b\in Q)\ (a \overset{\Lambda}{=} b \Longrightarrow a = b), \tag{4.3}$$

and then we can assume that Q is a subset of $Q^{\wedge}=Q^{+}/\overset{\Lambda}{=}$.

Thus, we have the following description of the class of (n,m)-subgroupoids of semigroups.

Theorem 4.4. *An (n,m)-groupoid $\underline{Q}=(Q;f)$ is an (n,m)-subgroupoid of a semigroup iff the presentation $\langle Q;\Lambda\rangle$ is proper, and then $\underline{Q}$ is an (n,m)-subgroupoid of $\underline{Q}^{\wedge}$. If this is satisfied, and if $\underline{Q}$ is an (n,m)-subgroupoid of a semigroup $\underline{S}=(S;\cdot)$, then there exists a unique homomorphism $\zeta\colon \underline{Q}^{\wedge}\to\underline{S}$, such that $\zeta(a)=a$ for every $a\in Q$.* □

The following proposition is a consequence of P.3.3. b) and T.4.4.

Proposition 4.5. *If $\min\{n,m\}\geq 2$, then every (n,m)-groupoid is an (n,m)-subgroupoid of a semigroup.* □

For the class of pure (n,m)-subgroupoids of semigroups, we have the following result:

Theorem 4.6. *If $\underline{Q}=(Q;f)$ is an (n,m)-groupoid, then the following conditions are equivalent:*

(i) $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{Q}^{\wedge}$;

(ii) $\underline{Q}$ is a pure (n,m)-subgroupoid of a semigroup;

(iii) $$(\forall a_1^m,b_1^m\in Q^m)(a_1^m\overset{\Lambda}{=}b_1^m\Longrightarrow a_1^m=b_1^m). \tag{4.4}$$

Proof. Clearly, (i) ⟹ (ii).

Let $\underline{Q}$ be a pure (n,m)-subgroupoid of a semigroup $\underline{S}=(S;\cdot)$, and let $a_1^m,b_1^m\in Q^m$ be such that $a_1^m\overset{\Lambda}{=}b_1^m$. By T.4.4., $\underline{Q}$ is an (n,m)-subgroupoid of $\underline{Q}^{\wedge}$, and there exists a homomorphism $\zeta\colon \underline{Q}^{\wedge}\longrightarrow\underline{S}$, such that $(\forall c\in Q)\zeta(c)=c$. Thus, we have $a_1a_2\cdots a_m=b_1\cdots b_m$ in $\underline{Q}^{\wedge}$, and this implies $a_1\bullet a_2\bullet\ldots\bullet a_m=b_1\bullet\ldots\bullet b_m$ in $\underline{S}$; hence we obtain that $a_1^m=b_1^m$. Thus, (ii) ⟹ (iii).

Assume now the condition (iii). If $a,b\in Q$ are such that $a\stackrel{\wedge}{=}b$, then we have $a^m \stackrel{\wedge}{=} b\,a^{m-1}$, and this, by (iii), implies $a^m=b\,a^{m-1}$, i.e. a=b. Therefore, by T.4.4, $\underline{Q}$ is an (n,m)-subgroupoid of $\underline{Q}^\wedge$. If $a_1^m, b_1^m \in Q^m$ are such that $a_1a_2\cdots a_m=b_1b_2\cdots b_m$ in $\underline{Q}^\wedge$, then $a_1^m \stackrel{\wedge}{=} b_1^m$, and this by (iii) implies $a_1^m=b_1^m$ in Q^+. i.e. $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{Q}^\wedge$. □

In the case $n-m=k\geq 1$ we have the following:

Proposition 4.7. *Let $n-m=k\geq 1$. An (n,m)-groupoid $\underline{Q}=(Q;f)$ is a pure (n,m)-subgroupoid of a semigroup iff $\underline{Q}$ is an (n,m)-semigroup.*

Proof. First suppose that $\underline{Q}$ is not an (n,m)-semigroup. Then, there exists $a_1^{n+k}\in Q^{n+k}$ and $i\in N_k$ such that

$$b_1^m = f(f(a_1^n)a_{n+1}^{n+k}) \neq f(a_1^i f(a_{i+1}^{i+n})a_{i+n+1}^{n+k}) = c_1^m.$$

Therefore, T.4.6. (iii) implies that $\underline{Q}$ is not a pure (n,m)-subgroupoid of a semigroup.

The converse follows from T.7.7.[1] □

The above discussion gives a complete description of all the (n,m)-subgroupoids of semigroups, except the (1,1+k)-subgroupoids of semigroups for $k\geq 1$. So, next we consider this case.

Suppose that $\underline{Q}=(Q;f)$ is a (1,1+k)-groupoid, where $k\geq 1$. For every $\alpha\in N$, define a set $\mathcal{P}_\alpha(f)$ of polynomial operations on Q with a degree α as follows.

First, $\mathcal{P}_0(f)=\{1_Q\}$ (where $1_Q: x\mapsto x$ is the identity transformation on Q), and $\mathcal{P}_1(f)=\{f\}$. Suppose that for every α: $1\leq\alpha<\beta$, $\mathcal{P}_\alpha(f)$ is a well-defined nonempty set of $(1,1+\alpha k)$-operations on Q. Then $h\in\mathcal{P}_\beta(f)$ iff there exist $g\in\mathcal{P}_{\beta-1}(f)$ and $i\in\{0,1,\ldots,(\beta-1)k\}$ such that

$$h(x)=x_1^{1+\beta k} \iff g(x)=x_1^i y x_{i+k+2}^{1+\beta k} \;\&\; f(y)=x_{i+1}^{i+k+1}. \tag{4.5}$$

Using the polynomial operations, we will describe the class of (1,1+k)-subgroupoids of semigroups and the class of pure (1,1+k)-subgroupoids of semigroups as well.

---

[1] *P.4.7. is stated in this form only for completeness. It is not used in the proof of T.7.7.*

Theorem 4.8. *A (1,1+k)-groupoid* $\underline{Q}=(Q;f)$ *is a (1,1+k)-subgroupoid of a semigroup iff for every positive integer* β *and polynomial operations* $g,h\in\mathcal{P}_\beta(f)$, *the following implication is satisfied:*

$$g(x) = h(y) \Longrightarrow x = y. \tag{4.6}$$

Proof. Let $(Q;f)$ be a $(1,1+k)$-subgroupoid of a semigroup and let, for an arbitrary β, the polynomial operations $g,h\in\mathcal{P}_\beta(f)$ and $x,y\in Q$ be such that $g(x)=h(y)$. Then $x \overset{\Delta}{=} y$, which implies that $x = y$, by T.4.4.

For the converse we need the following two lemmas.

Lemma 4.9. *If* $u,v,w\in Q^+$ *are such that* $v\vdash u$, $v\vdash w$, $u\neq w$, *then there exists* $v'\in Q^+$ *such that* $u\vdash v'$, $w\vdash v'$.

Proof. By $v\vdash u$, $v\vdash w$ it follows that

$$v = a_1^{i-1}a_ia_{i+1}^p = a_1^{j-1}a_ja_{j+1}^p,$$

$$u = a_1^{i-1}b_1^{1+k}a_{i+1}^p, \quad w = a_1^{j-1}c_1^{1+k}a_{j+1}^p,$$

where $f(a_i)=b_1^{1+k}$, $f(a_j)=c_1^{1+k}$. Since $u\neq w$, it follows that $i\neq j$, for example $i<j$. Putting

$$v' = a_1^{i-1}b_1^{1+k}a_{i+1}^{j-1}c_1^{1+k}a_{j+1}^p,$$

we obtain that $u\vdash v'$, $w\vdash v'$. □

Lemma 4.10. *Let* $u=a_1^s\in Q^s$, $v\in Q^+$. *There exist* $v_1,v_2,\ldots,v_t\in Q^+$ *such that*

$$u\vdash v_1\vdash v_2\vdash\cdots\vdash v_t = v$$

iff there exist integers $\alpha_1,\alpha_2,\ldots,\alpha_s\geq 0$ *and* $h_\nu\in\mathcal{P}_{\alpha_\nu}(f)$ *such that*

$$v = h_1(a_1)h_2(a_2)\cdots h_s(a_s).$$

Proof. We will prove this by induction on t. For $t=1$, we have

$$v = a_1^{i-1}b_1^{1+k}a_{i+1}^s = 1(a_1)\cdots 1(a_{i-1})f(a_i)1(a_{i+1})\cdots 1(a_s)$$

Let $v_{t-1} = h_1(a_1)h_2(a_2)\cdots h_s(a_s)$
$= \cdots c_1^{1+\delta k}\cdots$

$$v_t = v = h_1(a_1)\cdots h_{j-1}(a_{j-1})c_1^{i-1}d_1^{1+k}c_{i+1}^{1+\delta k}h_{j+1}(a_{j+1})\cdots h_s(a_s),$$

where $h_j(a_j) = c_1^{1+\delta k}$, $f(c_i)=d_1^{1+k}$. Then

$$v = h_1(a_1)\cdots h_{j-1}(a_{j-1})h_j'(a_j)h_{j+1}(a_{j+1})\cdots h_s(a_s),$$

where $h_j'(a_j)=c_1^{i-1}d_1^{1+k}c_{i+1}^{1+\delta k}$. □

Now, to complete the proof of T.4.8., assume that for every $g,h\in\mathcal{P}_\beta(f)$, $x,y\in Q$,

$$g(x) = h(y) \Longrightarrow x = y.$$

Let $x \overset{\Lambda}{=} y$, $x,y\in Q$. By T.4.4. it suffices to show that $x = y$. First, L.4.9. implies that there exist $u_1,\dots,u_t$, $v_1,\dots,v_t$ such that

$$x \vdash u_1 \vdash \cdots \vdash u_t = v_t \dashv v_{t-1} \dashv \cdots \dashv v_1 \dashv y.$$

Now, by L.4.10., there exist $g,h\in\mathcal{P}_t(f)$, such that

$$g(x) = u_t = v_t = h(y),$$

which implies $x = y$. □

The following theorem gives a description of the class of the pure (1,1+k)-subgroupoids of semigroups.

Theorem 4.11. *Let $\underline{Q}=(Q;f)$ be a (1,1+k)-groupoid, $k\geq 1$. Then, $\underline{Q}$ is a pure (1,1+k)-subgroupoid of a semigroup iff for every $\alpha_0,\dots,\alpha_k$, $\beta_0,\dots,\beta_k\geq 0$, such that $\alpha_0+\dots+\alpha_k = \beta_0+\dots+\beta_k$ and any $h_\nu\in\mathcal{P}_{\alpha_\nu}(f)$, $g_\nu\in\mathcal{P}_{\beta_\nu}(f)$, $\underline{Q}$ satisfies the following implication:*

$$h_0(x_0)\cdots h_k(x_k) = g_0(y_0)\cdots g_k(y_k) \Longrightarrow x_0^k = y_0^k. \qquad (4.7)$$

Proof. Let $\underline{Q}$ be a pure (1,1+k)-subgroupoid of a semigroup and let the assumption of the implication (4.7) holds. Then, by L.4.10., $x_0^k \overset{\Lambda}{=} y_0^k$ and this implies $x_0^k = y_0^k$. Thus, (4.7) is satisfied.

Conversely, suppose that (4.7) is satisfied in $\underline{Q}$, and $x_0^k,y_0^k\in Q^{k+1}$ be such that $x_0^k \overset{\Lambda}{=} y_0^k$. Then, the definition of $\overset{\Lambda}{=}$, L.4.9., and L.4.10. imply that there exist corresponding polynomial operations h_ν,g_ν such that

$$h_0(x_0)\cdots h_k(x_k) = g_0(y_0)\cdots g_k(y_k). \ \square$$

In the case $1 < n < m$, we do not know a satisfactory description of the class of pure (n,m)-subgroupoids of semigroups. The next theorem shows that such pure (n,m)-subgroupoids of semigroups do exist.

Theorem 4.12. *If $n < m$, then every free (n,m)-groupoid is a pure (n,m)-subgroupoid of a semigroup.*

Proof. Let $\underline{Q}=(\overline{B};f)$ be the free (n,m)-groupoid with a basis B. By T.4.6. we have to show that $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{Q}^{\wedge}$. We use the description of $\underline{Q}^{\wedge}$ given in §3. Thus $\underline{Q}^{\wedge}=(R;\bullet)$, where $R=\psi(\overline{B}^{+})$, and $x\bullet y=\psi(xy)$. Suppose that $u_1^m, v_1^m \in \overline{B}^m$ are such that $u_1\bullet u_2\bullet\ldots\bullet u_m = v_1\bullet v_2\bullet\ldots\bullet v_m$ in $(R;\bullet)$, i.e. $\psi(u_1^m)=\psi(v_1^m)$. Since $n < m$, $\psi(u_1^m)\neq u_1^m$ iff $u_\lambda=(\lambda,w_1^n)$, for $\lambda\in N_m$, and then $\psi(u_1^m) = w_1^n$. □

It is natural to ask the question about the existence of pure (n,m)-subgroupoids $\underline{Q}$ of semigroups, under the assumption that $\underline{Q}$ has a corresponding property $\mathcal{C}$. In the case when $n-m \geq 0$, the answer to this question give P.4.2. and P.4.7.

Here we will show that, for $2 \leq n \leq m$, there are no nontrivial commutative (n,m)-groupoids which are pure (n,m)-subgroupoids of semigroups.

Let $\underline{Q}=(Q;f)$ be a commutative pure (n,m)-subgroupoid of a semigroup $\underline{S}=(S;\cdot)$, where $2 \leq n \leq m$. In the case $n=m$, by P.4.2., we have $f(a_1^n)=a_1^n$ and so $f(a_2a_1a_3^n)=a_2a_1a_3^n$, which implies that $a_1^n=a_2a_1a_3^n$, i.e. $a_1=a_2$. Since a_1 and a_2 are arbitrary, the equality $a_1=a_2$ implies that $|Q|=1$.

Now suppose that $2 \leq n < m$. If $b_1,\ldots,b_m\in Q$, then

$$b_1b_2\ldots b_m = (b_1b_2\ldots b_n)b_{n+1}\ldots b_m = \\ = (b_2b_1\ldots b_n)b_{n+1}\ldots b_m = b_2b_1\ldots b_m$$

implies that $b_1=b_2$ for arbitrary $b_1,b_2\in Q$. Hence $|Q|=1$.

Thus we have

Proposition 4.13. *If $2 \leq n \leq m$, then a commutative (n,m)-groupoid $\underline{Q}=(Q;f)$ is a pure (n,m)-subgroupoid of a semigroup iff $|Q|=1$.* □

(Note that every (1,1+k)-groupoid is commutative, so that in this case, P.4.8. and P.4.11. can be applied.)

§5. THE GENERAL ASSOCIATIVE LAW

The general associative law is true for v.v. semigroups too. We will prove this fact in details. First we will introduce the notion of polynomial operation.

Throughout this section we will assume that $\underline{Q}=(Q;f)$ is a given (n,m)-groupoid, where $n-m=k \geq 1$.

Let $g, g_1, g_2, \ldots, g_t$ be v.v. operations on a set Q, such that g is an (n,m)-operation, and g_ν is an (n_ν, m_ν)-operation for every $\nu \in N_t$, and let the following equalities be satisfied:

$$p = n_1 + n_2 + \ldots + n_t, \qquad n = m_1 + m_2 + \ldots + m_t.$$

Define a (p,m)-operation $h=g(g_1^t)$ by:

$$h(x_1^p) = g(g_1(\underline{x}_1)g_2(\underline{x}_2)\cdots g_t(\underline{x}_t)), \tag{5.1}$$

where $x_1^p = \underline{x}_1\underline{x}_2\ldots\underline{x}_t$, $\underline{x}_i \in Q^{n_i}$.

For every positive integer α define a set $\mathcal{P}_\alpha(\underline{Q}) = \mathcal{P}_\alpha(f)$ of <u>polynomial operations</u>[1], with a <u>degree</u> α, inductively, in the following way. First put

$$\mathcal{P}_0(f) = \{1_Q\}, \quad \mathcal{P}_1(f) = \{f\}. \tag{5.2}$$

(As before, 1_Q: $x \mapsto x$ is the identity transformation on Q; further on, we will write 1 instead of 1_Q.)

Suppose that the number $\beta \geq 2$ is such that for every α: $0 \leq \alpha < \beta$ the set $\mathcal{P}_\alpha(f)$ of v.v. operations is well-defined. Then the set $\mathcal{P}_\beta(f)$ of v.v. operations on Q will be defined in the following way: $h \in \mathcal{P}_\beta(f)$ iff there exist v.v. operations g, g_ν on Q such that $g \in \mathcal{P}_\alpha(f)$, $g_\nu \in \mathcal{P}_{\alpha_\nu}(f)$, $\beta = \alpha + \sum_{\nu=1}^{t} \alpha_\nu$, $0 < \alpha < \beta$ and

$$h = g(g_1^t). \tag{5.3}$$

[1] *The definition of* $\mathcal{P}_\alpha(f)$ *in this section differs from the definition of* $\mathcal{P}_\alpha(f)$ *in §4, because here we have* $n > m$.

Proposition 5.1. *For every integer $\alpha \geq 0$, the set $\mathcal{P}_\alpha(f)$ is nonempty. If $\alpha > 0$ and $h \in \mathcal{P}_\alpha(f)$, then h is an $(m+\alpha k, m)$-operation.*

Proof. Let $\alpha \geq 0$. Define a set $\{f^{(s)} \mid s \geq 1\}$, of v.v. operations on Q, such that $f^{(s)} \in \mathcal{P}_s(f)$, by induction on s. First, for $s=1$, put $f^{(1)}=f$. Suppose that $f^{(\alpha)} \in \mathcal{P}_\alpha(f)$ is a well-defined $(m+\alpha k, m)$-operation on Q and define a v.v.o. $f^{(\alpha+1)}$ by:

$$f^{(\alpha+1)} = f(f^{(\alpha)}{}_1^k). \tag{5.4}$$

Then $f \in \mathcal{P}_1(f)$, $1 \in \mathcal{P}_0(f)$ and $f^{(\alpha)} \in \mathcal{P}_\alpha(f)$ imply that $f^{(\alpha+1)} \in \mathcal{P}_{\alpha+1}(f)$ and that $f^{(\alpha+1)}$ is an $(m+(\alpha+1)k, m)$-operation on Q. Thus, for every $\alpha \geq 0$, the set $\mathcal{P}_\alpha(f)$ is nonempty.

Next we show the second part of the proposition, using the induction again. Suppose that

> $g \in \mathcal{P}_\alpha(f)$, $g_\nu \in \mathcal{P}_{\alpha_\nu}(f)$, $h=g(g_1^t) \in \mathcal{P}_\beta(f)$, $\beta=\alpha+\alpha_1+\ldots+\alpha_t$, where $0<\alpha<\beta$, g is an $(m+\alpha k, m)$-operation, and g_ν is an (n_ν, m_ν)-operation on Q.

If $\alpha_\nu=0$, then $g_\nu=1$ which means that $n_\nu=1=m_\nu$, and for $\alpha_\nu>0$ we have $n_\nu=m+\alpha_\nu k$, $m_\nu=m$. Therefore, h is a (p,m)-operation, where

$$p = n_1+n_2+\ldots+n_t, \quad m+\alpha k = m_1+m_2+\ldots+m_t.$$

Let i integers from the sequence $\alpha_1, \alpha_2, \ldots, \alpha_t$ be positive, and the other t-i be zeros. Without loss of generality, we may assume that $\alpha_1, \alpha_2, \ldots, \alpha_i > 0$, $\alpha_{i+1}=\ldots=\alpha_t=0$. Then we have:

$$p = (m+\alpha_1 k)+\ldots+(m+\alpha_i k)+(t-i) = mi+(\alpha_1+\ldots+\alpha_i)k+(t-i)$$

and $m+\alpha k=mi+(t-i)$, which implies that

$$p = m+\alpha k+(\alpha_1+\ldots+\alpha_i)k = m+\beta k \quad (\beta=\alpha+\alpha_1+\ldots+\alpha_i).$$

Thus, h is an $(m+\beta k, m)$-operation. □

Proposition 5.2. *If $g \in \mathcal{P}_\alpha(f)$, then $\mathcal{P}_\beta(g) \subseteq \mathcal{P}_{\alpha\beta}(f)$.*

Proof. The statement is obviously true for $\beta \in \{0,1\}$ and so we will assume that $\beta \geq 2$.

Let $\ell \in \mathcal{P}_\beta(g)$. Then, there exist integers $\gamma, \gamma_1, \ldots, \gamma_t$ such that $0<\gamma<\beta$, $\beta=\gamma+\gamma_1+\ldots+\gamma_t$, $\ell=h(h_1^t)$, $h \in \mathcal{P}_\gamma(g)$, $h_\nu \in \mathcal{P}_{\gamma_\nu}(g)$. By the inductive hypothesis it follows that $h \in \mathcal{P}_{\alpha\gamma}(f)$, $h_\nu \in \mathcal{P}_{\alpha\gamma_\nu}(f)$ and

$$\alpha\beta = \alpha\gamma + \alpha\gamma_1 + \ldots + \alpha\gamma_t,$$

and thus $\ell \in \mathcal{P}_{\alpha\beta}(f)$. □

For the proof of the <u>general associative law</u> (GAL) we need the following lemma:

Lemma 5.3. *Let $(Q;f)$ be an (n,m)-semigroup. Then*

$$f^{(\gamma+\delta)} = f^{(\gamma)}(\overset{r}{1}\, f^{(\delta)}\, \overset{\gamma k-r}{1}) \qquad (5.5)$$

for every $\gamma,\delta \geq 1$, and $0 \leq r \leq \gamma k$.

Proof. The proof will be done by induction. Since $(Q;f)$ is an (n,m)-semigroup, it follows that $\mathcal{P}_2(f)=\{f^{(2)}\}$, i.e. $f^{(1+1)}=f^{(2)}=f(\overset{r}{1}\, f\, \overset{k-r}{1})$.

1) Let (5.5) be satisfied for $1,\delta$. Then

$$\begin{aligned} f^{(1+\delta+1)} &= f(f^{(1+\delta)}\overset{k}{1}) = f(f(\overset{r}{1}\, f^{(\delta)}\overset{k-r}{1})\overset{k}{1}) \\ &= f^{(2)}(\overset{r}{1} f^{(\delta)}\overset{2k-r}{1}) = f(\overset{r}{1} f(f^{(\delta)}\overset{k-r}{1})) = f(\overset{r}{1} f^{(\delta+1)}\overset{k-r}{1}). \end{aligned}$$

2) Let (5.5) be satisfed for $\gamma,1$. Then by 1):

$$\begin{aligned} f^{(\gamma+1+1)} &= f(\overset{k}{1} f^{(\gamma+1)}) = f(\overset{k}{1} f^{(\gamma)}(\overset{r-k}{1}\, f\, \overset{\gamma k-r+k}{1})) \\ &= f(\overset{k}{1} f^{(\gamma)})(\overset{r}{1} f \overset{(\gamma+1)k-r}{1}) = f^{(\gamma+1)}(\overset{r}{1} f \overset{(\gamma+1)k-r}{1}), \end{aligned}$$

for $r \geq \gamma k$; and

$$\begin{aligned} f^{(\gamma+1+1)} &= f(f^{(\gamma+1)}\overset{k}{1}) = f(f^{(\gamma)}(\overset{r}{1} f \overset{\gamma k-r}{1})\overset{k}{1}) \\ &= f(f^{(\gamma)}\overset{k}{1})(\overset{r}{1}\, f \overset{(\gamma+1)k-r}{1}) = f^{(\gamma+1)}(\overset{r}{1}\, f \overset{(\gamma+1)k-r}{1}), \end{aligned}$$

for $r < \gamma k$.

3) Let (5.5) be satisfied for γ,δ. Then

$$\begin{aligned} f^{(\gamma+\delta+1)} &= f^{(\gamma+\delta)}(\overset{r}{1}\, f \overset{(\gamma+\delta)k-r}{1}) = \\ &= f^{(\gamma)}(\overset{r}{1}\, f^{(\delta)}\overset{\gamma k-r}{1})(\overset{r}{1}\, f \overset{(\gamma+\delta)k-r}{1}) = \\ &= f^{(\gamma)}(\overset{r}{1}\, f^{(\delta)}(f\, \overset{\delta k}{1})\overset{\gamma k-r}{1}) = f^{(\gamma)}(\overset{r}{1}\, f^{(\delta+1)}\overset{\gamma k-r}{1}). \ □ \end{aligned}$$

Theorem 5.4. (GAL). *An (n,m)-groupoid $Q=(Q;f)$ is an (n,m)-semigroup iff, for every integer $\alpha \geq 0$, the set $\mathcal{P}_\alpha(f)$ has exactly one element.*

Proof. The definition of (n,m)-semigroups implies that $\underline{Q}$ is an (n,m)-semigroup iff $\mathcal{P}_2(f)=\{f^{(2)}\}$, where $f^{(s)}$ is defined in the proof of P.5.1. Therefore, if for every α, $\mathcal{P}_\alpha(f)$ is a one-element set, then $\underline{Q}$ is an (n,m)-semigroup.

Conversely, let $\underline{Q}$ be an (n,m)-semigroup. Obviously, $|\mathcal{P}_\alpha(f)|=1$ for $\alpha \leq 2$. We will show, by induction, that $\mathcal{P}_\beta(f)=\{f^{(\beta)}\}$ for every $\beta \geq 1$, which will complete the proof.

Let $h \in \mathcal{P}_\beta(f)$, i.e. $h=g(g_1^t)$, where $g \in \mathcal{P}_\alpha(f)$, $g_\lambda \in \mathcal{P}_{\alpha_\lambda}(f)$, $\beta=\alpha+\alpha_1+\ldots+\alpha_t$, $0<\alpha<\beta$. The inductive hypothesis implies that $g=f^{(\alpha)}$ and: $\alpha_\lambda=0 \Longrightarrow g_\lambda = 1$, $\alpha_\lambda > 0 \Longrightarrow g_\lambda = f^{(\alpha_\lambda)}$. Since $0<\alpha<\beta$, there exists an integer λ, such that $\alpha_\lambda > 0$. Let $r+1$ be the least positive integer, such that $\alpha_{r+1}=\tau\neq 0$. Then $g_{r+1}=f^{(\tau)}$ and $g_1=g_2=\ldots=g_r=1$. Using L.5.3., we have:

$$\begin{aligned} h = g(g_1^t) &= f^{(\alpha)}(\overset{r}{1} f^{(\tau)} g_{r+2}^t) = \\ &= f^{(\alpha)}(\overset{r}{1} f^{(\tau)}(\overset{\alpha k-r}{1})(\overset{r+m+\tau k}{1} g_{r+2}^t) = \\ &= f^{(\alpha+\tau)}(\overset{r+m+\tau k}{1} g_{r+2}^t). \end{aligned}$$

Applying the same procedure a finite number of times, we obtain that $h=f^{(\beta)}(\overset{\beta k+m}{1})=f^{(\beta)}$. □

As a consequence of GAL and P.5.2. we have

Proposition 5.5. *If $(Q;f)$ is an (n,m)-semigroup, then $(Q;f^{(s)})$ is an $(sk+m,m)$-semigroup, for every $s \geq 1$.* □

Using GAL and the usual induction, we obtain

Proposition 5.6. *If $(Q;f)$ is a commutative (n,m)-semigroup, then $(Q;f^{(s)})$ is a commutative $(m+sk,m)$-semigroup, for every $s \geq 1$.* □

Now we give the following characterization for the cancellative v.v. semigroups.

Theorem 5.7. *If $(Q;f)$ is an (n,m)-semigroup, then the following conditions are equivalent:*

(i) $(Q;f)$ is cancellative;

(ii) $(Q;f^{(s)})$ *is a cancellative* $(m+sk,m)$*-semigroup, for every* $s \geq 1$;

(iii) $(Q;f^{(s)})$ *is a cancellative* $(m+sk,m)$*-semigroup, for some* $s \geq 1$;

(iv) $(Q;f)$ *satisfies the following implication:*

$$f(a_1^k\ x_1^m) = f(a_1^k\ y_1^m) \text{ or } f(x_1^m\ b_1^k) = f(y_1^m\ b_1^k) \implies x_1^m = y_1^m;$$

(v) there exist $i,s \geq 1$, $i \in N_{sk-1}$, $sk \geq 2$, *such that the following implication in* $(Q;f)$ *is true:*

$$f^{(s)}(a_1^i x_1^m a_{i+1}^{sk}) = f^{(s)}(a_1^i y_1^m a_{i+1}^{sk}) \implies x_1^m = y_1^m.$$

Proof. (i) $\implies$ (ii). Let $(Q;f)$ be cancellative and let $t \geq 0$ be an integer such that, for every s: $1 \leq s < t$, the v.v. semigroup $(Q;f^{(s)})$ is cancellative. Suppose that the equality

$$f^{(t)}(a_1^i x_1^m a_{i+1}^{tk}) = f^{(t)}(a_1^i y_1^m a_{i+1}^{tk}) \tag{5.6}$$

is true, where $0 \leq i \leq tk$, and let $\alpha \leq \min\{i,k\}$. By the equality (5.6),

$$f(a_1^\alpha f^{(t-1)}(a_{\alpha+1}^i x_1^m a_{i+1}^{(t-1)k-i+\alpha}) a_{(t-1)k-i+\alpha+1}^{tk}) =$$

$$= f(a_1^\alpha f^{(t-1)}(a_{\alpha+1}^i y_1^m a_{i+1}^{(t-1)k-i+\alpha}) a_{(t-1)k-i+\alpha+1}^{tk});$$

now, using the facts that $(Q;f)$ is cancellative and that $(Q;f^{(t-1)})$ is cancellative, we obtain that $x_1^m=y_1^m$, i.e. the equality (5.6) implies that $x_1^m=y_1^m$. Therefore the $(m+tk,m)$-semigroup $(Q;f^{(t)})$ is cancellative.

(ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (i). Suppose that, for some $s \geq 2$, the $(m+sk,m)$-semigroup $(Q;f^{(s)})$ is cancellative and let

$$f(a_1^i x_1^m a_{i+1}^k) = f(a_1^i y_1^m a_{i+1}^k), \tag{5.7}$$

where $i \in \{0,1,\dots,k\}$. Then, for arbitrary $b_1,\dots,b_{(s-1)k} \in Q$,

$$f^{(s)}(b_1^{(s-1)k} a_1^i x_1^m a_{i+1}^k) = f^{(s)}(b_1^{(s-1)k} a_1^i y_1^m a_{i+1}^k),$$

by which, since $(Q;f^{(s)})$ is cancellative, we obtain that $x_1^m=y_1^m$. Thus (5.7) implies that $x_1^m=y_1^m$, i.e. $(Q;f)$ is cancellative.

It is clear that (i) $\implies$ (iv). We will show that (iv) $\implies$ (i).

We can assume that $k \geq 2$, for if $k=1$, then (iv) is the condition (b.2) of §2, i.e. $(Q;f)$ is cancellative.

Let (iv) be true and let

$$f(\underline{a}\ \underline{x}\ \underline{b}) = f(\underline{a}\ \underline{y}\ \underline{b}),$$

where $\underline{a}\underline{b} \in Q^k$, $\underline{x},\underline{y} \in Q^m$. Then:

$$f^{(2)}(\underline{b}\ \underline{a}\ \underline{x}\ \underline{b}\ \underline{a}) = f^{(2)}(\underline{b}\ \underline{a}\ \underline{y}\ \underline{b}\ \underline{a}),\text{ or}$$

$$f(f(\underline{b}\underline{a}\underline{x})\underline{b}\underline{a}) = f(f(\underline{b}\underline{a}\underline{y})\underline{b}\underline{a}),$$

by which, after two applications of (iv), one obtains $\underline{x}=\underline{y}$. Thus (iv) $\Longrightarrow$ (i).

Since the implication (ii) $\Longrightarrow$ (v) is obviously true, it remains to show that (v) implies some of the conditions (i)-(iv).

Suppose first that the following condition is true:

(v′) $k \geq 2$ and there exists $i \in N_{k-1}$ such that the following implication in $(Q;f)$ is true:

$$f(a_1^i x_1^m a_{i+1}^k) = f(a_1^i y_1^m a_{i+1}^k) \Longrightarrow x_1^m = y_1^m.$$

We will show that (v′) $\Longrightarrow$ (iv). First, one can show by induction on s that, by (v′), the following implication is true:

$$f^{(s)}(a_1^{si} x_1^m a_{si+1}^{sk}) = f^{(s)}(a_1^{si} y_1^m a_{si+1}^{sk}) \Longrightarrow x_1^m = y_1^m. \qquad (5.8)$$

Let $f(\underline{x}\underline{a})=f(\underline{y}\underline{a})$, $d(\underline{a})=k$, $d(\underline{x})=d(\underline{y})=m$, and choose $\underline{b}$ and $\underline{c}$ such that:

$$d(\underline{b}\underline{x}\underline{a}\underline{c}) = sk+m,\quad d(\underline{b})=si,\quad d(\underline{a}\underline{c})=sk-si.$$

Then one can obtain the following equality:

$$f^{(s)}(\underline{b}\underline{x}\underline{a}\underline{c}) = f^{(s)}(\underline{b}\underline{y}\underline{a}\underline{c}).$$

Applying (5.8), we have $\underline{x}=\underline{y}$. By symmetry, we have also $f(\underline{a}\underline{x})=f(\underline{a}\underline{y}) \Longrightarrow \underline{x} = \underline{y}$. Thus we showed that (v′) $\Longrightarrow$ (iv), i.e. (v′) $\Longrightarrow$ (i).

Now, since $f^{(s)}$ satisfies the condition (v′), (when k is replaced by sk), we conclude that $(Q;f^{(s)})$ is cancellative, i.e. (v) $\Longrightarrow$ (iii). □

The next theorem gives a characterization of v.v. groups, using the operations $f^{(s)}$.

Theorem 5.8. *If $(Q;f)$ is an (n,m)-semigroup, then the following conditions are equivalent:*

(i) $(Q;f)$ is a v.v. group.

(ii) $(Q;f^{(s)})$ is a v.v. group, for each $s \geq 1$.

(iii) $(Q;f^{(s)})$ is a v.v. group, for some $s \geq 1$.

(iv) There exist $s \geq 1$ satisfying $sk \geq 2$, and $i \in N_{sk-1}$, such that the equation

$$f^{(s)}(\underline{a}\ \underline{x}\ \underline{b}) = \underline{c} \tag{5.9}$$

is solvable for given $\underline{a} \in Q^{i}$, $\underline{b} \in Q^{sk-i}$, $\underline{c} \in Q^{m}$.

(v) There exists $s \geq 1$, such that $sk \geq 2$ and for each $i \in N_{sk-1}$, the equation (5.9) is solvable.

(vi) For each $s \geq 1$, satisfying $sk \geq 2$, there exists $i \in N_{sk-1}$, such that the equation (5.9) is solvable.

(vii) For each $s \geq 1$, satisfying $sk \geq 2$, and each $i \in N_{sk-1}$, the equation (5.9) is solvable.

Proof. It is obvious that: (ii) ==> (iii), (vii) ==> (v), (vi) ==> (iv), (vii) ==> (vi), and (v) ==> (iv).

(i) ==> (ii): Let $s \geq 1$, $\underline{a}_1, \ldots, \underline{a}_s \in Q^{k}$, $\underline{c} \in Q^{m}$. Then there exist $\underline{x}_1, \ldots, \underline{x}_s \in Q^{m}$ such that $f(\underline{x}_s \underline{a}_s) = \underline{c}$, $f(\underline{x}_{s-1} \underline{a}_{s-1}) = \underline{x}_s, \ldots, \ldots, f(\underline{x}_1 \underline{a}_1) = \underline{x}_2$. This implies that $f^{(s)}(\underline{x}_1 \underline{a}_1 \ldots \underline{a}_s) = \underline{c}$. Symmetrically, there exists $\underline{y} \in Q^{m}$ such that $f^{(s)}(\underline{a}_1 \ldots \underline{a}_s \underline{y}) = \underline{c}$. This, together with P.5.5., implies that $(Q;f^{(s)})$ is a v.v. group.

(iii) ==> (i): Let $\underline{a} \in Q^{k}$, $\underline{c} \in Q^{m}$, $a \in Q$, and let $\underline{b} = \overset{(s-1)k}{a}$. Since $(Q;f^{(s)})$ is a v.v. group, there exists $\underline{d} \in Q^{m}$, such that $f^{(s)}(\underline{a}\underline{b}\underline{d}) = \underline{c}$. This implies that the equation $f(\underline{a}\underline{x}) = \underline{c}$ has a solution. Symmetrically, the equation $f(\underline{y}\underline{a}) = \underline{c}$ has a solution. This, together with the assumption that $(Q;f)$ is a v.v. semigroup, implies that $(Q;f)$ is a v.v. group.

(ii) ==> (vii): Let $s \geq 1$, $sk \geq 2$ and $i \in N_{sk-1}$. Let $\underline{a} \in Q^{i}$, $\underline{b} \in Q^{sk-i}$, $\underline{c} \in Q^{m}$, $a \in Q$, $\underline{u} = \overset{sk-i}{a}$, $\underline{v} = \overset{i}{a}$. Then there exists $\underline{w} \in Q^{m}$, such that $f^{(s)}(\underline{a}\underline{u}\underline{w}) = \underline{c}$, and there exists $\underline{z} \in Q^{m}$, such that $f^{(s)}(\underline{z}\underline{v}\underline{b}) = \underline{w}$. Now, $f^{(s)}(\underline{a} f^{(s)}(\underline{u}\underline{z}\underline{v})\underline{b}) = \underline{c}$, i.e. the equation $f^{(s)}(\underline{a}\underline{x}\underline{b}) = \underline{c}$ is solvable.

(iv) $\Longrightarrow$ (v). Let s and i be the numbers which exist by (iv). The proof of (v) will be devided into three parts. Let $j\in N_{sk-1}$, $a_1^j\in Q^i$, $b_1^{sk-j}\in Q^{sk-j}$, $\underline{c}\in Q^m$, $a\in Q$.

(a) If $j<i$, then (iv) implies that there exist $\underline{d},\underline{g}\in Q^m$ such that $f^{(s)}(a_1^j \overset{i-j}{a} \underline{d}\, b_{i-j+1}^{sk-j})=\underline{c}$ and $f^{(s)}(\overset{i}{a}\, \underline{g} \overset{sk-2i+j}{a} b_1^{i-j})=\underline{d}$. Then $f^{(s)}(a_1^j h_1^m b_1^{sk-j})=\underline{c}$ for $h_1^m=f^{(s)}(\overset{2i-j}{a}\underline{g}\overset{sk-2i+j}{a})$.

(b) If $0<t=j-i\leq i$, then (a) and (iv) imply that there exist $\underline{d},\underline{g}\in Q^m$ such that $f^{(s)}(a_1^t \underline{d} \overset{i}{a} b_1^{sk-j})=\underline{c}$ and $f^{(s)}(a_{t+1}^j \underline{g} \overset{sk-i}{a})=\underline{d}$. Then $f^{(s)}(a_1^j h_1^m b_1^{sk-j})=\underline{c}$, for $h_1^m=f^{(s)}(\underline{g}\overset{sk}{a})$.

(c) The general case follows from (a) and (b).

(v) $\Longrightarrow$ (iii). Let s be the number which exists by (v). Let $\underline{a}\in Q^{sk}$, $\underline{c}\in Q^m$. Then $\underline{a}=b\underline{d}$ for some $b\in Q$, $\underline{d}\in Q^{sk-1}$. Let $u\in Q$ and $\underline{v}=\overset{sk-1}{u}$. Then (v) implies that there exist $\underline{\alpha},\underline{\beta}\in Q^m$ such that $f^{(s)}(b\underline{\alpha}\underline{v})=\underline{c}$ and $f^{(s)}(\underline{d}\underline{\beta}u)=\underline{\alpha}$. So, $f^{(s)}(\underline{\beta}\overset{sk}{u})$ is a solution of the equation $f^{(s)}(\underline{a}\underline{x})=\underline{c}$. Symmetrically, the equation $f^{(s)}(\underline{x}\underline{a})=\underline{c}$ is solvable. □

By using GAL, we can consider v.v. semigroups as v.v. algebras with infinitely many v.v. operations and these algebras we call „poly-(n,m)-semigroups". The notion of poly-(n,m)-semigroup will be introduced here, and it will be used in the construction of free v.v. semigroups (§6).

Let P be a non-empty set and let n,m,k be as above, i.e. $n-m=k\geq 1$. If

$$g:\ P^{(n,m)} \longrightarrow P^m, \text{ where } P^{(n,m)} = \bigcup_{s\geq 1} P^{m+sk},$$

is a mapping, then $P=(P;g)$ is called a poly-(n,m)-groupoid.

From a given (n,m)-groupoid we can obtain a poly-(n,m)-groupoid as follows.

Let $\underline{Q}=(Q;f)$ be an (n,m)-groupoid and let π be a choice of one and only one polynomial operation $\pi_s\in\mathcal{P}_s(f)$. For this π, define a mapping

$$f^{\pi}:\ Q^{(n,m)} \longrightarrow Q^m \text{ by: } f^{\pi}(a_1^{m+sk}) = \pi_s(a_1^{m+sk}).$$

Then we obtain a poly-(n,m)-groupoid $\underline{Q}^{\pi}=(Q;f^{\pi})$ which is said to be induced by the (n,m)-groupoid $\underline{Q}$.

In general, since the set $\mathcal{P}_s(f)$ may have more than one element, it is possible a given (n,m)-groupoid to induce more than one poly-(n,m)-groupoid.

For the choice $\pi_s=f^{(s)}$, we use the notations $f^{\#}$ and $\underline{Q}^{\#}$ instead of f^{π} and $\underline{Q}^{\pi}$ respectively. (Here, $f^{(s)}$ is as in (5.4).)

Note that there are poly-(n,m)-groupoids that are not induced by an (n,m)-groupoid.

On the other hand, if $\underline{P}=(P;g)$ is a poly-(n,m)-groupoid, then one can obtain an (n,m)-groupoid $\underline{P}_{\#}=(P;g_{\#})$, where

$$g_{\#}(a_1^n) = g(a_1^n),$$

i.e. $g_{\#}$ is the restriction of g on P^n. Therefore, the (n,m)-groupoid $\underline{P}_{\#}$ is called a restriction of the poly-(n,m)-groupoid $\underline{P}$.

Obviously, if $\underline{Q}=(Q;f)$ is an (n,m)-groupoid, then

$$(f^{\pi})_{\#}(a_1^n) = f^{\pi}(a_1^n) = f(a_1^n),$$

i.e.

$$(\underline{Q}^{\pi})_{\#} = \underline{Q},$$

for any choice π.

However, if $\underline{P}$ is a poly-(n,m)-groupoid, then it may happen $(\underline{P}_{\#})^{\pi} \neq \underline{P}$. We will give below (P.5.10) a sufficient condition for the equality $(P_{\#})^{\pi}=\underline{P}$.

A poly-(n,m)-groupoid $\underline{P}=(P;g)$ is called a poly-(n,m)-semigroup iff the following equality

$$g(a_1^j g(b_1^{m+rk}) a_{j+1}^{sk}) = g(a_1^j b_1^{m+rk} a_{j+1}^{sk}) \tag{5.10}$$

is an identity in $\underline{P}$, for every $a_{\nu}, b_{\lambda} \in P$, $r,s \geq 1$ and $j \in \{0,1,\dots,sk\}$.

By GAL we obtain the following:

Proposition 5.9. *Let $\underline{Q}=(Q;f)$ be an (n,m)-semigroup. Then:*

(i) $\underline{Q}^{\#}$ is the unique poly-(n,m)-groupoid induced by $\underline{Q}$;

(ii) $(\underline{Q}^{\#})_{\#} = \underline{Q}$;

(iii) $\underline{Q}^{\#}$ is a poly-(n,m)-semigroup. □

The following proposition is also true.

Proposition 5.10. *Let* $\underline{P}=(P;g)$ *be a poly-(n,m)-semigroup. Then:*

(i) the restriction $\underline{P}_{\#}$ *is an (n,m)-semigroup;*

(ii) $(\underline{P}_{\#})^{\#} = \underline{P}$. □

The notion of a cancellative (n,m)-groupoid and of a poly-(n,m)-group as well can be introduced in an obvious way. Namely, a poly-(n,m)-groupoid $\underline{P}=(P;g)$ is said to be cancellative iff the following quasi-identity

$$g(a_1^j x_1^m a_{j+1}^{sk}) = g(a_1^j y_1^m a_{j+1}^{sk}) \Longrightarrow x_1^m = y_1^m, \qquad (5.11)$$

holds in $\underline{P}$ for every $s \geq 1$ and every $j \in \{0,1,\ldots,sk\}$.

A poly-(n,m)-groupoid $\underline{P}=(P;g)$ is called a poly-(n,m)-group iff $\underline{P}$ is a poly-(n,m)-semigroup and any equations on x_1^m, y_1^m of the form

$$g(a_1^{sk} x_1^m) = b_1^m = g(y_1^m a_1^{sk}), \qquad (5.12)$$

are solvable in $\underline{P}$ for every $s \geq 1$.

The following proposition is true:

Proposition 5.11. *Let* $\underline{Q}$ *be an (n,m)-groupoid.*

(i) $\underline{Q}$ *is a cancellative (n,m)-semigroup iff* $\underline{Q}^{\#}$ *is a cancellative poly-(n,m)-semigroup;*

(ii) $\underline{Q}$ *is an (n,m)-group iff* $\underline{Q}^{\#}$ *is a poly-(n,m)-group.* □

By P.5.9. and P.5.10., it is not necessary to make any distinction between an (n,m)-semigroup and its induced poly-(n,m)-semigroup because there is no essential difference between them. However there is a reasonable motivation for introducing poly-(n,m)-semigroups, as we will see in §6 in the construction of free v.v. semigroups.

The above discussion allows us to write $[a_1^{m+sk}]$ instead of $g(a_1^{m+sk})$ in both cases: for (n,m)-semigroups (s=1) and for poly-(n,m)-semigroups ($s \geq 1$, arbitrary). In this notation we will admit also s=0, setting $[a_1^m]=a_1^m$.

We will use this notation in the following:

Proposition 5.12. *Let $(Q;f)$ be a cancellative (n,m)-semigroup and let $\underline{a}\underline{x}\underline{b}$, $\underline{a}\underline{y}\underline{b}$, $\underline{a}'\underline{x}\underline{b}'$, $\underline{a}'\underline{y}\underline{b}'\in Q^{(n,m)}$. Then:*

$$[\underline{a}\underline{x}\underline{b}] = [\underline{a}\underline{y}\underline{b}] \Longrightarrow [\underline{a}'\underline{x}\underline{b}'] = [\underline{a}'\underline{y}\underline{b}'].$$

Proof. Let $[\underline{a}\underline{x}\underline{b}]=[\underline{a}\underline{y}\underline{b}]$ and choose an arbitrary element $e\in Q$. Let $\underline{a}\in Q^{(s-1)k-r}$ for $0 \le r < k$. Then $[\underline{a}\underline{x}\underline{b}\overset{sk}{e}] = [\underline{a}\underline{y}\underline{b}\overset{sk}{e}]$ implies that $[\underline{x}\underline{b}\overset{k-r}{e}] = [\underline{y}\underline{b}\overset{k-r}{e}]$. Let $\underline{a}'\in Q^{(t-1)k+\ell}$ for $0 \le \ell < k$. Then $[\underline{a}'\underline{x}\underline{b}\overset{k-r}{e}\overset{k-\ell}{e}] = [\underline{a}'\underline{y}\underline{b}\overset{k-r}{e}\overset{k-\ell}{e}]$, and $[\overset{k-r-\ell}{e}\underline{a}'\underline{x}\underline{b}\overset{2k}{e}] = [\overset{2k-r-\ell}{e}\underline{a}'\underline{y}\underline{b}\overset{2k}{e}]$. This implies that $[\overset{2k-r-\ell}{e}\underline{a}'\underline{x}\underline{b}] = [\overset{2k-r-\ell}{e}\underline{a}'\underline{y}\underline{b}]$. Because $\underline{a}\underline{x}\underline{b}$, $\underline{a}\underline{y}\underline{b}$, $\underline{a}'\underline{x}\underline{b}'$, $\underline{a}'\underline{y}\underline{b}'\in Q^{(n,m)}$, the same procedure leads to $[\underline{a}'\underline{x}\underline{b}']=[\underline{a}'\underline{y}\underline{b}']$. □

§6. FREE VECTOR VALUED SEMIGROUPS

In this section we will give a construction of free (n,m)-semigroups. Although the construction will make sense for $m=1$ too, we will assume that $m \ge 2$, because the description of free n-semigroups (i.e. $(n,1)$-semigroups) with a basis B is well known.

Namely, if $m=1$ and $n \ge 3$, then the subset D of B^+ defined by

$$D = \{b_1^{\alpha k+1} \mid \alpha \ge 1,\ \ b_\nu \in B\} \tag{6.1}$$

is a free n-semigroup with a basis B, the operation in D being the usual concatenation of sequences.

First we will state the following:

Proposition 6.1. *Let B be a non-empty set and $(\overline{B};f)$ be the free (n,m)-groupoid with a basis B. If $\approx$ is the least congruence on $(\overline{B};f)$ such that the corresponding quotient (n,m)-groupoid $(\overline{B}/\approx;f)$ is an (n,m)-semigroup, then $(\overline{B}/\approx;f)$ is a free (n,m)-semigroup with a basis B.*

Proof. Let $(Q;g)$ be an (n,m)-semigroup, $\xi\colon B \longrightarrow Q$ be a mapping and $\overline{\xi}\colon (\overline{B};f) \longrightarrow (Q;g)$ be the unique homomorphism which is an extension of ξ. Then $(\overline{B}/\ker\overline{\xi};f)$ is an (n,m)-semigroup.

This implies that $\approx \subseteq \ker\bar{\xi}$. If we define $\eta\colon \bar{B}/\approx \longrightarrow Q$ by $\eta(u^{\approx}) = \bar{\xi}(u)$, then we obtain that η is an extension of ξ and a homomorphism of (n,m)-semigroups. Moreover, η is unique with respect to these properties. ▯

Using the characterisation of (n,m)-semigroups as poly-(n,m)-semigroups (P.5.9., P.5.10.), we can give a description of the free (n,m)-semigroups by a corresponding factorization of free poly-(n,m)-groupoids. Namely:

Proposition 6.2. *Let $\underline{F}(B)$ be a free poly-(n,m)-groupoid with a basis B. If $\approx$ is the least congruence on $\underline{F}(B)$ such that $\underline{F}(B)/\approx$ is a poly-(n,m)-semigroup, then $\underline{F}(B)/\approx$ is a free (n,m)-semigroup with a basis B.* ▯

(Here, the notion of: a congruence, a homomorphism and a free poly-(n,m)-groupoid, we will not define explicitely. However, we will give a description of a free poly (n,m)-groupoid as well.)

In the first section (§1) we gave a description of a free (n,m)-groupoid $(\bar{B};f)$ with a given basis B. By a similar discussion one obtains the following description of a free poly-(n,m)-groupoid with a basis B.

Proposition 6.3. *Let B be a nonempty set and let*

$$C_0 = B,\ C_{p+1} = C_p \cup N_m \times C_p^{(n,m)\,1)},\quad F(B) = \bigcup_{p\geq 0} C_p$$

Define a mapping $f\colon F(B)^{(n,m)} \longrightarrow F(B)^m$ by:

$$f(u_1^{m+sk}) = v^m \iff (\forall i \in N_m)\ \ v_i = (i, u_1^{m+sk}).$$

Then $\underline{F}(B) = (F(B);f)$ is a free poly-(n,m)-groupoid with a basis B. ▯

Below we will give a more explicit description of the congruences in $(\bar{B};f)$ and $\underline{F}(B)$, which are denoted by a same symbol $\approx$.

First we define a relation $\overset{o}{\vdash}$ in $\bar{B}$ and a relation in F(B) with the same notation $\overset{o}{\vdash}$. Namely, the relation $\overset{o}{\vdash}$ in $\bar{B}$ is defined by:

---

1) *See §5 for the definition of $C_p^{(n,m)}$, p. 34: $P^{(n,m)}$.*

$$u \overset{o}{\longmapsto} v \iff u = (i, u_1^j (1, u_{j+1}^{j+n}) \ldots (m, u_{j+1}^{j+n}) u_{j+n+1}^{n+k}),$$

$$v = (i, (1, u_1^n), \ldots, (m, u_1^n) u_{n+1}^{n+k})$$

for some $j \in N_k$.

If $u, v \in F(B)$, then:

$$u \overset{o}{\longmapsto} v \iff [u = (i, x'(1,y) \ldots (m,y) x''), \quad v = (i, x'yy'')]$$

where $i \in N_m$, $x', x'' \in F(B)^*$, $x'x'' \in F(B^+)$ and $y \in F(B)^{(n,m)}$. Suppose that $\overset{\alpha}{\longmapsto}$ is defined in $\overline{B}'$ and in $F(B)$ as well. Then in each one of these cases we set

$$u \overset{\alpha+1}{\longmapsto} v \iff u = (i, x'u'x''), \quad v = (i, x'v'x''),$$

where $u' \overset{\alpha}{\longmapsto} v'$. Now,

$$u \longmapsto v \iff (\exists \alpha) \quad u \overset{\alpha}{\longmapsto} v. \qquad (6.2)$$

If $\sim$ is the symetric extension of the relation $\longmapsto$, then the relation $\approx$ is the reflexive and transitive extension of .

In other words,

$$u \sim v \iff u \longmapsto v \text{ or } v \longmapsto u,$$

and

$u \approx v$ iff there exist $t \geq 0$ and $u_1, u_2, \ldots, u_t$ such that $u_0 = u$, $u_t = v$, $u_{i-1} \sim u_i$ for $i \in \{1, 2, \ldots, t\}$.

The following problem rises naturally in the both cases: given two elements $u, v \in \overline{B}$ or $u, v \in F(B)$, find an effective procedure for answering the question whether or not $u \approx v$. One of the possibilities to solve this problem is the notion of reduced elements.

It is natural to say (in both cases) that an element u is reduced iff there is no element v such that $u \longmapsto v$. Directly from the definition of $\longmapsto$ in the two cases it follows that every sequence of elements $u_0, u_1, \ldots$, such that $u_i \longmapsto u_{i+1}$ is finite. This implies that, for every w, after a finite number of steps, one can get an element u, such that $w \approx u$ and u is reduced. If, in addition, the following proposition is true:

„u,v are reduced and $u \approx v \Longrightarrow u = v$",

then the following proposition would be true:

„$(\forall w)(\exists! u$, reduced) $w \approx u$".

Therefore, to answer the question whether or not $w \approx v$, one would find the reduced representatives of w and v, and $w \approx v$ iff these representatives are equal.

In the first case (in P.6.1.) it is possible two distinct reduced elements to be equivalent (and thus this procedure can not be used), as the following example shows.

Example 6.4. Let n=3 and m=2. If $a_\nu \in B$, and

$$w = (1,a_1(1,a_2(1,a_3^5)(2,a_3^5))(2,a_2(1,a_3^5)(2,a_3^5))) \in \overline{B},$$

then:

$$w \approx (1,a_1(1,(1,a_2^4)(2,a_2^4)a_5)(2,(1,a_2^4)(2,a_2^4)a_5)) \approx$$

$$\approx (1,(1,a_1(1,a_2^4)(2,a_2^4))(2,a_1(1,a_2^4)(2,a_2^4))a_5) \approx$$

$$\approx (1,(1,(1,a_1^3)(2,a_1^3)a_4)(2,(1,a_1^3)(2,a_1^3)a_4)a_5) = u.$$

We also have:

$$w \approx (1,(1,a_1^2(1,a_3^5))(2,a_1^2(1,a_3^5))(2,a_3^5)) = v$$

and this implies that $u \approx v$. It is clear that u and v are reduced. Thus two different reduced elements in $\overline{B}$ are equivalent. We note that such examples we have for any n,m of $m \geq 2$.

Fortunately, this is not possible in F(B), i.e. two distinct reduced elements are not equivalent. Thus, to obtain the free poly-(n,m)-semigroup with a basis B, it is necessary to give a description of the reduced elements and the procedure for obtaining reduced representatives of given elements as well.

First we define a notion of a length of an element $x \in X$, in notation: $|x|$, where $X \in \{\overline{B}, F(B), \overline{B}^+, F(B)^+\}$, by induction, as follows (see also p. 20):

$|b| = 1$ for $b \in B$;

$|(i,u_1^{m+rk})| = \sum_{\nu=1}^{m+rk} |u_\nu|$ for $(i,u_1^{m+rk}) \in C_{p+1}$; and

$|u_1^t| = \sum_{\nu=1}^{t} |u_\nu|$ for $u_1^t \in \bar{B}^+$ or $F(B)^+$.

Denote by $S(B)=S$ the set of all the reduced elements of $F(B)$.

Now we will define a mapping ψ: $S^+ \to S^+$ as follows.

1) If $x \in S^\alpha$ and $1 \le \alpha \le m-1$, then $\psi(x)=x$.

Assume that $\psi(y) \in S$ is well defined for every $y \in S^+$ such that $|y| < |x|$ and $\psi(y)$ satisfies the following condition:

$$\psi(y) \ne y \Longrightarrow m < d(y) < d(\psi(y)) \text{ and } |\psi(y)|<|y|. \quad (6.4)$$

Now, if x has a form $x=x'(1,z)\ldots(m,z)x''$, where $x',x'' \in S^*$, $(\nu,z) \in S$ and x' has the least possible dimension, then we define $\psi(x)$ by:

2) $\psi(x) = \psi(x'zx'')$.

And, if x has not such a form, then we put:

3) $\psi(x) = x$.

The assumption (6.4) implies that if $\psi(x)$ is defined by 2), then:

$$|\psi(x)| < |x| \text{ and } d(\psi(x)) > d(x),$$

and this implies that ψ: $S^+ \to S^+$ is well defined mapping such that (6.4) holds for every $y \in S^+$.

By induction on length, the following statement can be easily shown.

Proposition 6.5. *If $x',x'',z \in S^*$, $(\nu,y) \in S$, $x \in S^+$, $i,\alpha,\beta \in N_m$, $\alpha \ne 1$, $\beta \ne m$, then:*

(i) $\psi(x'(1,y)\ldots(m,y)x'') = \psi(x'yx'')$,

(ii) $\psi(x'xx'') = \psi(x'\psi(x)x'')$,

(iii) $\psi^2 = \psi$,

(iv) $\psi(x) \ne x \Longrightarrow m < d(x) < d(\psi(x))$ and $|\psi(x)| < |x|$,

(v) $d(\psi(x)) \equiv d(x) \pmod k$,

(vi) $\psi(yx) \ne (i,y)z$, $\psi(xy) \ne z(i,y)$,

(vii) $\psi((\alpha,y)x) = (\alpha,y)\psi(x)$, $\psi(x(\beta,y)) = \psi(x)(\beta,y)$,

(viii) $(i,x) \in S$ iff $x \in F(B)^{(n,m)}$ and $\psi(x)=x$. □

Now, we are ready to prove the main result of this section.

First, if $u_1^{m+sk} \in S^{(n,m)}$ and $i \in N_m$, then $v_i = (i, \psi(u_1^{m+sk})) \in S$, and thus we can define a poly-(n,m)-groupoid $\underline{S}=(S;g)$ by:

$$g(u_1^{m+sk}) = v_1^m \text{ iff } (\forall i \in N_m) \quad v_i = (i, \psi(u_1^{m+sk})). \tag{6.5}$$

Theorem 6.6. *$\underline{S}$ is a free poly-(n,m)-semigroup with a basis B.*

Proof. By P.6.5. (i), (iii), (viii) it can be easily seen that $\underline{S}$ is a poly-(n,m)-semigroup and it is clear that B is a generating subset of S.

Assume that $\underline{Q}=(Q;f)$ is a poly-(n,m)-semigroup and $\xi: b \mapsto \bar{b}$ a mapping from B into Q. Then, there is a unique homomorphism $\bar{\xi}: \underline{F}(B) \longrightarrow \underline{Q}$. Denote by ζ the restriction of $\bar{\xi}$ on S. We will show that ζ is a homomorphism from $\underline{S}$ in $\underline{Q}$, and this will complete the proof.

Let $g(u_1^{m+sk}) = v_1^m$, i.e. $v_i = (i, \psi(u_1^{m+sk})$, and $\zeta(u_\nu) = \bar{u}_\nu$, $\zeta(v_\lambda) = \bar{v}_\lambda$. If $(1, u_1^{m+sk}) \in S$, then we have $\psi(u_1^{m+sk}) = u_1^{m+sk}$ and thus

$$\bar{v}_i = \zeta(i, u_1^{m+sk}) = f_i(\bar{u}_1^{m+sk}), \text{ i.e. } f(\bar{u}_1^{m+sk}) = \bar{v}_1^m.$$

Assume that: $\quad u_1^{m+sk} = u_1^j(1, w_1^{m+rk}) \ldots (m, w_1^{m+rk}) u_{j+m+1}^{m+sk}$.

Then:

$$v_i = (i, \psi(u_1^j w_1^{m+rk} u_{j+m+1}^{m+sk})),$$

and by induction we have:

$$\begin{aligned} \bar{v}_i &= f_i(\bar{u}_1^j \bar{w}_1^{m+rk} \bar{u}_{j+m+1}^{m+sk}) \\ &= f_i(\bar{u}_1^j f(\bar{w}_1^{m+rk}) \bar{u}_{j+m+1}^{m+sk}) \\ &= f_i(\bar{u}_1^{m+sk}). \end{aligned}$$

Thus, we showed that

$$g(u_1^{m+sk}) = v_1^m \Longrightarrow f(\bar{u}^{m+sk}) = \bar{v}_1^m. \ \square$$

As a corollary we obtain the following desired result:

Proposition 6.7. *If $u, v \in S \subset F(B)$ are such that $u \approx v$, then $u = v$.* $\square$

Note that we do not make difference between a free poly-(n,m)-semigroup and a free (n,m)-semigroup. Therefore, $\underline{S}(B)$ is a free (n,m)-semigroup.

Theorem 6.8. *Let $\underline{S}=(S;g)$ be a free (n,m)-semigroup, with a basis B of cardinality β, and $m \geq 2$. If α is a cardinal such that $\alpha \leq max\{\beta, \aleph_0\}$, then there exists an (n,m)-subsemigroup $\underline{T}$ of $\underline{S}$, which is a free (n,m)-semigroup with a basis C of cardinality α.*

Proof. Clearly, it is enough to show that if $B=\{b\}$ is a one-element set, then $\underline{S}$ has a free (n,m)-subsemigroup with an infinite basis.

Namely, if $\underline{S}$ is defined as in T.6.6., and if

$$a_r = (1, b^{m+rk}),$$

then the (n,m)-subsemigroup $\underline{T}$ of $\underline{S}$ generated by $A=\{a_r \mid r \geq 1\}$ is a free (n,m)-semigroup with a basis A. □

We note that the above result, in the case $m = 1$ holds only if $\beta \geq 2$.

Theorem 6.9. *Every free (n,m)-semigroup is cancellative.*

Proof. Let $\underline{S}=(S;g)$ be the free poly-(n,m)-semigroup with a basis B, defined as above. We will show that the following implication holds:

$$\psi(xy) = \psi(xz) \text{ or } \psi(yx) = \psi(zx) \Longrightarrow \psi(y) = \psi(z) \qquad (6.6)$$

for any x,y,z, and this will imply the desired result that $\underline{S}$ is cancellative.

Assume that $\psi(xy)=\psi(xz)$. We will show that $\psi(y)=\psi(z)$, by induction on $|xyz|$. First if $\psi(xy)=xy$, $\psi(xz)=xz$, then $y=z$. By P.6.5. (ii) we have

$$\psi(\psi(x)y) = \psi(x\psi(y)) = \psi(\psi(x)z) = \psi(x(\psi(z)),$$

and thus we can assume:

$$\psi(x) = x, \quad \psi(y) = y, \quad \psi(z) = z, \text{ and } \quad \psi(xy) \neq xy.$$

Therefore, we have:

$$x = x'(1,x")\ldots(\beta,x"), \quad y = (\beta+1,x")\ldots(m,x")y',$$

for some $\beta \in N_{m-1}$.

Then, one of the following conditions holds:

$$a)\ \psi(xz) = xz, \quad b)\ z = (\beta+1,x'')\dots(m,x'')z'.$$

In the case a) we would have:

$$x'(1,x'')\dots(\beta,x'')z = \psi(x'x''y')$$

and this would imply:

$$(1,x'')\dots(\beta,x'')z = \psi(x''y'),$$

which is impossible by P.6.5. (vi).

If b) holds, then we have:

$$\psi(x'x''y') = \psi(x'x''z'),$$

and this (by the induction) implies $\psi(y')=\psi(z')$, hence (by P.6.5. (vii)):

$$\begin{aligned}\psi(y) &= (\beta+1,x'')\dots(m,x'')\psi(y')\\ &= (\beta+1,x'')\dots(m,x'')\psi(z')\\ &= \psi(z).\end{aligned}$$

Thus, $\psi(xy)=\psi(xz) \Longrightarrow \psi(y)=\psi(z)$, and by symmetry: $\psi(yx)=\psi(zx) \Longrightarrow \psi(y)=\psi(z)$. This completes the proof of (6.6).

Assume that

$$g(u_1^{sk}v_1^m) = g(u_1^{sk}w_1^m),$$

where u_ν, v_λ, $w_\lambda \in S$, $s \geq 1$, i.e.

$$\psi(u_1^{sk}v_1^m) = \psi(u_1^{sk}w_1^m).$$

By (6.6), this implies $\psi(v_1^m)=\psi(w_1^m)$, hence by P.6.5. (iv), either $v_1^m=\psi(v_1^m)=\psi(w_1^m)=w_1^m$, or $\psi(v_1^m)=y=\psi(w_1^m)$, where $v_1^m=(1,y)(2,y)\cdots(m,y)= =w_1^m$. □

Let $\underline{S}=(S;[\ ])$ be the free (n,m)-semigroup with a basis B as above. (Here we denote g by $[\ ]$). Denote by $\hat{S}$ the set $\psi(S^+)$, i.e.

$$\hat{S} = \{x \in S^+ \mid \psi(x) = x\}.$$

Define a (binary) operation $\bullet$ on $\hat{S}$ by:

$$x \bullet y = \psi(xy)$$

By P.6.5 (ii) and (6.6) it can be easily shown that the following statement is true:

Proposition 6.10. *$\underline{\hat{S}} = (\hat{S};\bullet)$ is a cancellative semigroup generated by S.* □

Theorem 6.11. *$\underline{\hat{S}}$ is the universal semigroup for the (n,m)-semigroup $\underline{S}$.*

Proof. We have to show that $\underline{\hat{S}}=\langle S;\Lambda\rangle$, where

$$\Lambda = \{(u_1^n, v_1^m) \mid [u_1^n] = v_1^m \text{ in } \underline{S}\}.$$

First, it is clear that the embedding from S in $\hat{S}$ is a realization of (S,Λ) in $\underline{\hat{S}}$.

Let $\xi: u \mapsto \bar{u}$ be a realization of (S,Λ) in a semigroup $\underline{H}=(H;\circ)$. We are looking for a homomorphism $\zeta: \underline{\hat{S}} \to \underline{H}$, which is an extension of ξ.

Consider first the homomorphism $\xi^+: S^+ \to \underline{H}$, defined as in §3, i.e. by:

$$\xi^+(u_1^\alpha) = \bar{u}_1 \circ u_2 \circ \ldots \circ \bar{u}_\alpha,$$

for every $u_1^\alpha \in S^\alpha$. By induction on lengths and dimensions we will show that:

$$(\forall x \in S^+)\, \xi^+(x) = \xi^+(\psi(x)). \tag{6.7}$$

We have only to consider the case when $\psi(x) \neq x$.

If $x=(1,y)\ldots(m,y)$, where, $(\nu,y)\in S$, $y=u_1^{m+rk}$, then:

$$\xi^+(x) = \overline{(1,y)} \circ \overline{(2,y)} \circ \ldots \circ \overline{(m,y)}$$

and:

$$\xi^+(\psi(x)) = \xi^+(y) = \bar{u}_1 \circ \ldots \circ \bar{u}_{m+rk} = \xi^+(x),$$

for $[u_1^{m+rk}] = v_1^m$, where $v_\lambda=(\lambda,y)$.

Assume now that, $x=x'(1,y)\ldots(m,y)x''$, where $x',x''\in S^*$, $x'x''\in S^+$, $(\nu,y)\in S$. Then we have:

$$\begin{aligned}\xi^+(\psi(x)) &= \xi^+(x'yx'') = \xi^*(x')\xi^+(y)\xi^*(x'')\\ &= \xi^*(x')\xi^+((1,y)(2,y)\ldots(m,y))\xi^*(x'')\\ &= \xi^+(x'(1,y)\ldots(m,y)x'')\\ &= \xi^+(x),\end{aligned}$$

and this completes the proof of (6.7).

If $x,y\in\hat{S}$, then:

$$\zeta(x\bullet y) = \zeta(\psi(xy)) = \xi^{+}(\psi(xy)) = \xi^{+}(xy) = \xi^{+}(x)\circ\xi^{+}(y) = \zeta(x)\circ\zeta(y),$$

and this implies that $\zeta\colon \underline{\hat{S}} \longrightarrow \underline{H}$ is a homomorphism which is an extension of ξ. □

If we consider $\underline{S}$ as a poly-(n,m)-semigroup, then it is natural to consider the following presentation:

$$\Lambda' = \{(u_1^{m+sk}, v_1^m) \mid [u_1^{m+sk}] = v_1^m \text{ in } \underline{S}\}.$$

Then, we have $\langle S;\Lambda\rangle = \langle S;\Lambda'\rangle = \hat{S}$, and this statement is a corollary from the following more general

Proposition 6.12. *Let $\underline{P}=(P;g)$ be a poly-(n,m)-semigroup and Λ,Λ' sets of semigroup relations on P defined by:*

$$\Lambda = \{(a_1^n, b_1^m) \mid g(a_1^n) = b_1^m\}$$

$$\Lambda' = \{(a_1^{m+sk}, b_1^m) \mid g(a_1^{m+sk}) = b_1^m\}.$$

Then: $\langle P;\Lambda\rangle = \langle P;\Lambda'\rangle$. □

§7. UNIVERSAL COVERINGS OF VECTOR VALUED SEMIGROUPS

Here we will give a more precise description of the universal semigroup $\underline{Q}^{\wedge}$ of an (n,m)-semigroup $\underline{Q}=(Q;f)$, defined in §3. We recall that, as in §5, if $u=a_1^{sk+m}\in Q^{sk+m}$, then

$$[u] = f^{(s)}(a_1^{sk+m})\in Q^m$$

for every $s\geq 0$. The relations $\vdash$, $\sim$ and $\stackrel{\Lambda}{=}$ are defined as in §3 with $\Lambda=\Lambda_Q$.

Proposition 7.1. *If $u\in Q^m$, $v\in Q^{+}$, then*

$$u \stackrel{\Lambda}{=} v \text{ iff } v\in Q^{sk+m} \text{ and } [v]=u, \text{ for some } s\geq 0.$$

Proof. Let $v\in Q^{sk+m}$ and $[v]=u\in Q^m$. If $s = 0$, then $u = v$, and clearly $u \stackrel{\Lambda}{=} v$; if $s = 1$, then the definition of Λ_Q implies $u \stackrel{\Lambda}{=} v$. Suppose that $v=a_1^{sk+m}$, $s \geq 2$ and $[v]=u$. Then $[v]=[wa_{k+m+1}^{sk+m}]$, where $w=[a_1^{k+m}]\in Q^m$ and $v \stackrel{\Lambda}{=} wa_{k+m+1}^{sk+m}$. Since $[wa_{k+m+1}^{sk+m}]=u$, by induction on s we have $wa_{k+m+1}^{sk+m} \stackrel{\Lambda}{=} u$, i.e. $u \stackrel{\Lambda}{=} v$.

Assume now that $u\in Q^m$, $v\in Q^+$ and $u \stackrel{\Lambda}{=} v$. Then there exist $u_o,\dots,u_t\in Q^+$, $t\geq 0$, such that $u=u_o$, $v=u_t$ and $u_{i-1} \sim u_i$ for $i\in N_t$. By P.3.3. c) we have that $d(u_{i-1}) \equiv d(u_i) \equiv m(\mathrm{mod}k)$ for every $i\in N_t$, and so it is enough to prove that $[u_{i-1}]=[u_i]$. But, the last equality is true by the definition of $\vdash$ and the GAL. □

<u>Proposition 7.2</u>. *If $u\in Q^\alpha$ and $\alpha \geq m$ then there exists a unique $\beta\in\{0,1,\dots,k-1\}$ such that $\alpha-m \equiv \beta(\mathrm{mod}k)$ and $u \stackrel{\Lambda}{=} v$ for some $v\in Q^{m+\beta}$.*

<u>Proof</u>. Let $\alpha=m+\gamma k+\beta$, $0 \leq \beta < k$, and suppose that $u=u'u''$, where $u'\in Q^{m+\gamma k}$, $u''\in Q^\beta$. Then, by P.7.1., $u' \stackrel{\Lambda}{=} [u']$, which implies $u \stackrel{\Lambda}{=} v$, where $v=[u']u''\in Q^{m+\beta}$. □

As a consequence of P.7.1. and P.7.2., we have that:

$$u,v\in Q^m \text{ and } u \stackrel{\Lambda}{=} v \text{ imply } u = v.$$

i.e.

$$Q^m \subseteq Q^\wedge.$$

Thus, by P.3.3. and the above remark, we have the following description of the universal semigroup $\underline{Q}^\wedge$.

<u>Theorem 7.3</u>. *The universal semigroup $\underline{Q}^\wedge$ of an (n,m)-semigroup $\underline{Q}$ has a carrier $Q^\wedge$ represented as a disjoint union of the form*

$$Q \cup Q^2 \cup \dots \cup Q^m \cup Q_{m+1} \cup \dots \cup Q_{n-1} \tag{7.1}$$

where $Q_{m+\beta}=Q^{m+\beta}/\stackrel{\beta}{=}$ and $\stackrel{\beta}{=}$ is the restriction of $\stackrel{\Lambda}{=}$ on $Q^{m+\beta}$ for every $\beta\in N_{k-1}$. □

Note that, by using the multiplicative notation • for the operation on $Q^\wedge$, we have that

$$a_1\dots a_i \bullet b_1\dots b_j = \begin{cases} a_1\dots a_i b_1\dots b_j & \text{if } i+j < n \\ [a_1^i b_1^{n-i}]\bullet b_{n-i+1}\dots b_j & \text{if } i+j \geq n \end{cases} \tag{7.2}$$

We will denote by $Q^\vee$ the subset

$$Q^m \cup Q_{m+1} \cup \dots \cup Q_{n-1}$$

of $Q^\wedge$ and we say that $Q^\vee$ is the <u>universal envelope</u> of $\underline{Q}$. It is clear that:

Proposition 7.4. *$Q^{\vee}$ is an ideal in $\underline{Q}^{\wedge}$.* □

Note that the set $N_{n-1}=N_{k+m-1}=\{1,\dots,m,m+1,\dots,n-1\}$ is a cyclic semigroup generated by 1, of an index m and a period k, with respect to the operation ⊕ defined by:

$$\alpha\oplus\beta = \begin{cases}\alpha+\beta & \text{if } \alpha+\beta \le n-1\\ \alpha+\beta-tk & \text{if } m+tk \le \alpha+\beta < m+(t+1)k\end{cases} \qquad (7.3)$$

The following proposition follows directly from (7.2) and (7.3).

Proposition 7.5. *The map $\|\ \|: Q^{\wedge} \longrightarrow N_{n-1}$ defined by $\|u\| = \alpha$ iff $u\in Q^{\alpha}$ or $u\in Q_{\alpha}$, is a homomorphism from $\underline{Q}^{\wedge}$ onto $(N_{n-1};\oplus)$.* □

If $m \le \ell k < m+k$, then ℓk is the neutral element in the subgroup $Z_k'=\{m,m+1,\dots,m+k-1\}$ of $(N_{n-1};\oplus)$. This, and P.7.5. imply:

Corollary 7.6. *$Q_{\ell k}$ is a subsemigroup of $\underline{Q}^{\vee}$ and so of $\underline{Q}^{\wedge}$.* □

T.7.3. implies that the following is true:

Theorem 7.7. *Every (n,m)-semigroup $\underline{Q}$ is a pure (n,m)-subgroupoid of its universal semigroup $\underline{Q}^{\wedge}$.*

(In this case we say that $\underline{Q}$ is a pure (n,m)-subsemigroup of $\underline{Q}^{\wedge}$.) □

This result is a generalization of Post's theorem for polyadic groups ([4], [6], [41]) and that is why we reffer to it as Post Theorem.

Further on, according to T.7.7., the semigroup $\underline{Q}^{\wedge}$ will be called a universal covering of the (n,m)-semigroup $\underline{Q}$.

We note that, if $\underline{P}=(P;g)$ is a poly-(n,m)-semigroup, then the semigroup $\langle P;\ \Gamma(\underline{P})\rangle$, where

$$\Gamma(\underline{P}) = \{(a_1^{m+sk}, b_1^m) \mid g(a_1^{m+sk}) = b_1^m,\ s\ge 1,\ a_\nu, b_\lambda \in P\},$$

coincides with the universal covering $\underline{Q}^{\wedge}$ of the restriction $\underline{Q}=\underline{P}_{\#}$ of $\underline{P}$.

A semigroup $\underline{S}=(S;\cdot)$ is said to be a covering of an (n,m)-groupoid $\underline{Q}$ iff $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{S}$ and $\underline{S}$ is generated by Q. Every covering of an (n,m)-semigroup $\underline{Q}$ is a homomorphic image of the universal covering $\underline{Q}^{\wedge}$, i.e.

Proposition 7.8. *If a semigroup* $\underline{S}$ *is a covering of an (n,m)-semigroup* $\underline{Q}$, *then the inclusion* $a \mapsto a$ *of Q into S can be uniquely extended to a homomorphism of* $\underline{Q}^{\wedge}$ *into* $\underline{S}$. □

Proposition 7.9. *If the universal envelope* $\underline{Q}^{\vee}$ *is a cancellative semigroup, then* $\underline{Q}$ *is a cancellative (n,m)-semigroup. In this case, if* $a_1^{m+i}, b_1^{m+i} \in Q^{m+i}$, $0 \le i < k$, *then the following conditions are equivalent:*

(i) $a_1 \ldots a_{m+i} = b_1 \ldots b_{m+i}$ *in* Q_{m+i};

(ii) the equality

$$[c_1^{sk-i} a_1^{m+i}] = [c_1^{sk-i} b_1^{m+i}] \tag{7.4}$$

holds in $\underline{Q}$ *for every* $s \ge 1$ *and every* $c_\nu \in Q$;

(iii) there exist $s \ge 1$ *and* $c_\nu \in Q$ *such that the equality (7.4) holds in* $\underline{Q}$.

Proof. Let $u \in Q^k$, $v,w \in Q^m$ and suppose that $[uv]=[uw]$. Then in $\underline{Q}^{\vee}=(Q^{\vee};\bullet)$ we have $u\bullet v=[uv]=[uw]=u\bullet w$, which implies $v=w$. Similarly, $[vu]=[wu]$ implies $v=w$.

It is clear that (i) ⟹ (ii), (ii) ⟹ (iii).

Suppose that for some $s \ge 1$ and some $c_\nu \in Q$ the equality (7.4) holds in $\underline{Q}$. Then we have in $\underline{Q}^{\vee}$

$$c_1\bullet\ldots\bullet c_{sk-i}\bullet a_1\ldots a_{m+i} = c_1\bullet\ldots\bullet c_{sk-i}\bullet b_1\ldots b_{m+i}$$

and multiplying by $d_1\ldots d_{m+i}$,

$$[d_1^{m+i} c_1^{sk-i}]\bullet a_1\ldots a_{m+i} = [d_1^{m+i} c_1^{sk-i}]\bullet b_1\ldots b_{m+i}.$$

The last equality implies $a_1\ldots a_{m+i}=b_1\ldots b_{m+i}$ in $\underline{Q}^{\vee}$, i.e. in $\underline{Q}^{\wedge}$.□

Next we will show that every cancellative (n,m)-semigroup admits a cancellative covering.

Theorem 7.10. *Let* $\underline{Q}$ *be a cancellative (n,m)-semigroup and define a relation* $\approx$ *on* Q^+ *by*

$$u \approx v \iff (\exists w \in Q^+)[uw] = [vw]. \tag{7.5}$$

Then $\approx$ *is a congruence on* Q^+ *and* $\underline{Q}^{\sim} = Q^+/\approx$ *is a cancellative covering of* $\underline{Q}$.

Proof. If $u\in Q^{\alpha}$, $v\in Q^{\beta}$ and $u \approx v$ then $\alpha \equiv \beta\,(\mathrm{mod}\,k)$; namely, if $w\in Q^{\gamma}$ and $[uw]=[vw]$, then $\alpha+\gamma \equiv \beta+\gamma\ (\mathrm{mod}\,k)$.

From P.5.12., it follows that $u \approx v$ iff $d(u) \equiv d(v)\,(\mathrm{mod}\,k)$ and $[w_1uw_1]=[w_1vw_2]$ for every $w_1,w_2\in Q^+$ such that $d(w_1uw_2)\equiv m\ (\mathrm{mod}\,k)$. Now, it is easy to see that $\approx$ is a congruence on Q^+ and that

$$wu \approx wv \text{ or } uw \approx vw \text{ implies } u \approx v.$$

Thus, the factor semigroup $\underline{Q}^{\sim} = Q^+/\approx$ is cancellative.

We can assume that $Q\subseteq Q^{\sim} = Q^+/\approx$, since

$$a,b\in Q \text{ and } a \approx b \text{ implies } [\overset{n}{a}]=[ba^{n-1}] \text{ in } \underline{Q},$$

i.e. $\overset{m}{a}=\overset{m}{b}\,\overset{m-1}{a}$, after cancelling. Hence, $a=b$.

Let $[a_1^n]=b_1^m$ in $\underline{Q}$. Then $a_1^n \approx b_1^m$, i.e. $a_1a_2\ldots a_n=b_1\ldots b_m$ in $\underline{Q}^{\sim}$. This means that $\underline{Q}$ is an (n,m)-subsemigroup of $\underline{Q}^{\sim}$. In fact, $\underline{Q}$ is a pure (n,m)-subsemigroup of $\underline{Q}^{\sim}$, since

$$a_1^m \approx b_1^m \text{ implies } [c_1^k a_1^m] = [c_1^k b_1^m] \text{ in } \underline{Q},$$

and the cancellativity of $\underline{Q}$ implies $a_1^m = b_1^m$.

It is clear that $\underline{Q}^{\sim}$ is generated by Q. □

Example 7.11. Let $(Q;[\ ])$ be a constant (n,m)-semigroup defined as in E.2.6.1), i.e. there is an $a_1^m\in Q^m$ such that

$$[x_1^n] = a_1^m$$

for all $x_\nu\in Q$. Then:

$$u \overset{\Lambda}{=} v \iff u = v \text{ or } (u=u'a_1^m u'',\ v=v'a_1^m v''), \tag{7.6}$$

where $u,v\in Q^+$, $u'u''$, $v'v''\in Q^+$ (and $\Lambda=\Lambda_{\underline{Q}}$). Namely, let $u=b_1^i a_1^m b_{i+1}^j$, $v=c_1^p a_1^m c_{p+1}^j$ for some $b_\nu,c_\lambda\in Q$, $i,p\geq 0$. Then, the fact that $x_1^n \overset{\Lambda}{=} y_1^n \overset{\Lambda}{=} a_1^m$ for every $x_\nu,y_\lambda\in Q$ implies

$$\begin{aligned} u &\overset{\Lambda}{=} b_1^i x_1^n b_{i+1}^j \overset{\Lambda}{=} b_1^i x_1^{n-i} x_{n-i+1}^n b_{i+1}^j \\ &\overset{\Lambda}{=} c_1^p y_1^{n-p} x_{n-i+1}^n b_{i+1}^j \overset{\Lambda}{=} c_1^p y_1^n c_{p+1}^j \\ &\overset{\Lambda}{=} c_1^p a_1^m c_{p+1}^j \overset{\Lambda}{=} v. \end{aligned}$$

Conversely, let $u \overset{\Lambda}{=} v$ and $u \neq v$. Then we can apply some of the defining relations from Λ iff u and v contain a_1^m as a subword.

As a consequence of (7.6) we have that

$$Q_\alpha = \{u\in Q^+ \mid |u|=\alpha,\ a_1^m \text{ is not a subword of } u\} \cup \{a_1^m\ {}^{\alpha-m}_{b}\}$$

for every α: $m<\alpha<n$, where b is a fixed element of Q. Hence, the multiplication in Q^ can be given by

$$c_1^s \bullet d_1^t = \begin{cases} c_1^s d_1^t, & \text{if } s+t<n \text{ and } a_1^m \text{ is not a subword of } c_1^s d_1^t \\ a_1^m\ \overset{r}{b}, & \text{otherwise,} \end{cases}$$

where $0\le r<k$, $r \equiv s+t-m$ (modk).

It can be easily seen that if $|Q|=q<\infty$, then $|Q\hat{}|<\infty$ as well, and moreover

$$|Q\hat{}| = k-2+\frac{q^{m+1}-1}{q-1}+\frac{kq^{k+1}-(k+1)q^k+1}{(q-1)^2}$$

for $q\ge 2$, and $|Q\hat{}|=n-1$ for $q=1$.

Example 7.12. Let (Q;[ ]) be a left zero (n,m)-semigroup defined as in E.2.6. 2). Then we have

$$u \overset{\Lambda}{=} v \iff u = v \text{ or } (d(u)=d(v)) \text{ and } (\exists c_1^m \in Q^m)(u=c_1^m u',\ v=c_1^m v') \qquad (7.7)$$

where $u,v\in Q^+$, $u',v'\in Q^*$. Namely, the defining relations imply that if $u \overset{\Lambda}{=} v$, then the first m elements of u and v are the same. On the other hand, since $u \overset{\Lambda}{=} v$ imply $|u| \equiv |v|$ (modk), if $u=c_1^m b_1^j$, $v=c_1^m d_1^j$ $(j\ge 0)$ then

$$u \overset{\Lambda}{=} c_1^m b_1^j \overset{\Lambda}{=} c_1^m x_1^n b_1^j \overset{\Lambda}{=} c_1^m y_1^m d_1^j$$
$$\overset{\Lambda}{=} c_1^m d_1^j \overset{\Lambda}{=} v.$$

Now we have the following description of Q^:

$$Q_\alpha = \{u\in Q^+ \mid u=v\ \overset{\alpha-m}{b},\ |v|=m\},$$

where $m<\alpha<n$ and b is a fixed element of Q. The multiplication on Q^ is given by

$$c_1^s \bullet d_1^t = \begin{cases} c_1^s d_1^t, & \text{if } s+t\le m \\ c_1^m\ \overset{r}{b}, & \text{if } s\ge m \\ c_1^s d_1^{m-s}\ \overset{r}{b}, & \text{otherwise} \end{cases}$$

where $0\le r<k$, $r\equiv s+t-m$ (modk).

If $|Q|=q<\infty$ then $|Q\hat{}|<\infty$ as well, and

$$|Q^{\wedge}| = kq^{m-1} + \frac{q^m - 1}{q-1}$$

for $q \geq 2$, and $|Q^{\wedge}| = n-1$ for $q = 1$.

By duality, we have corresponding results for right zero (n,m)-semigroups too.

Example 7.13. Let $(Q;[\ ])$ be an (n,m)-rectangular band, defined in E.2.6. 3), where $Q=A\times B$, $\underline{A}$ is a left zero and $\underline{B}$ is a right zero (n,m)-semigroup. One can show that if $u,v\in Q^+$, $m < |u| = |v| < n$, then

$$u \overset{\Lambda}{=} v \iff u = (a_1,b_1)\dots(a_i,b_i),$$
$$v = (c_1,d_1)\dots(c_i,d_i),\ a_1^m = c_1^m,\ b_{i+1-m}^i = d_{i+1-m}^i,$$

where $a_\nu, c_\nu \in A$, $b_\lambda, d_\lambda \in B$. Also, if $|A| = \alpha < \infty$, $|B| = \beta < \infty$, then

$$|Q^{\wedge}| = |(A\times B)^{\wedge}| = k\alpha^m\beta^m - 1 + \frac{\alpha^m\beta^m - 1}{\alpha\beta - 1}$$

for $\alpha\beta \geq 2$, and $|Q^{\wedge}| = n-1$ for $\alpha\beta = 1$.

Example 7.14. A universal covering semigroup of a free (n,m)-semigroup $\underline{S}=(S;[\ ])$ is the semigroup $\underline{\hat{S}}=(\hat{S};\bullet)$ defined in §6. But $\underline{\hat{S}}$ has not the usual form:

$$\hat{S} = S \cup S^2 \cup \dots \cup S^m \cup S_{m+1} \cup \dots \cup S_{n-1}. \tag{7.8}$$

To get such a form we have to make a modification.

Define first a mapping $x \mapsto \overline{x}$ from $\hat{S}$ in S^+, in the following way. If $x\in\hat{S}$, then:

$$\overline{x} = \begin{cases} x & \text{if } 1 \leq d(x) < n \\ (1,y)\dots(m,y)z, & \text{where } x=yz,\ y\in S^{(n,m)},\ 0 \leq d(z) < k. \end{cases}$$

Denote by $S^{\wedge}$ the set $\{\overline{x} \mid x\in\hat{S}\}$, and define a (binary) operation $\bullet$ on $S^{\wedge}$ by:

$$\overline{x}\bullet\overline{y} = \overline{\psi(xy)}.$$

Then we get a semigroup $\underline{S}^{\wedge}=(S^{\wedge};\bullet)$ isomorphic with $\underline{\hat{S}}$, and morover (7.8) holds.

§8. VECTOR VALUED GROUPS AND THEIR COVERINGS

In this section we will give some characterizations of v.v. groups, mainly using their coverings.

Let n,m be given integers, $n-m=k\geq 1$. Recall that an (n,m)-semigroup (Q;[ ]) is an (n,m)-group if for each $\underline{a}\in Q^k$, $\underline{b}\in Q^m$, there exist $\underline{x},\underline{y}\in Q^m$ such that $[\underline{a}\underline{x}]=\underline{b}=[\underline{y}\underline{a}]$. The question about the existence of v.v. groups will be considered later, but we know that (n,m)-groups do exist; see E.2.7. Since an (n,m)-group (Q;[ ]) is also an (n,m)-semigroup, we have the universal covering semigroup $Q^\wedge=\langle Q;\wedge_Q\rangle$, (see §3 and §7), and by T.7.3:

$$Q^\wedge = Q\cup Q^2\cup\ldots\cup Q^{m-1}\cup Q^\vee,$$

where

$$Q^\vee = Q^m\cup Q_{m+1}\cup\ldots\cup Q_{n-1}$$

is the universal envelope of Q.

Proposition 8.1. *Let (Q;[ ]) be an (n,m)-semigroup. Then the following conditions are equivalent:*

(i) (Q;[ ]) is an (n,m)-group;

(ii) $\underline{Q}^\vee$ is a group.

Proof. (i) ⟹ (ii): Let $a_1\ldots a_{m+p}$, $b_1\ldots b_{m+q}\in Q$, and let $m+p=sk+r$ for $0\leq r<k$. Let $a\in Q$. Then for $a_1^{m+p}\,\overset{k-r}{a}\in Q^{(s+1)k}$ there exists $c_1^m\in Q^m$ such that $[a_1^{m+p}\,\overset{k-r}{a}\,c_1^m]=b_1^m$ (see T.5.8). This implies that

$$a_1a_2\ldots a_{m+p}\cdot \overset{k-r}{a}\cdot c_1\ldots c_m\cdot b_{m+1}\ldots b_{m+q} = b_1b_2\ldots b_{m+q} \text{ in } \underline{Q}^\vee.$$

Hence, the equations $\underline{a}\cdot\underline{x}=\underline{b}$ have solutions in $\underline{Q}^\vee$, and the proof that the equations $x\cdot a=b$ have solutions in $\underline{Q}^\vee$, is symmetrical. So, $\underline{Q}^\vee$ is a group.

(ii) ⟹ (i): Let $a_1^{rk}\in Q^{rk}$, $b_1^m\in Q^m$ be given, where $m\leq rk<n$. Then, there exists $c_1\ldots c_{m+p}\in\underline{Q}^\vee$, such that $a_1\ldots a_{rk}\cdot c_1\ldots c_{m+p}=b_1\ldots b_m$ in $\underline{Q}^\vee$. P.3.3 implies that $m\equiv m+p+rk\ (\mathrm{mod}\,k)$, i.e. p=0. P.7.1 implies that $[a_1^{rk}c^m]=b_1^m$. Hence the equations $[a_1^{rk}x_1^m]=b_1^m$, have solutions in Q, and symmetrically, the equations $[x_1^m a_1^{rk}]=b_1^m$ have solutions in Q. This, together with P.5.5., implies that $(Q;[\ ]^{(r)})$ is v.v. group, which together with T.5.8. implies that $\underline{Q}$ is a v.v. group. □

The proofs of the following two corollaries follow directly from P.8.1, P.3.3, C.7.6, T.5.8 and P.7.9.

Corollary 8.2. *Let $(Q;[\ ])$ be an (n,m)-group, and $m \leq \ell k < m+k$. Then $(Q_{\ell k};\cdot)$ is a normal subgroup of $\underline{Q}^{\vee}$. Moreover, the factor group $\underline{Q}^{\vee}/\underline{Q}_{\ell k}$ is a cyclic group of order k.* □

Corollary 8.3. *If $\underline{Q}$ is a v.v. group, then $\underline{Q}$ is a cancellative v.v. semigroup.* □

Corollary 8.4. *Let $(Q;[\ ])$ be an (n,m)-semigroup, $m \leq \ell k < m+k$, and $t=\ell k-m$. Then $(Q;[\ ])$ is an (n,m)-group iff for each $a_1^t \in Q^t$, $(Q^m;*)$ is a group, where $\underline{x}*\underline{y}=[\underline{x}a_1^t\underline{y}]$. Moreover, each $(Q^m;*)$ is isomorphic to $(Q_{\ell k};\cdot)$.* □

Now we give the proof of P.2.8.

Proof of P.2.8. The implication (i) $\Longrightarrow$ (ii) follows from the definition of v.v. group and C.8.3. The implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i) are obvious. The implication (i) $\Longrightarrow$ (iii) follows from T.5.8 and C.8.3.

We note that P.7.9 is applicable for v.v. groups, because the universal envelope of a v.v. group is a cancellative semigroup.

Next we have the following propositions.

Proposition 8.5. *If $\underline{H}$ is an (n,m)-subgroup[1)] of an (n,m)-group $\underline{Q}$, then $\underline{H}^{\wedge}$ is a subsemigroup of $\underline{Q}^{\wedge}$, and $\underline{H}^{\vee}$ is a subgroup of $\underline{Q}^{\vee}$.*

Proof. If $a_1 \ldots a_{m+i}=b_1 \ldots b_{m+i}$ in $\underline{Q}^{\wedge}$, for $a_\nu, b_\nu \in H$, then P.7.9. implies that $a_1 \ldots a_{m+i}=b_1 \ldots b_{m+i}$ in $\underline{H}^{\wedge}$. □

Proposition 8.6. *If $\underline{Q}$ is an (n,m)-group and $a_1^i \in Q^i$, $b_1^{m+i} \in Q^{m+i}$, $1 \leq i < k$, then for each $0 \leq j \leq i$, there exists a unique $x_1^m \in Q^m$ such that*

$$a_1 \ldots a_j \cdot x_1 \ldots x_m \cdot a_{j+1} \ldots a_i = b_1 \ldots b_{m+i} \text{ in } \underline{Q}^{\wedge}.$$

Proof. Let $c_1^{k-i} \in Q^{k-i}$ be an arbitrary element. Then the equation

---

[1)] *i.e. $H \subseteq Q$ and $\underline{H}$ is an (n,m)-group with respect to the (n,m)-operation of $\underline{Q}$.*

$$[c_1^{k-i}a_1^j x_1^m a_{j+1}^i] = [c_1^{k-i}b_1^{m+i}]$$

has a unique solution $x_1^m \in Q^m$. Now, the conclusion follows from P.7.9. □

As a consequence of the above propositions we have the following:

Corollary 8.7. *Let $(Q;[\ ])$ be an (n,m)-group. Then, for each $a\in Q$*

$$Q^{\vee} = Q^m \cup aQ^m \cup \ldots \cup \overset{k-1}{a} Q^m,$$

where $\overset{i}{a}Q^m = \{\overset{i}{a}x_1^m \mid x_1^m \in Q^m\}$. Moreover, the operation $\bullet$ on $Q^\wedge$ is given by:

$$(x_1\ldots x_r)\bullet(y_1\ldots y_s) = \begin{cases} x_1\ldots x_r y_1\ldots y_s & \text{if } r+s\le m \\ \overset{r+s-pk-m}{a} z_1^m & \text{if } m+pk\le r+s < m+(p+1)k, \end{cases}$$

where z_1^m is the unique element from Q^m such that

$$x_1\ldots x_r y_1\ldots y_s = \overset{r+s-pk-m}{a} z_1\ldots z_m$$

(see P.8.6.), i.e.

$$[\overset{(p+1)k+m-r-s}{a} x_1^r y_1^s] = [\overset{k}{a} z_1^m]. \ \square$$

Using P.7.5 and C.8.7 we obtain:

Corollary 8.8. *Let $(Q;[\ ])$ be an (n,m)-group. Then $\underline{Q}^{\vee}$ is isomorphic to $(Z_k\times Q^m;*)$ where*

$$(i,x_1^m)*(j,y_1^m) = (i+j,z_1^m),$$

z_1^m is defined as in C.8.7, and $i+j$ is in the group $(Z_k;+)$.

Proof. C.8.7 implies that $\underline{Q}^{\vee}$ is isomorphic to $(Z_k' \times Q^m, *')$ where

$$Z_k' = \{m,m+1,\ldots,m+k-1\},\ (i,x_1^m)*'(j,y_1^m) = (i\oplus j,z_1^m).$$

The conclusion follows from the fact that $(Z_k';\oplus)$ is isomorphic to $(Z_k;+)$. □

Now we consider the question about the existence of a covering semigroup $\underline{S}$ for a given v.v. group $\underline{Q}$, such that $\underline{S}$ is a group. We note that the universal covering semigroup $\underline{Q}^\wedge$ is not a group for $m\ge 2$, but the answer to the above question is positive.

Proposition 8.9. *The universal cancellative covering semigroup* $\underline{Q}^{\sim}$ *(defined in §7.) for a v.v. group* $\underline{Q}$ *is isomorphic to the universal envelope* $\underline{Q}^{\vee}$.

Proof. Suppose that $\underline{S}=(S;\bullet)$ is an arbitrary covering semigroup of $\underline{Q}$. Let $S^{\vee}=\{a_1\bullet\ldots\bullet a_t \mid t\geq m,\ a_{\nu}\in Q\}$. Then $S^{\vee}$ is an ideal of $\underline{S}$, and moreover, $\underline{S}^{\vee}$ is a homomorphic image of $\underline{Q}^{\vee}$. Hence $\underline{S}^{\vee}$ is a group. Now let $\underline{S}=\underline{Q}^{\sim}$ be the universal cancellative covering semigroup for $\underline{Q}$. Let $a\in Q$ be a fixed element. Then by C.8.5., $\underline{Q}^{\vee}=Q^m\cup aQ^m\cup\ldots\cup \overset{k-1}{a}Q^m$. Let $b\in Q$ and let $j\in\{1,\ldots,k-1\}$ such that k is a divisor of $n+j-1$. For $[\overset{n-1}{a}b]$ there exists a unique $b_1^m\in Q^m$ such that $[\overset{n-1}{a}b]=[\overset{n+j-1}{a}b_1^m]$. Hence

$$\overset{n-1}{a}b = \overset{n+j-1}{a}b_1\ldots b_m \text{ in } \underline{Q}^{\sim}, \text{ i.e. } b=a^jb_1\ldots b_m.$$

If $\overset{i}{a}x_1^m=\overset{j}{a}y_1^m$ in $\underline{Q}^{\sim}$, then $i=j$ and $x_1^m=y_1^m$ in $\underline{Q}^{\sim}$, and since $\underline{Q}$ is a group, $x_1^m=y_1^m$ in Q^m. Now, since Q generates $\underline{Q}^{\sim}$ and each element of Q is an image of an element of $\underline{Q}^{\vee}$, it follows that $\underline{Q}^{\sim}$ is a group. □

Corollary 8.10. *If* $\underline{S}$ *is a cancellative covering semigroup of a v.v. group, then* $\underline{S}$ *is a group.* □

Next we give the following:

Corollary 8.11. *If* $\underline{Q}$ *is a v.v. group, such that* $|Q|=q<\infty$, *then*

(a) $|Q_{m+i}| = |a^iQ^m|=q^m,\ i\in\{0,1,2,\ldots,k-1\}$;

(b) $|Q^{\vee}| = k\cdot q^m$; *and*

(c) $|Q^{\wedge}| = q+q^2+\ldots+q^{m-1}+k\cdot q^m$.

Proof. Follows from C.8.7. □

Corollary 8.12. *(Lagrange Theorem). Let* $\underline{H}$ *be an* (n,m)*-subgroupoid of an* (n,m)*-group* $\underline{Q}$, *and let* $|Q|=q<\infty$. *Then* $|H|=p$ *is a divisor of* q.

Proof. P.8.5 implies that $\underline{H}^{\vee}$ is a subgroup of $\underline{Q}^{\vee}$. Using C.8.11 and Lagrange Theorem for groups, we have that $k\cdot p^m$ is a divisor of $k\cdot q^m$, which implies that p is a divisor of q. □

Now, we give a description of the universal covering semigroup for a special kind of v.v. groups, by considering a few examples.

<u>Example 8.13</u>. Let $(Q;[\ ])$ be an (n,m)-group and $m \leq \ell k < m+k$. Suppose that the subgroup $\underline{Q}_{\ell k}$ of $\underline{Q}^{\vee}$ has a neutral element equal to $\overset{\ell k}{e}$ for $e \in Q$. Then

$$[x_1^m \overset{\ell k}{e}\ ] = [\ \overset{\ell k}{e} x_1^m] = x_1^m$$

for each $x_1^m \in Q^m$. C.8.7 implies that the universal envelope $Q^{\vee}$ has the form

$$Q^{\vee} = Q^m \cup eQ^m \cup \ldots \cup \overset{k-1}{e} Q^m,$$

and moreover, the multiplication on $Q^{\wedge}$ is given by:

$$x_1^r \bullet y_1^s = \begin{cases} x_1^r y_1^s & \text{if } r+s \leq m \\ \overset{t}{e}\left[\overset{\ell k-t}{e}\ x_1^r y_1^s\right] & \text{if } r+s \geq m, \end{cases}$$

where $0 \leq t = r+s-m-pk < k$.

<u>Example 8.14</u>. Let $\underline{G}$ be a group with a neutral element $e \in G$. Then $(G;[\ ])$, where $[x_1^m y_1^m] = (x_1 y_1, \ldots, x_m y_m)$ is a $(2m,m)$-group; see E.2.7. 1). Such $(2m,m)$-groups are called „<u>trivial</u>" <u>$(2m,m)$-groups</u>. In this case $\ell k = m$, and the multiplication in

$$G^{\wedge} = G \cup \ldots \cup G^m \cup eG^m \cup \ldots \cup \overset{m-1}{e} G^m$$

is given by:

$$x_1^r \bullet y_1^s = \begin{cases} x_1^r y_1^s & \text{if } r+s \leq m \\ \overset{r+s-m}{e}\left[\overset{2m-r-s}{e}\ x_1^r y_1^s\right] & \text{if } r+s > m, \end{cases}$$

$$e^r x_1^m \bullet y_1^s = \begin{cases} \overset{r+s}{e}\left[\overset{m-s}{e} x_1^m y_1^s\right] & \text{if } r+s < m \\ \overset{r+s-m}{e}\left[\overset{m-s}{e} x_1^m y_1^s\right] & \text{if } r+s \geq m, \end{cases}$$

$$x_1^r \bullet \overset{s}{e} y_1^m = \begin{cases} \overset{r+s}{e}\left[\overset{m-r-s}{e} x_1^r \overset{s}{e} y_1^m\right] & \text{if } r+s < m \\ \overset{r+s-m}{e}\left[\overset{2m-r-s}{e} x_1^r \overset{s}{e} y_1^m\right] & \text{if } r+s \geq m, \end{cases}$$

$$e^r x_1^m \bullet \overset{s}{e} y_1^m = \begin{cases} \overset{r+s}{e}\left[\overset{m-s}{e} x_1^m \overset{s}{e} y_1^m\right] & \text{if } r+s < m \\ \overset{r+s-m}{e}\left[\overset{2m-s}{e} x_1^m \overset{s}{e} y_1^m\right] & \text{if } r+s \geq m, \end{cases}$$

where $0 \leq r,\ s < m$.

Example 8.15. Let (G;[ ]) be the (4,2)-group given in E.2.7. 3). Then $G^{\wedge}=G\cup G^2\cup 0G^2$, and $x\cdot y=xy$, $x\cdot(yz)=(xy)\cdot z=0[0xyz]$, $(xy)\cdot(zt)=[xyzt]$, $x\cdot 0yz=[x0yz]$, $0xy\cdot z=[0xyz]$, $xy\cdot 0zt=0[yxzt]$, $0xy\cdot zt=0[xyzt]$ and $0xy\cdot 0zt=[yxzt]$.

Example 8.16. Let (G;[ ]) be an (n+1,n)-group. Then $G^{\wedge}=G\cup G^2\cup\ldots\cup G^n$ and the multiplication on $G^{\wedge}$ is given by:

$$x_1^r\cdot y_1^s=\begin{cases} x_1^r y_1^s & \text{if } r+s\le n\\ [x_1^r y_1^s] & \text{if } r+s>n.\end{cases}$$

At the end of this section, we give a corresponding generalization of Hosszú-Gluskin's theorem for some types of v.v. groups.

Theorem 8.17. *Let (G;[ ]) be an (sm,m)-group. Then there exist: a binary group $(G^m;\bullet)$, an element $c\in G^m$, and an automorphism θ of this group, such that for each $a_1,\ldots,a_s\in G^m$,*

$$[a_1\ldots a_s]=a_1\bullet\theta(a_2)\bullet\ldots\bullet\theta^{s-2}(a_{s-1})\bullet\theta^{s-1}(a_s)\bullet c, \tag{8.1}$$

where

$$\theta(c)=c \text{ and } \theta^{s-1}(b)=c\bullet b\bullet c^{-1}, \text{ for } b\in G^m. \tag{8.2}$$

Proof. Since (G;[ ]) is an (sm,m)-group, $(G^m;g)$, where

$$g(x_1^m x_{m+1}^{2m}\ldots x_{(s-1)m+1}^{sm})=[x_1^{sm}], \tag{8.3}$$

is an (s,1)-group. Then, Hosszú-Gluskin's theorem implies that there exist: a binary group $(G^m;\bullet)$, an element $c\in G^m$ and an automorphism θ of $(G^m;\bullet)$, satisfying (8.2) and

$$g(u_1^s)=u_1\bullet\theta(u_2)\bullet\ldots\bullet\theta^{s-1}(u_s)\bullet c \tag{8.4}$$

for each $u_\nu\in G^m$. Now, (8.1) follows from (8.4) and (8.3). □

§9. SOME CLASSES OF VECTOR VALUED GROUPS

In this section we consider some classes of v.v. groups, touching upon existence problems for them also.

The investigation of (n,m)-groups pushes forward naturally the cases n=2m and n=m+1, because for m=1 one obtains the class of groups in both cases. Further on, we will assume that m is a given positive integer such that $m \geq 2$.

Recall that any (2m,m)-group defined as in E.2.7. 1), is said to be trivial (see E.8.14). By [5], there exist also non-trivial (2m,m)-groups.

Note that if $\underline{G}=(G;[\ ])$ is a (2m,m)-semigroup and if we define a (binary) operation $\bullet$ on G^m by:

$$x_1^m \bullet y_1^m = [x_1^m y_1^m],$$

then we obtain a semigroup $(G^m;\bullet)$, called associated semigroup to $\underline{G}$.

Proposition 9.1. *A (2m,m)-semigroup $\underline{G}=(G;[\ ])$ is a (2m,m)-group iff its associated semigroup $(G^m;\bullet)$ is a group.*

In that case, the identity of the group $(G^m;\bullet)$ has a form $\overset{m}{e}$, where $e \in G$, and moreover the following equality in $\underline{G}$

$$[x_1^i \overset{m}{e} x_{i+1}^m] = x_1^m$$

holds for every $i \in \{0,1,\ldots,m\}$ and $x_1^m \in G^m$.

(We say that e is the identity of $\underline{G}$ and that $(G^m;\bullet)$ is the associated group to $\underline{G}$.)

Proof. The first part of the proposition follows easily as a consequence of C.8.4.

Let $\underline{G}=(G;[\ ])$ be a (2m,m)-group and let e_1^m be the identity of the associated group. If $x_1^m \in G^m$ and $0 \leq i \leq m$, then:

$$\begin{aligned}[x_1^i e_1^m x_{i+1}^m] &= e_1^m \bullet [x_1^i e_1^m x_{i+1}^m] = \\ &= [e_1^i [e_{i+1}^m x_1^i e_1^m] x_{i+1}^m] \\ &= [e_1^m x_1^m] = x_1^m;\end{aligned}$$

therefore

$$[e_1 e_1^m e_2^m] = e_1^m = [e_1^m e_1^m],$$

which implies $e_1 = e_2 = \ldots = e_m$ (=e). □

As a direct consequence of P.9.1 we obtain:

Corollary 9.2. *Let $\underline{G}=(G;[\ ])$ be a $(2m,m)$-group with the identity e and let $H \subseteq G$. Then H is a subgroup of $\underline{G}$ iff H^m is a subgroup of the associated group, and in that case $e \in H$.* □

(Here, the notion "a subgroup of a v.v. group" means "a v.v. subgroup of a v.v. group".) □

The existence of the identity e enables us to introduce the notion of a normal subgroup of a (2m,m)-group as a kernel of a homomorphism.

Proposition 9.3. *If $\xi: \underline{G} \to \underline{G}'$ is a homomorphism from the $(2m,m)$-group $\underline{G}=(G;[\ ])$ to the $(2m,m)$-group $\underline{G}'=(G';[\ ]')$, then*

$$H = Ker\xi = \{x \in G \mid \xi(x) = e'\} = \xi^{-1}(e')$$

is a subgroup of $\underline{G}$ with the following properties:

$$[x_1^{i-1} H^m x_i^m] = [x_1^m H^m], \qquad (9.1)$$

$$[x_1^m H^m] = [y_1^m H^m] \iff (\forall j \in N_m)[{}^m_j x\, H^m] = [{}^m_j y\, H^m] \qquad (9.2)$$

for every $x_\nu, y_\nu \in G$, $i \in N_m$.

(Here, e' is the identity of $\underline{G}'$, and $[x_1^{i-1} H^m x_i^m]$ has the usual meaning, i.e.

$$[x_1^{i-1} H^m x_i^m] = \{[x_1^{i-1} h_1^m x_i^m] \mid h_1^m \in H^m\}.)$$

Proof. Denote by $\bar{\xi}$ the homomorphism from $(G^m;\bullet)$ to $(G'^m;\bullet)$ induced by ξ, i.e.

$$\xi(x_1^m) = y_1^m \iff (\forall i \in N_m)\ y_i = \xi(x_i).$$

Then

$$Ker\bar{\xi} = H^m,$$

which implies that H^m is a normal subgroup of the group $(G^m;\bullet)$ and by P.9.2, H is a subgroup of $\underline{G}$; now, by the fact that H^m is a normal subgroup of G^m, one obtains that (9.1) is true.

Suppose that $x_1^m, y_1^m \in G^m$ are such that $[x_1^m H^m]=[y_1^m H^m]$. Then $\bar{\xi}(x_1^m)=\bar{\xi}(y_1^m)$, i.e. $\xi(x_j)=\xi(y_j)$ for every $j \in N_m$, which implies that

$\overline{\xi}(\overset{m}{x_j}) = \overline{\xi}(\overset{m}{y_j})$ for every $j \in N_m$. This implies that (9.2) is also true. □

Now suppose that H is a subgroup of the (2m,m)-group $\underline{G}=(G;[\ ])$ with the properties (9.1) and (9.2). Then H^m is a normal subgroup of the associated group $(G^m;\bullet)$. Consider the quotient group G^m/H^m and the subset G/H of G^m/H^m defined by:

$$G/H = \{\overset{m}{x}H^m \mid x \in G\}. \tag{9.3}$$

To shorten the notation we will write $\overline{x}$ instead of $\overset{m}{x}H^m$. Therefore:

$$\overline{x} = \overline{y} \text{ iff } \overset{m}{x}H^m = \overset{m}{y}H^m.$$

Consider the canonical mapping nat_H: $x \mapsto \overline{x}$ from G onto G/H. We will show that:

Proposition 9.4. *There exists a unique (n,m)-operation $[\ ]'$ on G/H such that nat_H is a homomorphism from $(G;[\ ])$ to $(G/H;[\ ]')$.*

Proof. By the fact that nat_H is a surjective mapping from G onto G/H it follows that there exists at most one operation $[\ ]'$ with the demanded properties.

To prove that such an operation exists it is necessary only to see that (9.2) implies directly that the following implication is true:

$$(\forall i \in N_{2m})\, \overline{x}_i = \overline{y}_i \text{ and } [x_1^{2m}] = u_1^m,\ [y_1^{2m}] = v_1^m \Longrightarrow (\forall j \in N_m)\ \overline{u}_j = \overline{v}_j. \tag{9.4}$$

By (9.4) one obtains that a (2m,m)-operation $[\ ]'$ on G/H is well-defined by:

$$[x_1^{2m}] = u_1^m \Longrightarrow [\overline{x}_1^{2m}]' = \overline{u}_1^m, \tag{9.5}$$

and also that nat_H: $(G;[\ ]) \longrightarrow (G/H;[\ ]')$ is a homomorphism. □

By P.9.4 one obtains the following:

Proposition 9.5. *If $\underline{G}=(G;[\ ])$ is a (2m,m)-group with the identity e, H is a subgroup of $\underline{G}$ with the properties (9.1) and (9.2) and if $(G/H;[\ ]')=\underline{G}/H$ is defined as above, then:*

(i) $\underline{G}/H$ is a group with the identity $\overline{e}$,

(ii) nat_H is a homomorphism, such that $Ker\,nat_H = H$. □

A subgroup of a (2m,m)-group $\underline{G}$ is said to be normal in $\underline{G}$ iff (9.1) and (9.2) hold. Then it is natural to say that $\underline{G}/H$ is a quotient group of $\underline{G}$ with respect to H.

As a summary of the above results, we obtain the following:

Theorem 9.6. *Let H be a subgroup of a (2m,m)-group* $\underline{G}$. *Then H is normal in* $\underline{G}$ *iff there exists a homomorphism* $\xi: \underline{G} \rightarrow \underline{G}'$ *such that* $H=Ker\xi$. □

By the definition of $\underline{G}/H$ directly one obtains also the following result:

Proposition 9.7. *If H is a normal subgroup in a (2m,m)-group* $\underline{G}$, *then* H^m *is a normal subgroup in* $(G^m;\bullet)$ *and the groups* $(G^m/H^m;\bullet)$, $((G/H)^m;\bullet)$ *are isomorphic.*

In other words, $(G^m/H^m;\bullet)$ *is the associated group of* $\underline{G}/H$. □

The following proposition for the trivial (2m,m)-groups is true:

Proposition 9.8. *Let* $\underline{G}$ *be a trivial (2m,m)-group induced by a group* $(G;\bullet)$. *Then:*

(i) The usual m-th Cartesian power of $(G;\bullet)$ *is the associated group of* $\underline{G}$.

(ii) H is a normal subgroup of $\underline{G}$ *iff H is a normal subgroup of* $(G;\bullet)$. □

The trivial (2m,m)-groups are not of a special interest. However, it is desirable to have corresponding abstract descriptions of this class of (2m,m)-groups, which are given by the following proposition (proved in [5]):

Proposition 9.9. *If* $\underline{G}$ *is a (2m,m)-group with the identity e, then the following conditions are equivalent:*

(i) $\underline{G}$ *is trivial.*

(ii) There exist binary operations $*_1, *_2, \dots, *_m$ *on G such that*

$$[x_1^{2m}] = y_1^m \iff (\forall j \in N_m)\ y_j = x_j *_j x_{m+j}$$

for every x_ν, y_λ.

(iii) For every $x, y \in G$, $x_1^m \in G^m$ and integers r, s, $i \in N_m$, $r \neq s$ the following equalities hold:

$$[\overset{m-i+1}{e}\, x_1^m\, \overset{i-1}{e}] = x_i^m x_1^{i-1},$$

$$[\overset{r-1}{e}\, x\, \overset{m-1}{e}\, y\, \overset{m-r}{e}]_s = e, \text{ for } s \neq r. \ \square$$

(We note also that there is no problem to formulate and prove corresponding theorems for isomorphisms for the class of (2m,m)-groups.)

Now we will consider the class of (m+1,m)-groups, where $m \geq 2$.

By P.5.5, an (m+1,m)-semigroup $\underline{G}=(G;[\ ])$ induces an (m+k,m)-semigroup for every $k \geq 2$, and thus a corresponding (2m,m)-semigroup. Moreover, $\underline{G}$ is an (m+1,m)-group iff it is a (2m,m)-group. This implies that to any (m+1,m)-group $\underline{G}=(G;[\ ])$ it is possible to join a corresponding associated group $(G^m;\bullet)$; in this case, it is the group $G^\vee$.

We will show below that the (m+1,m)-groups can be defined (like groups) by one (m+1,m)-operation, one unary and one nullary operation.

<u>Theorem 9.10</u>. *If $\underline{G}=(G;[\ ])$ is an (m+1,m)-semigroup, then the following statements are equivalent:*

(i) $\underline{G}$ is an (m+1,m)-group.

(ii) There exists an element $e \in G$ and a transformation $h: G \to G$ on G such that:

$$[x_1^i\, \overset{m}{e}\, x_{i+1}^m] = x_1^m \tag{9.6}$$

$$[x\, \overset{m}{e}] = [\overset{m}{e}\, x] \tag{9.7}$$

$$[x h(x) h^2(x) \ldots h^m(x)] = \overset{m}{e} \tag{9.8}$$

$$h^{m+1} = 1_G \tag{9.9}$$

for every $x \in G$, $x_1^m \in G^m$, $i \in \{0, 1, \ldots, m\}$.

<u>Proof</u>. Let $\underline{G}$ be an (m+1,m)-group. Then $\underline{G}$ induces a corresponding (2m,m)-group. So, if e is the identity of that (2m,m)-

group, then by P.9.1 one obtains that (9.6) holds, and clearly (9.7) is a consequence of (9.6), and the cancellativity in G.

If x is a given element of G, then there exists a unique $y_1^m \in G^m$ such that $[xy_1^m] = \overset{m}{e}$. Let us put $y_1 = h(x)$. Using the fact that $\overset{m}{e}$ is the identity of $(G^m;\bullet)$, and using also (9.6) and (9.7), we obtain

$$\overset{m}{e} = [xy_1^m] = [xy_1^m xy_1^m],$$

by which:

$$y_1^m = [y_1^m xy_1^m] = [y_1^m x]\bullet y_1^m,$$

i.e. $[y_1^m x] = \overset{m}{e}$. By the last equality we have $y_2 = h(y_1) = h^2(x)$. Similarly one obtains that $y_i = h^i(x)$ for every i, and also that $h^{m+1}(x) = x$.

Therefore, (i) ⟹ (ii).

Now suppose that the condition (ii) holds.

By (9.6) it follows that $\overset{m}{e}$ is an identity of the semigroup $(G^m;\bullet)$. Also, by (9.8) one obtains that for every $x_1^m \in G^m$:

$$[x_1^m h(x_m)h^2(x_m)\dots h^m(x_m)h(x_{m-1})\dots h^m(x_{m-1})\dots h(x_1)\dots h^m(x_1)] = e^m,$$

and this implies that $(G^m;\bullet)$ is a group. □

(Note, that (9.9) is a consequence of (9.6) to (9.8)).

If G is an (m+1,m)-group, and e is the identity of the corresponding (2m,m)-group, then we say also that e is the <u>identity of G</u>.

<u>Proposition 9.11</u>. *Let $\underline{G} = (G;[\ ])$ be an $(m+1,m)$-group and e be the identity of G. If for every $x_1^m \in G^m$ and some $i \in \{0,\dots,m\}$*

$$[x_1^i e x_{i+1}^m] = x_1^m,$$

then $|G| = 1$.

<u>Proof</u>. The cancellativity of G and the fact that e is the identity of G, implies that $[ey_1^m] = y_1^m$ for each $y_1^m \in G$. Now, $[\overset{m+1}{e}] = \overset{m}{e}$ implies that

$$\overset{m-1}{e}x = [\overset{m}{e}\, e\, \overset{m-1}{x}] = [\overset{m+1}{e}\, \overset{m-1}{x}] = [\overset{m}{e}\, \overset{m-1}{x}] = [\overset{m-1}{x}\, \overset{m}{e}] =$$

$$= [\overset{m-1}{x}\, \overset{m+1}{e}] = \overset{m-1}{x} e,$$

for each $x \in G$, i.e. $|G| = 1$. □

Proposition 9.12. *If G is a nonempty set, then the following statements are equivalent:*

(i) There exists an (m+1,m)-group $\underline{G}=(G;[\ ])$.

(ii) There exists a group $(G^m;\bullet)$ *and a mapping* $\xi: x \mapsto \overline{x}$ *from G into* G^m, *such that*

$$x_1^m = \overline{x}_1 \bullet \overline{x}_2 \bullet \ldots \bullet \overline{x}_m, \qquad (9.10)$$

for every $x_1^m \in G^m$.

(Clearly, every mapping $\xi: G \to G^m$ *with the property (9.10) is injective.)*

Proof. Let $\underline{G}=(G;[\ ])$ be an (m+1,m)-group and let $(G^m;\bullet)$ be the associated group of $\underline{G}$. If we set $\overline{x}=[x\overset{m}{e}]$, where e is the identity of $\underline{G}$, then we obtain that (9.10) holds. Thus: (i) $\Longrightarrow$ (ii).

Conversely, suppose that $(G^m;\bullet)$ is a group and $\xi: x \mapsto \overline{x}$ a mapping from G into G^m such that (9.10) holds. If we set

$$[x_0^m] = \overline{x}_0 \bullet \overline{x}_1 \bullet \ldots \bullet \overline{x}_m,$$

then we obtain an (m+1,m)-group for which the given group $(G^m;\bullet)$ is the associated group. □

An (m+1,m)-group $\underline{G}=(G;[\ ])$ is said to be trivial iff $|G|=1$. The question for the existence of nontrivial (m+1,m)-groups comes naturally.

Theorem 9.13. *If G is an infinite set, then there exists an (m+1,m)-group* $\underline{G}=(G;[\ ])$.

Proof. This proposition is a consequence of the main result of the paper [14], by which if B is a nonempty set, then a free (m+1,m)-group $\underline{F_m}(B)=(F_m(B);[\ ])$ with a basis B has the cardinality

$$|F_m(B)| = \max\{|B|, \aleph_0\}. \ \square$$

(We note that in the mentioned paper [14] a satisfactory combinatorial description of a free (m+1,m)-group $\underline{F_m}(B)$ is given.)

It remains the case when G is a finite set.

Theorem 9.14. *If* $m \geq 2$, *then there is no nontrivial finite (m+1,m)-group.*

This proposition is a direct consequence of the following result, which was proved by Prof. John Thompson (and he kindly provided us with that proof):

Proposition 9.15. *If $(H;\cdot)$ is a finite group such that there exists a subset S of H with the properties:*

$$S\cdot S = \{x\cdot y \mid x,y\in S\} = H \text{ and } |S|^2=|H|, \qquad (9.11)$$

then $|H|=1$. □

Professor Thomson's proof of P.9.15 is via the gorup algebra $\mathbb{C}[H]$ over the field of complex numbers, Wedderburn's theorem for a decomposition of $\mathbb{C}[H]$, and characters of finite groups.[1)]

It is possible to generalize P.9.15. to:

Proposition 9.16. *If $(H;\cdot)$ is a finite group and $S\subseteq H$ such that*

$$\underbrace{S\cdot\ldots\cdot S}_{m} = \{x_1\cdot\ldots\cdot x_m \mid x_\nu\in S\} = H \text{ and } |S|^m = |H|,$$

where $m\geq 2$, then $|H|=1$. □

The conclusion in T.9.14. comes easily as a consequence of P.9.16.

Indeed, let $\underline{G}=(G;[\ ])$ be a finite $(m+1,m)$-group. By P.9.12., if $(G^m;\bullet)$ is the associated group of G and if $S=\{[x\overset{m}{e}] \mid x\in G\}$, where e is the identity of $\underline{G}$, then

$$|S| = |G|,\ |S|^m = |G^m| = |G|^m \text{ and } \underbrace{S\cdot\ldots\cdot S}_{m} = G^m.$$

By P.9.16., if follows that $|G^m|=1$, i.e. $|G|=1$.

This completes the proof of T.9.14.

We note also that, as a consequence of T.9.13, one obtains the following generalization:

Corollary 9.17. *If G is an infinite set and n,m,k are positive integers such that $n-m=k\geq 1$, then there exists an (n,m)-group $\underline{G}=(G;[\ ])$.*

Proof. By T.9.13., there exists an $(m+1,m)$-group $\underline{G}=(G;[\ ])$ which by T.5.8. induces a corresponding (n,m)-group. □

---

1) See p. 72.

Next we are going to describe a method for production of examples of $(2m+s,m)$-groups, $s \geq 1$.

Let $(G;\cdot)$ be a group with a neutral element e. Suppose that there exists a homomorphism $*$ from the product group $(G^{m+s};\cdot)$ into the product group $(G^m;\cdot)$ such that:

(i) $(\overset{s}{e}x_1^m)^* = x_1^m$, (ii) $x_1^{m+s} \in \ker * \Longrightarrow x_2^{m+s}x_1 \in \ker *$.

We extend $*$ to a homomorphism (denoted again by $*$) $*$: $G^{t(m+s)} \longrightarrow G^m$ by:

$$(x_1^{t(m+s)})^* = (x_1^{m+s})^* \cdot (x_{m+s+1}^{2(m+s)})^* \dots (x_{(t-1)(m+s)+1}^{t(m+s)})^*.$$

Next, we extend $*$: $\bigcup\limits_{t\geq 1} G^{t(m+s)} \longrightarrow G^m$ to $*$: $\bigcup\limits_{\lambda\geq 0} G^{m+\lambda} = \overline{G} \longrightarrow G^m$ by:

$$(x_1^{t(m+s)+p})^* = (\overset{m+s-p}{e}x_1^p)^* \cdot (x_{p+1}^{t(m+s)+p})^*,$$

where $t \geq 0$, $0 \leq p < m+s$.

The following theorem is proved in [10]:

Theorem 9.18. *Let* $[\ ]: G^{2m+s} \longrightarrow G^m$ *be defined by:*

$$[x_1^{2m+s}] = (\overset{s}{e}x_1^m)^*(x_{m+1}^{m+s})^* = (x_1^{2m+s})^*.$$

Then $(G;[\ ])$ *is a* $(2m+s,m)$*-group.* □

The v.v. group in E.2.7. 2) is obtained by the above procedure. In this example, the homomorphism $*$: $G^{m+1} \longrightarrow G^m$ is given by $(xx_1^m)^* = (x_1-x,\dots,x_m-x)$.

Let us examine the universal covering semigroup for $(G;[\ ])$ as above.

From the assumptions about $(G;\cdot)$ and $*$, it follows that the neutral element in the group $(G^m;\cdot)$, where $x_1^m\cdot y_1^m = [x_1^m \overset{s}{e} y_1^m]$ (see C.8.4.), is of the form $\overset{m}{e}$. So, by E.8.13.,

$$G^\wedge = G \cup \dots \cup G^m \cup eG^m \cup \dots \cup \overset{m+s-1}{e} G^m,$$

and

$$x_1^r \cdot y_1^t = \begin{cases} x_1^r y_1^t & \text{if } r+t \leq m \\ \overset{p}{e}(\overset{m+s-p}{e} x_1^r y_1^t)^* & \text{if } r+t > m, \end{cases}$$

where $0 \leq p = r+t-m-q(m+s) < m+s$.

We will prove the following:

<u>Proposition 9.19</u>. *If $\underline{G}=(G;[\ ])$ is a (5,3)-group and if $1<|G|<+\infty$, then $|G|$ is an even number.*

<u>Proof</u>. Let $(G;[\ ])$ be a (5,3)-group and for $a\in G$, let $f(a)g(a)h(a)$ be the neutral element in the group $(G^3;\star)$ (see C.8.4.). It is easy to check out that $f(f(a))=g(a)$, $f(g(a))=h(a)$ and $f(h(a))=a$, i.e. $g=f^2$, $h=f^3$ and $f^4=1_G$. Moreover, there are only three possibilities:

$$a = f(a) = g(a) = h(a);\ \text{or}\ a = g(a) \neq f(a) = h(a)$$

$$\text{or}\ |\{a,f(a),g(a),h(a)\}| = 4.$$

If for some $a\in G$, $a=f(a)=g(a)=h(a)$, then $[\overset{5}{a}]\star[\overset{5}{a}]=[\overset{11}{a}]=[\overset{7}{a}]=\overset{3}{a}$. If $[\overset{5}{a}]=\overset{3}{a}$, then

$$[x_1^3\overset{4}{a}] = x_1^3\star\overset{3}{a} = x_1^3\star[\overset{5}{a}] = [x_1^3\overset{6}{a}]$$

implies that $x_1^3=[x_1^3\overset{2}{a}]$. Symmetrically, $[\overset{2}{a}x_1^3]=x_1^3$. Now, for $x_1^3=axa$, $[ax\overset{3}{a}]=[\overset{3}{a}xa]$ implies that $x=a$ for each $x\in G$, i.e. $|G|=1$. Hence, if $1<|G|<\infty$, then $[\overset{5}{a}]$ is an element of order 2 in $(G^3;\star)$, which implies that 2 is a divisor of $|G|$.

If $|G|<\infty$ and for some $a\in G$, $a\neq f(a)$, then there is a partition of G into (disjoint) subsets with 2 or 4 elements, which implies that 2 is a divisor of $|G|$, i.e. $|G|$ is an even number. □

• The above proof shows that finite sets with an odd number of elements (bigger than 1) do not admit a (5,3)-group structure.

We note that it is enough to consider the existence questions only for (n,m)-groups where n and m are relatively prime, as the following propositions states.

<u>Proposition 9.20</u>. *If there exists an (n,m)-group $\underline{G}=(G;[\ ])$ and if $t\geq 1$, then there exists an (nt,mt)-group $\underline{G}'=(G;[\ ]')$ too.*

<u>Proof</u>. If $\underline{G}=(G;[\ ])$ is an (n,m)-group, then $\underline{G}'=(G;[\ ]')$, where $[\ ]': G^{nt}\longrightarrow G^{mt}$ is defined by

$$[x_1^t y_1^t \ldots z_1^t]'_{rt+i} = [x_i y_i \ldots z_i]_{r+1},$$

$0\leq r\leq m-1$, $1\leq i\leq t$, is an (nt,mt)-group. □

Proposition 9.21. *If there exists a (tn,tm)-group $(G;[\ ])$, then there exists an (n,m)-group $(G^t;[\ ]')$ too.*

Proof. If $(G;[\ ])$ is a (tn,tm)-group, then $(G^t;[\ ]')$, where $[\ ]': (G^t)^n \to (G^t)^m$ is defined by

$$[x_1^t y_1^t \dots z_1^t]_i' = [x_1^t y_1^t \dots z_1^t]_i$$

is an (n,m)-group. □

§10. NOTES AND COMMENTS

The notion of vector valued operation is treated for the first time, in our knowledge, in [22] (aside from the use of vector valued functions in the analysis and its applications). Namely, the paper [22] is concerned with the problem of characterization for some algebras of partial v.v. operations. Similar questions are considered in [35], where a class of algebras with countably many partial binary operations is examined and it is shown that every such algebra is a subalgebra of an algebra of v.v. operations. Different questions connected to the composition algebras, especially their completeness, are treated in the extentive paper [17], which appeared several years ago. (Here, if $Op(A)$ is the set of v.v. operations on a set A, $\bullet$ is the usual composition and $\times$ the direct product of mappings, then $\mathcal{A} = (Op(A);\bullet,\times,1_A)$ is called a composition algebra on A. If $F \subseteq Op(A)$, then $(A;F)$ is a v.v. algebra. A special attention is given to the case when F is a generating set of the composition algebra $\mathcal{A}$.)

The definitions of v.v. groupoids and v.v. semigroups for the first time are given in [34]. The notion of weak v.v. quasigroups is given in [33], under the name "(n,m)-quasigroups", while the notion of v.v. quasigroups, defined in §2, for the first time is introduced in [2]. (Also, the notion of a partial v.v. quasigroup is given there, but in a more general content.) Several interpretations of v.v. quasigroups are given in [2], where the most interesting is the geometric one. A review of the known results about v.v. quasigroups is given in [27] (in this book). Therefore we will mention briefly here only that v.v. groupoids, v.v. quasigroups and their geometric interpretations are treated in the papers: [2], [20], [21], [23], [24], [25], [26], [29], [30], [31], [32], [38].

The paper [4] is entirely concerned with the v.v. semigroups. The notion of v.v. group for the first time is introduced in that paper. It is proved there Post Theorem for v.v. semigroups, and also several other results connected to this theorem. The question about the existence of nontrivial v.v. groups (i.e. v.v. groups with more than one element) is also considered in [4]. Examples of nontrivial (sm,m)-groups are given too, by using ordinary groups and a theorem (that, for each m, the free (m+1,m)-group is nontrivial) is stated, which implies that for each $n > m$, nontrivial (n,m)-groups do exist. Although the above theorem is true, which follows from the main result in [14], its proof given in [4] is not complete, i.e. we do not know a direct proof that the identity $x = y$ is not a consequence of the axioms for (m+1,m)-groups. At the end of [4], a list of problems is given, some of which are solved. For example, in [11], a satisfactory combinatorial description of free v.v. semigroups is given, and this made possible to prove more general v.v. variants of Post and Cohn-Rebane Theorems, which is done in [6] and [7].

The result about the non-existence of nontrivial finite (m+1,m)-groups came successively. Namely, at the beginning, in [10], some non-existence conditions for (m+1,m)-groups were obtained, which implied, for example, that the number of elements of a finite (3,2)-group had to be divisible by 6, and later in [12] an elementary proof that (3,2)-groups with 6 and 12 elements do not exist was given. At this moment, we do not have a general answer to the question: when do nontrivial finite (n,m)-groups exist, if m is not a divisor of n?

Besides this, the question about the existence of nontrivial v.v. groups in some classes of v.v. groups is of special interest. we note that [10] contains some answers to this question. It is known that there are no nontrivial commutative (n,m)-groups for $m \geq 2$ [4], but for some classes of v.v. groups it is usefull to weaken the commutativity condition uch that a large class of v.v. groups is obtained. Thus, in [15], (2m,m)-groups whose associated group is commutative are examined and examples of such "nontrivial" groups are obtained. Moreover, it is shown in [15] that finite (4,2)- and (6,3)-groups whose associated group is cyclic must be "trivial". The interesting question about the existence of finite "nontrivial" (2m,m)-groups with a prime number of elements, which is analogous to the fact that a finite group with prime number of elements must be cyclic, is still open.

Similarly as for the semigroups and groups, it is of interest to consider and examine continuous v.v. semigroups and v.v. groups. It is shown that continuous (3,2)-groups over the real numbers do not exist [28]. The question about some continuous v.v. groups is treated in [16].

We noted in the introduction that the presentation of semigroups is of use for the examination of v.v. groups, since to each v.v. groupoid $\underline{Q}$ we associate its universal semigroup $\underline{Q}^{\wedge}$ via a corresponding presentation. Here we have the similar algorithmical problems as for the usual semigroups and groups [40], [42], [43]. These problems are not examined in details in this work but we note that a sufficiently effective reduction in the semigroup presentation (see page 17) gives an algorithm for solving the word problem in this presentation. Thus, if $\underline{A} = (A;F)$ is a "sufficiently effective" partial v.v. algebra, then the semigroup that contains $\underline{A}$, in the proof of Cohn-Rebane Theorem (page 19) is also "sufficiently effective". We note that the construction of the free (n,m)-semigroups (considered as poly-(n,m)-semigroups) in §6 is also effective. It was mentioned in §6 (by E.6.4) that the given reduction in the free (n,m)-groupoids $\overline{B}$ was not good, but it is possible to alter the definition of reduced elements and to obtain an effective construction of free (n,m)-semigroups via free (n,m)-groupoids.

In [9], presentations of v.v. semigroups are examined; descriptions of free v.v. semigroups in some varieties (for example, the variety of commutative v.v. semigroups) are obtained; and corresponding Post Theorems for these varieties are proven.

The general associative law is characterized by the fact that $|\mathcal{P}_\alpha(f)| = 1$, for each α (see page 29). This suggests a generalization of v.v. semigroups to a more general class of v.v. algebras, namely v.v. associatives. If F is a set of v.v. operations on $\underline{Q}$, then it is possible, in a similar manner, to define sets $\mathcal{P}_{n,m}(F)$ of polynomial operations on Q. An algebra (Q;F) is called a v.v. associative if $|\mathcal{P}_{n,m}(F)| \le 1$ for each $n,m \ge 1$. A special kind of v.v. associatives is considered in [1].

A part of the results, stated and proved in this work, are published earlier, mainly in the following papers: [34], [4], [5], [10], [11], [12], [13], [7], [8], [1]. In many cases, new simpler proofs are given here. (For example, GAL is supposed in all of these papers, but an explicit proof is given for the first time in this work.) In the main text, we usually do not quote the source where a corresponding result is given for the first time. On the other hand, many results are stated in this work for the first time. They are: 1.1, 1.2; 2.1, 2.2, 2.4, 2.12, 2.13; 3.3, 3.7; 4.12, 4.13; 5.1-5.4, 5.7-5.12; 6.4, 6.8-6.12; 7.5, 7.6, 7.9-7.14; 8.2, 8.4-8.16; 9.10, 9.11, 9.12, 9.14, 9.16, 9.19, 9.20, 9.21.

Finally, we make a note of a terminological (and historical) nature with respect to the terms "Post Theorem" and "Cohn-Rebane Theorem". First, in his extentive work [41], Post proved the following result (stated here in our terminology): "Every (n,1)-group is an (n,1)-subsemigroup of a group". Later on, in several papers (see, for ex., [38]) there is a proof of the Post Theorem for (n,1)-semigroups and also for some classes of (n,1)-semigroups. Any result of this kind we call "Post Theorem".

In the beginning of the sixties, the Soviet matematician Rebane proved a result which (again in terms of this work) could be stated in the following way: "If F is a set of finitary operations on a set Q and if Λ is the set of semigroup defining relations, defined by:

$$\Lambda = \{(b, fa_1^n) \mid b = f(a_1^n),\ a_\nu, b \in Q\},$$

then the presentation $\langle Q \cup F; \Lambda \rangle$ is pure". In the meantime, a similar result appeared in the monograph [36]. Results of this kind are known usually as "Cohn-Rebane Theorems" ([46]). Other vector valued variants of Post Theorems and Cohn-Rebane Theorems are given in [9].

We give here *Prof.* John Thomson's proof of P.9.15, in the same form as it was provided to us.

"Let A be the group algebra H over the field of complex numbers. Then by Wedderburn's theorem, $A=A_1 \oplus \ldots \oplus A_d$, where each A_i is a full matrix algebra of f_i by f_i matrices over C. For each subset T of H, set $[T] = \sum_{t \in T} t$ (this is an element of A). The hypothesis imply that $[S] \cdot [S] = \sum_{g \in H} g = N$, say. Each element a in A is uniquely $a=a_1+\ldots+a_d$, $a_i \in A_i$, and the maps $\chi_i : H \to C$, $g \longmapsto \mathrm{trace}_{A_i} g_i$, $i=1,\ldots,d$, are the irreducible characters of H. Choose notation so that $\chi_1 = 1_H$, the trivial character which assumes the value 1 at each group element. Then it is a basic and easy result that $N_i=0$ if $i>1$. Then $[S]=[S]_1+\ldots+[S]_d$, and if $i>1$, then $[S]_i^2=0$, so that $[S]_i$ is a nilpotent f_i by f_i matrix for each $i>1$, whence $\chi_i([S])=0$, $i>1$. On the other hand, $\chi_i([S]) = \sum_{s \in S} \chi_i(s)$. Hence

$$\sum_{i=1}^{d} \chi_i(1) \cdot \chi_i([S]) = \chi_1(1) \cdot \chi_1([S]) = \mathrm{card}S = \mathrm{card}H \cdot \delta_{1,S},$$

where $\delta_{1,S}=1$ if $1 \in S$, 0 otherwise. This is so since $\sum_i \chi_i(1)\chi_i$ is the character of the regular representation of H, so vanishes at each element of H - {1} and has value cardH at the identity element of H. So $n=n^2 \cdot \delta_{1,S}$, whence n=1."

B I B L I O G R A P H Y

I

[1] Celakoski N., Janeva B.: *Vector valued associatives*, Proc. Conf. "Algebra and Logic", Cetinje 1985, 15-21

[2] Čupona Ǵ., Ušan J., Stojaković Z.: *Multiquasigroups and some related structures*, Maced. Acad. Sci. and Arts, Contributions, Sect. Math. Techn. Sci. I.2 (1980), 5-12

[3] Čupona Ǵ., Stojaković Z., Ušan J.: *On finite multiquasigroups*, Publ. de l'Inst. math. Belgrade, 29(43), (1981), 53-59

[4] Čupona Ǵ.: *Vector valued semigroups*, Semigroup Forum, Vol. 26 (1983), 65-74

[5] Čupona Ǵ., Dimovski D.: *On a class of vector valued groups*, Proc.Conf. "Algebra and Logic", Zagreb (1984), 29-38

[6] Čupona Ǵ., Markovski S., Janeva B.: *Post Theorem for vector valued semigroups*, Proc. Conf. "Algebra and Logic", Cetinje 1985, 33-46

[7] Čupona Ǵ., Markovski S.: *Cohn-Rebane Theorem for vector valued Algebras*, Maced. Acad. Sci. and Arts, Contributions, VI/2, Sect. Math. Techn. Sci. (1985), 143-149

[8] Čupona Ǵ., Markovski S.: *Vector valued subgroupoids of semigroups*, Maced. Acad. Sci. and Arts, Contributions, Sect. Math. Techn. Sci. (1986), VII/2, 5-12

[9] Čupona Ǵ., Markovski S., Dimovski D., Janeva B.: *Introduction to combinatorial theory of vector valued semigroups*, (this volume)

[10] Dimovski D.: *Some existence conditions for vector valued groups*, God. Zb. Math. fac. Skopje 33-34 (1982-83), 99-103

[11] Dimovski D.: *Free vector valued semigroups*, Proc. Conf. "Algebra and Logic", Cetinje 1985, 55-62

[12] Dimovski D.: *On (3,2)-groups*, Proc. Conf. "Algebra and Logic", Cetinje 1985, 63-71

[13] Dimovski D.: *Examples of (2m+s,m)-groups*, Maced. Acad. Sci. and Arts, Contributions, Sect. Math. Techn. Sci. (1987) 103-114

[14] Dimovski D.: *Free (m+1,m)-groups*, (this volume)

[15] Dimovski D., Ilić S.: *Commutative (2m,m)-groups* (this volume)

[16] Dimovski D., Trenčevski K.: *1-dimensional (4,2)-Lie groups*, (this volume)

[17] Linn R.: *Composition completeness of algebras with (m,n)-place operations* (in German), Mitt. Math. Sem. Gissen No. 160 (1983), 115

[18] Markovski S.: *On Hosszú-Gluskin Theorem*, Proc. Conf. "Algebra and Logic", Sarajevo 1987

[19] Paunić D.: *Algebarske n-arne strukture*, Ph. D. thesis, Novi Sad 1987

[20] Polonijo M.: *On C^{n+1}-systems and $[n,m]$-groupoids*, Proc. Third Alg. Conf. Belgrade (1982), 111-116

[21] Polonijo M.: *Medial multiquasigroups*, Maced. Acad. Sci. and Arts. Contributions, Sec. Math. Techn. Sci. III/2 (1982), 31-39

[22] Schweizer B., Sklar A.: *The algebra of multiplace vector valued functions*, Bull. Amer. Math. Soc. (1967), 510-515

[23] Stojaković Z., Paunić D.: *Nonlinear multiquasigroups*, Zbor. Rad. PMF Novi Sad 10 (1980), 145-148

[24] Stojaković Z.: *On bisymmetric $[n,m]$-groupoids*, Zbor. Rad. PMF Novi Sad 12 (1982)

[25] Stojaković Z.: *On a class of bisymmetric $[n,m]$-groupoids*, Proc. Third Alg. Conf. Belgrade (1982), 139-143

[26] Stojaković Z., Paunić D.: *Identities on multiquasigroups*, Proc. Symp. "n-ary Structures",Skopje 1982, 195-200

[27] Stojaković Z.: *Vector valued quasigroups*, (this volume)

[28] Trenčevski K.: *Non-existence for some classes of continuous (3,2)-groups*, Proc. Conf. "Algebra and Logic", Cetinje 1985, 205-208

[29] Walter F.: *Aus der Theorie der Polyquasigrouppen*, Beiträge Algebra Geom. No. 18 (1984), 23-40

[30] Одобеску С.С.: Изотопия мультиопераций, Иссл. по теор. квазигрупп и лупп, Кишинев (1973), 127-132

[31] Одобеску С.С., Сандик М.Д.: Обратимые мультиоперации, Иссл. по теор. квазигрупп и лупп, Кишинев (1973), 133-139

[32] Сандик, М.Д.: Обратимые мультиоперации и подстановки, Acta Sci. Math. 39 (1977), 153-162

[33] Стојменовски К.: За [m,n]-квазигрупите, Год. Зб. Мат. фак. Скопје 28 (1978), 33-37

[34] Трпеновски Б., Чупона Ѓ.: [m,n]-групоиди, Билтен ДМФ СРМ Скопје 21, (1970), 19-29

[35] Чупона Ѓ.: Една класа делумни алгебри, Год. Зб. ПМФ Скопје 22, (1972), 5-37

II

[36] Cohn P.M.: *Universal algebra*, Harper & Row, London, 1965

[37] Havel V.: *General nets and their associated groupoids*, Proc. Symp. n-ary Structures, Skopje 1982, 242

[38] Glazek K.: *Bibliography of n-groups (polyadic groups) and some group like n-ary systems*, Proc. Symp. "n-ary Structures", Skopje 1982, 253-289

[39] Lallement G.: *Semigroups and Combinatorial Applications*, John Wiley & Sons. New York, 1979

[40] Magnus W., Karras A., Solitar D.: *Combinatorial group theory*, Wiley, New York, 1966

[41] Post E.L.: *Polyadic groups*, Trans. of the Amer. Math. Soc. Vol. 48 (1940), 208-350

[42] Rotman J.J.: *The theory of groups*, Allyn and Bacon, Boston, 1965

[43] Shoenfield J.R.: *Mathematical Logic*, Addison-Wesley, 1967

[44] Белоусов В.Д.: Алгебраические сети и квазигруппы, Кишинев 1971

[45] Белоусов В.Д.: n-арные квазигруппы, Кишинев

[46] Курош А.Г.: Общая алгебра, Лекции 1969-1970 учебного года, Москва 1974

[47] Ребане Ю.К.: Об одном представлении произвольных универсальных алгебр в полугруппах (unprinted)

[48] Ребане Ю.К.: О представлении универсальных алгебр в коммутативных полугруппах, Сиб. Мат. Журн. 7 (1966), 878-885

[49] Ušan J.: *Kvazigrupe*, Novi Sad 1979

[50] Ušan J.: *Uvod v teorijo kvazigrup*, I. del, Ljubljana 1986

I N D E X

SYMBOLS AND NOTATIONS