SOME CLASSES OF VECTOR VALUED ASSOCIATIVES

N. Celakoski, S. Markovski, B. Janeva

Abstract. The notion of a (K,m)-associative or a vector valued associative is introduced in [2] and conditions when a (K,m)-associative is a (K,m)-subassociative of a vector valued semigroup are stated there. In this paper we investigate some properties of cancellative and surjective (K,m)-associatives and give a combinatorial description of free vector valued associatives.

§1. m-DIMENSIONAL VECTOR VALUED ASSOCIATIVES

Let A be a nonempty set. By A^s will be denoted the s-th Cartesian power of the set A, i.e. $A^s=\{(a_1,\dots,a_s) \mid a_i\in A\}$. The elements of A^s will also be denoted by $a_1\dots a_s$ or shortly by a_1^s. If $a_1=\dots=a_s=a$, then we will write a^s instead of a_1^s.

If n,m are positive integers, then a mapping $f:A^n \to A^m$ is called an (n,m)-operation or a vector-valued (shortly: v.v.) operation with the length $\delta f=n$, the dimension $\rho f=m$, and the index $\iota f=\delta f-\rho f$. (In cases when parenthesis are more convenient we will write: $\delta(f)$, $\rho(f)$, $\iota(f)$. The pair (A;f) is called an (n,m)-groupoid (or a v.v. groupoid).

The set of all vector valued operations on a set A is denoted by Op(A); we note that the identity mapping on A, $1=1_A$, is in Op(A). The partial (binary) operation "composition" on Op(A), denoted multiplicatively, and the (binary) operation "direct product" on Op(A) denoted by ×, are defined as usual. That is, the composition is defined by:

$$g,h\in Op(A),\ \rho g=\delta h \Longrightarrow (\forall a_i\in A)(hg)(a_1^{\delta g})=h(g(a_1^{\delta g})), \tag{1.1}$$

where

$$\delta(hg)=\delta g,\ \rho(hg)=\rho h,\ \iota(hg)=\iota h+\iota g, \tag{1.2}$$

and the direct product is defined by:

$$g,h\in Op(A) \implies (\forall a_i, b_j \in A)(g\times h)(a_1^{\delta g} b_1^{\delta h}) = g(a_1^{\delta g})h(b_1^{\delta h}), \tag{1.3}$$

where

$$\delta(g\times h)=\delta g+\delta h,\ \rho(g\times h)=\rho g+\rho h,\ \iota(g\times h)=\iota g+\iota h. \tag{1.4}$$

Let F be a nonempty subset of Op(A), such that for all $f\in F$, $\delta f > \rho f \geq 1$. Then the pair (A;F) is called an F-algebra or a v.v. algebra.

Let (A;F) be an F-algebra and let $\mathcal{P}=\mathcal{P}(F)$ be the subset of Op(A) which is defined inductively by:

(i) $F\cup\{1\}\subseteq \mathcal{P}$,

(ii) $g, g_i \in \mathcal{P},\ \delta g = \sum_{i=1}^{p} \rho g_i \implies g(g_1\times\ldots\times g_p)\in\mathcal{P}$.

Every element of $\mathcal{P}$ is called a polynomial operation on (A;F).

We will prove the following:

Proposition 1.1. *Let $h\in Op(A)$, $h\neq 1$. Then $h\in\mathcal{P}(F)$ iff there exist $r, i_\nu, j_\nu \in N$[1], $f_\lambda\in F$, such that*

$$h = f_0(1^{i_1}\times f_1\times 1^{j_1})(1^{i_2}\times f_2\times 1^{j_2})\ldots(1^{i_r}\times f_r\times 1^{j_r}). \tag{1.5}$$

(Here 1^t is an abbreviation for $\underbrace{1\times 1\times\ldots\times 1}_{t}$; 1^0 is the "empty symbol". The right hand side of (1.5) will be called a "canonical form of h". More precisely, if (1.5) is satisfied, then the triple of sequences $(f_0^r;\ i_1^r;\ j_1^r)$ is called a canonical form of h.)

Proof. It is clear that if $g_1, g_2, g_3 \in Op(A)$ are such that $\delta g_1=\rho g_2$, then the following equalities are true:

$$g_2\times g_3 = (g_2\times 1^{\rho g_3})(1^{\delta g_2}\times g_3) \tag{1.6}$$

and

$$\begin{aligned} g_1g_2\times g_3 &= (g_1\times 1^{\rho g_3})(g_2\times g_3),\\ g_3\times g_1g_2 &= (g_3\times g_1)(1^{\delta g_3}\times g_2). \end{aligned} \tag{1.7}$$

---

[1] $N=\{0,1,2,\ldots\}$

Let $h \in \mathcal{P}(F)$, $h \neq 1$. If $h \in F$, then we can put $r=0$, $f_o=h$. Thus, we can assume that $h=g(g_1 \times \ldots \times g_p)$, where $g, g_\nu \in \mathcal{P}(F)$, $g \neq 1$ and $g_i \neq 1$ for some $i \in N_p$ [1]. Moreover, we can also assume that g, and each g_ν such that $g_\nu \neq 1$, admit the corresponding canonical forms. By a finite number of applications of (1.6) and (1.7) we can obtain a canonical form of h. □

Proposition 1.2. $\mathcal{P}(F) = \mathcal{P}(\mathcal{P}(F))$. □

An F-algebra (A;f) is called an F-associative iff any two polynomial operations in $\mathcal{P} = \mathcal{P}(F)$ with the same lenght and dimension are equal, i.e.

$$g,h \in \mathcal{P},\ \delta g=\delta h,\ \rho g=\rho h \Longrightarrow g=h. \tag{1.8}$$

As a consequence of P.1.2 we have:

Proposition 1.3. *A v.v. algebra (A;F) is an F-associative iff, for every $G \subseteq \mathcal{P}(F)$, $G \neq \{1\}$, (A;G) is a G-associative.* □

By the definition of a v.v. algebra (A;F) we have that $1 \notin F$ and $1 \in \mathcal{P} = \mathcal{P}(F)$.

Let $F_m = \{f \in F \mid \rho f = m\}$, $\mathcal{P}_m = \{h \in \mathcal{P} \mid \rho h = m\}$,

$\iota F_m = \{\iota f \mid f \in F_m\}$, $\iota\mathcal{P}_m = \{\iota h \mid h \in \mathcal{P}_m\}$.

The following proposition is proved in [2]:

Proposition 1.4. *$\iota\mathcal{P}_m$ is a subsemigroup of the additive semigroup of nonnegative integers (N;+) generated by the set ιF_m, i.e.*

$$\iota\mathcal{P}_m = \langle \iota F_m \rangle. \quad \square \tag{1.9}$$

Further on we will assume that $m \geq 2$ [2] is a fixed integer and that $\rho f=m$ for every $f \in F$. Thus $F=F_m$, $\mathcal{P}_m = \mathcal{P} \setminus \{1\} = \mathcal{P}'$,

$$\iota\mathcal{P}' = \langle \iota F \rangle. \tag{1.9'}$$

In this case we say that each F-algebra (A;F) is an m-dimensional F-algebra, or shortly an (F,m)-algebra.

---

1) *$N_t = \{1,2,\ldots,t\}$, for every integer $t > 0$.*

2) *If m=1, corresponding considerations are given, for example, in [1], [6].*

An m-dimensional F-algebra which is an F-associative is called an m-dimensional F-associative or an (F,m)-associative (and also an m-dimensional v.v. associative).

Proposition 1.5. *Let A be a nonempty set, K be a subsemigroup of* $(N;+)$, $0 \notin K$, $m \geq 2$ *and* $F=\{f_k \mid f_k : A^{k+m} \to A^m, k \in K\}$. *Then, the v.v. algebra* $(A;F)$ *is an* (F,m)*-associative iff for every* $n,t \in K$ *and every* α $(0 \leq \alpha \leq n)$ *the following equality holds:*

$$f_n(1^{\alpha} \times f_t \times 1^{n-\alpha}) = f_{n+t}. \tag{1.10}$$

Moreover, $\mathcal{P}(F)=F$.

Conversely, let $(A;G)$ *be a* (G,m)*-associative and let* $H=\mathcal{P}(G)$, $K=\iota H \setminus \{0\}$. *Then* $H=\{f_k \mid f_k : A^{k+m} \to A^m, k \in K\}$ *and (1.10) holds as well.*

Proof. Let (1.10) be true and let $g,h \in \mathcal{P}(F)$, $\delta g=\delta h$. We will represent g and h in canonical forms:

$$g = f_{k_0}(1^{i_1} \times f_{k_1} \times 1^{j_1}) \ldots (1^{i_r} \times f_{k_r} \times 1^{j_r}),$$

$$h = f_{\ell_0}(1^{p_1} \times f_{\ell_1} \times 1^{q_1}) \ldots (1^{p_s} \times f_{\ell_s} \times 1^{q_s}),$$

where,

$$i_\nu + j_\nu = k_0+k_1+\ldots+k_{\nu-1}, \quad p_\lambda + q_\lambda = \ell_0+\ell_1+\ldots+\ell_{\lambda-1}$$

for $1 \leq \nu \leq r$, $1 \leq \lambda \leq s$. Here, by $\delta g=\delta h$, we have

$$i_r+j_r+k_r+m = p_s+q_s+\ell_s+m,$$

i.e.

$$k_0+k_1+\ldots+k_r = \ell_0+\ell_1+\ldots+\ell_s. \tag{1.11}$$

By successive applications of (1.10) we obtain

$$g = f_{k_0+k_1}(1^{i_2} \times f_{k_2} \times 1^{j_2}) \ldots (1^{i_r} \times f_{k_r} \times 1^{j_r}) = \ldots =$$

$$= f_{k_0+k_1+\ldots+k_r},$$

$$h = f_{\ell_0+\ell_1}(1^{p_2} \times f_{\ell_2} \times 1^{q_2}) \ldots (1^{p_s} \times f_{\ell_s} \times 1^{q_s}) = \ldots =$$

$$= f_{\ell_0+\ell_1+\ldots+\ell_s},$$

which by (1.11) implies that g=h, i.e. (A;F) is an (F,m)-associative. Clearly, if (A;F) is an (F,m)-associative, then (1.10) holds.

Now, let (A;G) be a (G,m)-associative, $H=\varphi(G)$, $K=\iota H\setminus\{0\}$. By (1.9) we have that K is a subsemigroup of (N;+) and by P.1.3, (A;F) is an (H,m)-associative. If $k\in K$, then there is an $f_k \in H$ such that $\iota f_k = k$, i.e. H is as stated and, by the preceding, (1.10) holds. □

m-dimensional v.v. associatives can be also observed as structures with a so called "poly-operation". Here, a (K,m)-operation is called a mapping $f:A^{K+m} \to A^m$, where A is a non-empty set, K is a subsemigroup of (N;+), $0\notin K$ and

$$A^{K+m} = \bigcup_{k\in K} A^{k+m}$$

(see also [3], §5). The restriction $f_k = f|_{A^{k+m}}$ is a (k+m,m)-operation for every $k\in K$. We say that (A;f) is a (K,m)-groupoid. A (K,m)-groupoid (A;f) is called a (K,m)-semigroup iff for every $a_\nu, b_\lambda \in A$, for every $n,t\in K$ and for every $\alpha: 0\le\alpha\le n$, the following equality holds:

$$f(a_1^\alpha f(b_1^{t+m}) a_{\alpha+1}^n) = f(a_1^\alpha b_1^{t+m} a_{\alpha+1}^n). \tag{1.12}$$

By P.1.5 and (1.12) it follows that a (K,m)-groupoid (A;f) is a (K,m)-semigroup iff (A;f) is an (F,m)-associative, where

$$F = \{f_k \mid f_k \text{ is the restriction of } f \text{ on } A^{k+m},\ k\in K\}. \tag{1.13}$$

Therefore, we will not make any difference between (K,m)-semigroups and (F,m)-associatives, F being defined by (1.13). Hence, as usual, we can denote the poly-operation f by [ ], i.e. for every $a_\nu \in A$, $k\in K$, we can write

$$f(a_1^{k+m}) = [a_1^{k+m}].$$

We note that every v.v. operation $f:A^n \to A^m$ can be considered as an m-tuple of n-operations (i.e. of (n,1)-operations) $f_i:A^n \to A$, $i\in N_m$, defined by:

$$f(a_1^n) = b_1^m \iff f_i(a_1^n) = b_i,\ i\in N_m.$$

We say that f_i is the i-th component of f.

Using components, v.v. associatives can be also considered as a variety of universal algebras and, therefore, the standard universal algebraic concepts such as: subalgebra, homomorphism, congruence etc. make sense for v.v. associatives. Namely, the equality (1.10) can be written componentwise in the following form:

$$f_{r,i}(x_1^\alpha f_{s,1}(y_1^{s+m})\dots f_{s,m}(y_1^{s+m})x_{\alpha+1}^r) = f_{r+s,i}(x_1^\alpha y_1^{s+m} x_{\alpha+1}^r) \qquad (1.10')$$

or in a mixed (component-vector) form:

$$f_{r,i}(x_1^\alpha f_s(y_1^{s+m})x_{\alpha+1}^r) = f_{r+s,i}(x_1^\alpha y_1^{s+m} x_{\alpha+1}^r) \qquad (1.10'')$$

where $f_{k,i}$ denotes the i-th component of the operation f_k.

To every poly-operation [ ] we can associate component mappings $[\ ]_i : A^{K+m} \to A$, defined by:

$$[a_1^p] = b_1^m \iff [a_1^p]_i = b_i,\ i\in N_m.$$

Here the restrictions of $[\ ]_i$ over A^{k+m}, for every $k\in K$, are usual k+m-operations.

§2. VECTOR VALUED ASSOCIATIVES AND VECTOR VALUED SEMIGROUPS

Vector valued associatives are closely related to the vector valued semigroups. We will consider here this connection.

First we note that an (F,m)-associative (A;F) depends in fact on the set $J=\iota F$ of indices of the operations in F, as it is seen by (1.9′). Therefore an (F,m)-associative will be also called a (J,m)-associative and will be denoted by (A;J). Moreover, if $K \leq (N;+)$ (i.e. K is the subsemigroup of (N;+)) generated by J, K=<J>, then an F-algebra (A;F) is a (J,m)-associative iff the $\mathcal{P}'$-algebra $(A;\mathcal{P}')$ is a (K,m)-associative (see P.1.2).

Henceforth, no difference will be made between (A;F) and $(A;\mathcal{P}')$, i.e. we will consider any (J,m)-associative as a (K,m)-associative, where K=<J>. Also, if K=<L>, every (K,m)-associative will be considered as an (L,m)-associative (see P.1.3).

Therefore, we can consider only (K,m)-associative, where K is a subsemigroup of (N;+), $0\notin K$. In that case, if $M \leq K$, then a given (K,m)-associative (A;K) induces a corresponding (M,m)-associative (A;M) which is called an M-restriction of (A;K). Thus:

Proposition 2.1. *If $L \leq K \leq (N;+)$, then an L-restriction of every (K,m)-associative is an (L,m)-associative.* □

We will often use the notation $[x_1^{k+m}]^k$ instead of $g(x_1^{k+m})$, where g is a fixed polynomial operation with the index $k=\iota g$.

Clearly, if (A;J) is a (J,m)-associative, then for any $k \in J$, $(A;[\ ]^k)$ is an (m+k,m)-semigroup, which is said to be induced by the given (J,m)-associative.

On the other hand, every (m+d,m)-semigroup (A;[ ]) can be considered as a (K,m)-associative, where $K=\{sd \mid s \geq 1\}$ (i.e. $K=\langle d \rangle$) is the semigroup generated by d. Namely setting

$$(\forall k \in K)\, [\![x_1^{k+m}]\!] = [x_1^{k+m}],$$

we obtain by the general associative law ([3]) that $(A;[\![\ ]\!])$ is a (K,m)-associative. (Thus an (m+d,m)-semigroup is in fact a $(\langle d \rangle,m)$-associative.)

Let $K \leq L \leq (N;+)$. A (K,m)-associative $\underline{A}=(A;[\![\ ]\!])$ is called a (K,m)-subassociative of an (L,m)-associative $\underline{B}=(B;[\ ])$ iff $A \subseteq B$ and

$$(\forall a_\nu \in A)(\forall k \in K)\ [\![a_1^{k+m}]\!] = [a_1^{k+m}]. \tag{2.1}$$

We will also say that $\underline{B}$ is an extension of $\underline{A}$. In particular, if $L=\langle d \rangle$, then $(A;[\![\ ]\!])$ is called a (K,m)-subassociative of the (m+d,m)-semigroup (B;[ ]); in this case $d \mid GCD(K)$.

The following statements are proved in [2]:

Proposition 2.2. *A (K,m)-associative $(A;[\![\ ]\!])$ is a (K,m)-subassociative of an (m+d,m)-semigroup iff (A;[ ]) is a (K,m)-subassociative of an (m+1,m)-semigroup.* □

Proposition 2.3. *A (K,m)-associative is a (K,m)-subassociative of an (m+1,m)-semigroup iff $d=GCD(K) \in K$.* □

The next example shows that if $d=GCD(K)\notin K$, the class of (K,m)-subassociatives of $(m+1,m)$-semigroups is a proper subclass of the class of (K,m)-associatives.

Example 2.4. ([2]). Let $A=\{a,b,c\}$, $a\neq b\neq c\neq a$ and let J be a set of positive integers such that $d=GCD(J)\notin J$. If p is the least element of J, then the set $L=J\setminus\{\alpha p \mid \alpha \geq 1\}$ is nonempty; let q be the least element of L.

Define a set $F=\{f_k \mid k\in J\}$ of vector valued operations on A in the following way:

$$(\forall k\in J)\,\delta f_k = m+k, \quad \rho f_k = m \text{ and}$$

$$f_k(x_1^{m+k}) = \begin{cases} b^m, & \text{if } k=q \text{ and } x_1^{m+k}=c^{m+k} \\ a^m, & \text{otherwise} \end{cases}$$

Then: a) $(A;F)$ is a (J,m)-associative and b) this (J,m)-associative is not a (J,m)-subassociative of an $(m+1,m)$-semigroup.

We will state here two problems.

Let L,K be subsemigroups of $(N;+)$ such that $L\subset K$.

(i) Under what conditions an (L,m)-associative is an L-restriction of a (K,m)-associative?

Particularly, under what conditions an (L,m)-associative is an L-restriction of a vector valued semigroup?

(ii) Under what conditions an (L,m)-associative can be extended to a (K,m)-associative?

Specially, under what conditions an (L,m)-associative has an extension which is an L-restriction of a vector valued semigroup?

§3. CANCELLATIVE VECTOR VALUED ASSOCIATIVES

We will consider here some properties of cancellative v.v. associatives which are generalizations of the corresponding properties for J-associatives ([1], [6]).

A (K,m)-associative $\underline{A}=(A;[\ ])$ is said to be left cancellative iff for every $k\in K$ and for every a_ν, x_j, $y_p \in A$

$$[a_1^k x_1^m] = [a_1^k y_1^m] \implies x_1^m = y_1^m \tag{3.1}$$

and right cancellative iff

$$[x_1^m a_1^k] = [y_1^m a_1^k] \implies x_1^m = y_1^m. \tag{3.2}$$

A (K,m)-associative is cancellative iff it is left and right cancellative.

Proposition 3.1. *If $\underline{A}=(A;[\ ])$ is a (K,m)-associative, then the following conditions are equivalent:*

(i) $\underline{A}$ is cancellative.

(ii) For every $k\in K$, $i\in N_{k+1}$, $a_\nu, x_\lambda, y_\mu \in A$, the following implication is true

$$[a_1^{i-1} x_1^m a_i^k] = [a_1^{i-1} y_1^m a_i^k] \implies x_1^m = y_1^m \tag{3.3}$$

(iii) There exists a $k\in K$, $k\geq 2$, such that for any $a_\nu, x_\lambda, y_\mu \in A$ the implications (3.1) and (3.2) hold.

(iv) There exists a $k\in K$, $k\geq 2$, and an $i\in N_k$, $i\geq 2$, such that for every $a_\nu, x_\lambda, y_\mu \in A$ (3.3) holds.

Proof. (i) $\implies$ (ii). Assume that $\underline{A}$ is cancellative and $k\in K$, $i\in N_{k+1}$ are such that $[a_1^{i-1}x_1^m a_i^k]=[a_1^{i-1}y_1^m a_i^k]$. Then we have $[a_i^k a_1^{i-1} x_1^m a_i^k a_1^{i-1}]=[a_i^k a_1^{i-1} y_1^m a_i^k a_1^{i-1}]$, i.e. $[a_i^k a_1^{i-1}[x_1^m a_i^k a_1^{i-1}]]=$ $=[a_i^k a_1^{i-1}[y_1^m a_i^k a_1^{i-1}]]$. Thus, by (3.1) we obtain first

$$[x_1^m a_i^k a_1^{i-1}] = [y_1^m a_i^k a_1^{i-1}],$$

and by (3.2), $x_1^m=y_1^m$.

It is clear that (ii) $\implies$ (iii). We will prove first that (iii) $\implies$ (iv), and then (iv) $\implies$ (i).

Let $k\geq 2$ be a given fixed element of K, and let (3.1) and (3.2) be true. If $i\geq 2$ is such that $[a_1^{i-1}x_1^m a_i^k]=[a_1^{i-1}y_1^m a_i^k]$, then $[a_i^k a_1^{i-1}[x_1^m a_i^k a_1^{i-1}]]=[a_i^k a_1^{i-1}[y_1^m a_i^k a_1^{i-1}]]$ implies, by (3.1) and (3.2), that $x_1^m=y_1^m$.

It remains only to prove (iv) $\Longrightarrow$ (i). Assume that $k \in K$, $k \geq 2$, $i \in N_k$, $i \geq 2$ are such that (3.3) holds, for any $a_\nu, x_\lambda, y_\mu \in A$. Then, by ([3], T.5.7) the $(m+k,m)$-semigroup induced by $\underline{A}$ is cancellative. Let s be an arbitrary element of K. Then, by ([3], T.5.7), the $(m+sk,m)$-semigroup is cancellative as well, and this implies that the corresponding $(m+s,m)$-semigroup induced by $\underline{A}$ is cancellative. □

We note that in the same manner as in ([3], P.5.12), the following implication could be proved:

Proposition 3.2. *If $\underline{A}=(A;[\ ])$ is a cancellative (K,m)-associative, then for all $i,j,p,q,r \geq 0$, such that $i+j+p-m, q+j+r-m \in K$,*

$$[a_1^i x_1^j b_1^p] = [a_1^i y_1^j b_1^p] \Longrightarrow [c_1^q x_1^j d_1^r] = [c_1^q y_1^j d_1^r],$$

where $a_\nu, b_\nu, c_\nu, d_\nu, x_\nu, y_\nu \in A$. □

§4. SURJECTIVE VECTOR VALUED ASSOCIATIVE

A (K,m)-associative $\underline{A}=(A;[\ ])$ is surjective iff

$$[A^{K+m}] = A^m. \tag{4.1}$$

(Here: $[A^{K+m}]=\{[x^{k+m}] \mid x_\nu \in A,\ k \in K\}$.)

Proposition 4.1. *If $\underline{A}=(A;[\ ])$ is a (K,m)-associative, then the following statements are equivalent:*

(i) $[A^{k+m}] = A^m$ for every $k \in K$;

(ii) $[A^{k+m}] = A^m$ for some $k \in K$;

(iii) $\underline{A}$ is surjective.

Proof. Clearly, we have (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii), and, thus, we have to show that (iii) $\Longrightarrow$ (i). Since the subsemigroup K of $(N;+)$ is finitely generated ([5]), let $\{k_1,k_2,\dots,k_r\}$ be a generating subset of K. Then, assuming A is surjective, it is sufficient to show that $(\forall i \in N_r)\ [A^{k_i+m}]=A^m$. (Namely, if $[A^{k_i+m}]=A^m$, for every $i \in N_r$, and k is an arbitrary element of K, then $k=\alpha_1 k_1+\alpha_2 k_2+\dots+\alpha_r k_r$ and by using induction on the number of appearences of the generators of K, we have

$[A^{k+m}] = [A^{\alpha_1 k_1+\ldots+\alpha_r k_r+m}] = [[A^{k_1+m}]A^{(\alpha_1-1)k_1+\ldots+\alpha_r k_r}] =$

$= [A^m A^{(\alpha_1-1)k_1+\ldots+\alpha_r k_r}] = A^m.)$

Let k_i be a fixed element of $\{k_1,k_2,\ldots,k_r\}$. We will prove that $[A^{k_i+m}]=A^m$. If $a_1^m \in A^m$, then by the surjectivity of $\underline{A}$, there exist $b_\nu \in A$, such that $a_1^m=[b_1^{\nu_1 k_1+\ldots+\nu_r k_r+m}]$, where $\nu_j \geq 0$, $\nu_1+\ldots+\nu_r \geq 0$. By the same reason there exist $c_\lambda \in A$ such that $b_1^m = [c^{\lambda_1 k_1+\ldots+\lambda_r k_r+m}]$, where $\lambda_j \geq 0$, $\lambda_1+\ldots+\lambda_r \geq 0$, and thus

$$a_1^m = [c^{\lambda_1 k_1+\ldots+\lambda_r k_r+m}\, b_{m+1}^{\nu_1 k_1+\ldots+\nu_r k_\nu+m}].$$

Continuing this procedure one obtains that

$$a_1^m \in [A^{\varepsilon_1 k_1+\ldots+\varepsilon_r k_r+m}],$$

where at least one ε_s is such that $\varepsilon_s=k_i+p$, $p \geq 0$. Hence,

$$\varepsilon_i k_i + \varepsilon_s k_s = k_i+(\varepsilon_i+k_s-1)k_i+k_s p,$$

i.e. for some $k \in K$ we have

$$\varepsilon_1 k_1+\ldots+\varepsilon_r k_r+m = k_i+k+m.$$

Therefore

$$a_1^m \in [A^{k_i}[A^{k+m}]] \subseteq [A^{k_i}A^m] = [A^{k_i+m}],$$

i.e. $A^m \subseteq [A^{k_i+m}]$. Obviously, $[A^{k_i+m}] \subseteq A^m$, and so $[A^{k_i+m}]=A^m$. □

Here we note that a direct product of any nonempty collection of surjective (K,m)-associatives is a surjective (K,m)-associative. Clearly, a homomorphic image of a surjective (K,m)-associative is a surjective (K,m)-associative.

A (K,m)-associative $\underline{A}=(A;[\ ])$ is called a (K,m)-group iff for every $k \in K$, the $(m+k,m)$-semigroup $(A;[\ ]^k)$ is an $(m+k,m)$-group.

The notion of an $(m+k,m)$-group is defined, for example, in [3] and it is shown there that every $(m+k,m)$-group is cancellative. Thus every (K,m)-group is a surjective and cancellative (K,m)-associative. We will show the following:

Proposition 4.2. *Let K be a subsemigroup of (N;+) and d=GCD(K). Then a (K,m)-associative is a (K,m)-group iff it is a K-restriction of an (m+d,m)-group.*

Proof. Let $\underline{A}$=(A;[ ]) be an (m+d,m)-group. Let $[\ ]^s$ be the (m+sd,m)-operation induced by [ ]. The (<d>,m)-associative induced by $\underline{A}$ is a (<d>,m)-group, and thus, for all $s \geq 1$ $(A;[\ ]^s)$ is an (m+sd,m)-group. Then the K-restriction $(A;\{[\ ]^s \mid sd \in K\})$ is a (K,m)-group.

Suppose now that $\underline{A}=(A;[\![\ ]\!])$ is a (K,m)-group. We first note that there exists $p \in K$ such that

$$(x \in K \wedge x \geq p) \Longleftrightarrow (\exists r \in N)\ x = p + rd$$

(see [5]). We choose p to be the least element with the above property, and in this case the subset $K_*=\{p,p+d,p+2d,\ldots\}$ of K is called the regular part of K.

We will define an (m+d,m)-operation [ ] on A as follows.

Let $a_1^{m+d} \in A^{m+d}$ and $0 \leq i \leq d$. By P.4.1 there exist $x_\nu \in A$ such that $a_{i+1}^{i+m} = [x_1^{m+k}]$, $k, k+d \in K$, and we put

$$[a_1^{m+d}] = [\![a_1^i x_1^{m+k} a_{i+m+1}^{m+d}]\!].$$

The operation [ ] is well defined, since for all i, $0 \leq i \leq d$, we have:

$$[\![y_1^{m+t} a_{m+1}^{m+d}]\!] = [\![a_1^i z_1^{m+s} a_{i+m+1}^{m+d}]\!] \tag{4.2}$$

where $a_1^m=[\![y_1^{m+t}]\!]$, $a_{i+1}^{i+m}=[\![z_1^{m+s}]\!]$, $t,s,t+d,s+d \in K$; $y_\nu, z_\nu \in A$. Namely, for any $c_\nu \in A$,

$$[\![c_1^p y_1^{m+t} a_{m+1}^{m+d}]\!] = [\![c_1^p [\![y_1^{m+t}]\!] a_{m+1}^{m+d}]\!] = [\![c_1^p a_1^m a_{m+1}^{m+d}]\!] =$$

$$= [\![c_1^p a_1^i [\![z_1^{m+s}]\!] a_{i+m+1}^{m+d}]\!] = [\![c_1^p a_1^i z_1^{m+s} a_{i+m+1}^{m+d}]\!],$$

which imply (4.2) by the left cancellativity of $\underline{A}$.

Now we will show that the (m+d,m)-operation [ ] is associative. Namely,

$$[[a_1^{m+d}] b_{m+1}^{m+d}] = [\![a_1^i y_1^{p+m} a_{m+i+1}^{m+d} b_{m+1}^{m+d}]\!],$$

where $a_{i+1}^{i+m} = [\![y_1^{m+p}]\!]$,

$$[a_1^i[a_{i+1}^{m+d}b_{m+1}^{m+i}]b_{m+i+1}^{m+d}] = [\![a_1^i y_1^{m+p} a_{m+i+1}^{m+d} b_{m+1}^{m+i} b_{m+i+1}^{m+d}]$$

for i: $0 \le i \le d$, and thus

$$[[a_1^{m+d}]b_{m+1}^{m+d}] = [a_1^i[a_{i+1}^{m+d}b_{m+1}^{m+d}]b_{m+i+1}^{m+d}].$$

It is clear that each $(k+m,m)$-operation ($k\in K$) of $(A;[\![\ ]\!])$ is induced by the operation $[\ ]$. It remains to show that

$$(\forall a_1^d\in A^d,\ b_1^m\in A^m)(\exists x_1^m, y_1^m\in A^m)\,[a_1^d x_1^m] = b_1^m = [y_1^m a_1^d]. \qquad (4.3)$$

Let $c_1,\ldots,c_p$ be fixed elements of A. Then the equations

$$[\![a_1^d c_1^p u_1^m]\!] = b_1^m \text{ and } [\![v_1^m c_1^p a_1^d]\!] = b_1^m$$

have solutions on u_1^m, v_1^m in the (K,m)-group $(A;[\![\ ]\!])$ and then

$$[\![c_1^p u_1^m]\!] = x_1^m,\quad [\![v_1^m c_1^p]\!] = y_1^m$$

are solutions of the equations (4.3). Hence $(A;[\ ])$ is a $(d+m,m)$-group. □

§5. FREE VECTOR VALUED ASSOCIATIVES

The fact that the class of (K,m)-associatives can be characterized as a variety of universal algebras defined by a set of identities implies that every non-empty set B is a basis of a free (K,m)-associative. Here we give a convenient description of free (K,m)-associatives, following the ideas of [4].

First, we introduce some concepts and notations.

If X is a nonempty set, then X^+ denotes the set of all finite sequence of elements of X, i.e. $X^+ = \bigcup_{i\ge 1} X^i$. The set X^+ with the operation concatenation of sequences is a free semigroup with a basis X. If 1 denotes the empty sequence, then $X^* = X^+ \cup \{1\}$ is a free monoid with a basis X. If $K \subseteq N$, then we put $X^{K+m} = \bigcup_{k\in K} X^{k+m}$.

Now let B be a nonempty set, K a subsemigroup of $(N;+)$, $0\notin K$, and $m \ge 2$. We define a sequence of sets $B_0, B_1, B_2, \ldots$ in the following inductive way:

$$B_0 = B,\ B_{p+1} = B_p \cup N_m \times B_p^{K+m} \qquad (5.1)$$

and we put

$$\overline{B} = \bigcup_{p\geq o} B_p . \qquad (5.2)$$

Note that $u\in\overline{B}$ iff $u\in B$ or $u=(i,x)$, for some $i\in N_m$ and $x\in\overline{B}^{K+m}$.

We define a mapping from $\overline{B}^+$ into N, named a norm and denoted by $|\ |$, in the following inductive way:

$$b\in B \implies |b| = 1;$$

$$u = (i,x)\in\overline{B} \implies |u| = |x| + 1;$$

$$x,y\in\overline{B}^+ \implies |xy| = |x| + |y|.$$

An element $u\in\overline{B}$ is said to be reducible if $u=(i,x)$ and $x=u_1^{k+m}$, $u_\nu\in\overline{B}$, $k\in K$, u_λ is reducible for some λ, or $x=x'(1,y)\ldots$ $\ldots(m,y)x''$, where $x'x''\in\overline{B}^+$, $(1,y),\ldots,(m,y)\in\overline{B}$.

If $u\in\overline{B}$ is not reducible, then we say that u is reduced. The set of all reduced elements of B will be denoted by R.

Using induction on norm, we define a mapping $\psi:\overline{B} \to R$ which we will call a reduction.

(i) $u\in R \implies \psi(u)=u.$

Let $u\in\overline{B}\setminus R$ and suppose that for every $v\in\overline{B}$, such that $|v| < |u|$, $\psi(v)$ is well-defined element of R and the following condition is satisfied:

$$\psi(v) \neq v \iff |\psi(v)| < |v| \iff v\in\overline{B}\setminus R. \qquad (5.3)$$

Assume that $u=(i,u_1^{k+m})$, where $u_\lambda\in\overline{B}$, $k\in K$. Then $|u_j| < |u|$ for every $j\in N_{k+m}$, and by the inductive hypothesis, $\psi(u_j)\in R$ is defined and (5.3) is satisfied for $v=u_j$.

If $x=u_1^p$, $u_\nu\in\overline{B}$ and $\psi(u_\nu)$ are defined, then we will write

$$\psi(x) = \psi(u_1)\ldots\psi(u_p). \qquad (5.4)$$

If $\psi(u_j)\neq u_j$ for some j, then we set

(ii) $\psi(u) = \psi(i,\psi(u_1^{k+m})).$

Here, $\psi(u)$ is well-defined by (ii) since, by (5.3), $|\psi(u_j)| < |u_j|$ and thus

$$|(i,\psi(u_1^{k+m}))| = 1 + \sum_\nu|\psi(u_\nu)| < 1 + \sum_\nu|u_\nu| = |u|.$$

If we have $\psi(u_j)=u_j$ for every j, then by (5.3) every u_j is reduced and thus $u_1^{k+m}=x'(1,y)\ldots(m,y)x''$ for sóme $x'x''\in\overline{B}^{+j}$. Assume that x' has the least possible norm. Then we put

(iii) $\psi(u) = \psi(i,x'yx'')$.

Since the choice of x' is unique and $|x'yx''| < |x'(1,y)\ldots(m,y)x''| = |u|$, it follows by the inductive hypothesis that $\psi(u)$ is well defined by (iii). Moreover, (5.3) is satisfied if we replace v by u.

We will state some properties of the reduction ψ.

Proposition 5.1. *The following statements are true:*

(a) $\psi(u) \neq u \iff |\psi(u)| < |u| \iff u\in\overline{B}\setminus R$.

(b) $\psi(\psi(u)) = \psi(u)$, *for every* $u\in\overline{B}$.

(c) $\psi(i,xyz) = \psi(i,x\psi(y)z)$, *for every* $xyz\in\overline{B}^{K+m}$ *and every* $y\in\overline{B}$.

(d) $\psi(i,x(1,y)\ldots(m,y)z)=\psi(i,xyz)$, *for every* $xyz,y\in\overline{B}^{K+m}$.

Proof. (a) is proved in the above definition of ψ. This, and the fact that $\psi(u)\in R$ implies (b).

To prove (c) we first note that we can assume that $\psi(y)\neq y$. Let $x=u_1^{\alpha}$, $z=v_1^{\beta}$, where $u_\nu, v_\lambda\in\overline{B}$, $\alpha,\beta\geq 0$. If $\psi(u_\nu)=u_\nu$, $\psi(v_\lambda)=v_\lambda$ for every pair (ν,λ), then (c) is true by (ii), and thus we can assume that there is a pair (ν,λ) such that $\psi(u_\nu)\neq u_\nu$ or $\psi(v_\lambda)\neq v_\lambda$. If $x'=\psi(u_1)\psi(u_2)\ldots\psi(u_\alpha)$, $z'=\psi(v_1)\ldots\psi(v_\beta)$, then we have $|x'z'| < |xz|$, and (i) and induction on norm imply:

$$\psi(i,xyz) = \psi(i,x'\psi(y)z') = \psi(i,x'yz') = \psi(i,xyz).$$

It remains to show (d). Let $\psi(xyz)=xyz$ and let x has the least possible norm. Then (d) follows by (iii). If $\psi(xyz)\neq xyz$, then (a) implies that $\psi(x)\psi(y)\psi(z)$ has smaller norm than xyz; then by (ii) and (c), using induction on the norm, we have:

$$\begin{aligned}\psi(i,xyz) &= \psi(i,\psi(x)\psi(y)\psi(z)) = \psi(i,\psi(x)(1,\psi(y))\ldots(m,\psi(y))\psi(z)) = \\ &= \psi(i,\psi(x)\psi(1,\psi(y))\ldots\psi(m,\psi(y))\psi(z)) = \\ &= \psi(i,\psi(x)\psi(1,y)\ldots\psi(m,y)\psi(z)) = \\ &= \psi(i,x(1,y)\ldots(m,y)z).\end{aligned}$$

Now, consider the case $\psi(xyz)=xyz$ when the norm of x is not the least possible. But then $x=x'(1,t)\ldots(m,t)x''$, where x' has the least possible norm and then by (iii) and the inductive hypothesis, we have:

$$\psi(i,x'(1,t)\ldots(m,t)x''(1,y)\ldots(m,y)z) =$$
$$= \psi(i,x'tx''(1,y)\ldots(m,y)z) = \psi(i,x'tx''yz) =$$
$$= \psi(i,x'(1,t)\ldots(m,t)x''yz) = \psi(i,xyz). \quad \square$$

The set R of all reduced elements of $\overline{B}$ can be written now in the form

$$R = \{u\in\overline{B} \mid \psi(u) = u\}.$$

Let L be a subsemigroup of K. We define a subset R_L of R by induction on norm as follows. First, $B\subseteq R_L$, i.e. $|u| = 1$ implies $u\in R_L$. If $u=(i,u_1^{k+m})\in R$ then

$$u\in R_L \text{ iff } k\in L \text{ and } u_\nu\in R_L \text{ for every } \nu.$$

Thus, $R_K = R$.

Define in R a (K,m)-operation [ ] by:

$$(\forall k\in K)(\forall u_\nu,v_\lambda\in R)([u_1^{k+m}] = v_1^m \iff (\forall i\in N_m)v_i = \psi(i,u_1^{k+m})). \qquad (5.5)$$

<u>Proposition 5.2</u>. *The (K,m)-groupoid (R;[ ]) is a (K,m)-associative. For every subsemigroup L of K, R_L is an (L,m)-subassociative of (R;[ ]) and (R_L;[ ]) is generated by B.*

<u>Proof</u>. First, by P.5.1 (b) it follows that [ ] is a well defined (K,m)-operation on R. Also by P.5.1 it can be easily shown that (R;[ ]) is a (K,m)-associative.

In addition, the definitions of [ ] and R_L imply that if $u\in R_L$ for every $\nu\in N_{m+\ell}$, where $\ell\in L$, and $[u_1^{\ell+m}]=v_1^m$, then $v_\nu\in R_L$ for every $\lambda\in N_m$. Thus, R_L is an (L,m)-subassociative of (R;[ ]).

The conclusion that B is a generating subset of R_L can be also obtained in a usual obvious way. $\square$

<u>Proposition 5.3</u>. *Let L and K be as above and let (A;⟦ ⟧) be an (L,m)-associative. If $\xi:b \to \overline{b}$ is a mapping from B into A, then there exists a unique homomorphism $\overline{\xi}:(R_L;[\ ]) \to (A;⟦\ ⟧)$ which is an extension of ξ.*

Proof. By P.5.2, it is enough to show the existence of such a homomorphism $\overline{\xi}$. We define $\overline{\xi}$ by induction on norm. Thus, if $u \in B$, then $\overline{\xi}(b)=\xi(b)$. Assume that $u=(i,u_1^{\ell+m}) \in R_L$ is such that $\overline{\xi}(u_\nu)=b_\nu \in A$ is well defined for every $\nu \in N_{m+\ell}$. Then, we define $\overline{\xi}(u)$ by:

$$\overline{\xi}(u) = a_i, \text{ where } [\![b_1^{\ell+m}]\!] = a_1^m \text{ in } (A;[\![\ ]\!]).$$

Therefore, we have an extension $\overline{\xi}:R_L \to A$ of $\overline{\xi}$, and $\overline{\xi}:(R_L;[\ ]) \to (A;[\![\ ]\!])$ is a homomorphism by the definition of $[\ ]$ and $\overline{\xi}$. □

As a corollary from P.5.2 and P.5.3 we obtain the following

Theorem 5.4. *The (K,m)-associative (R;[ ]) is a free (K,m)-associative with a basis B. If (F;[ ]) is a free (K,m)-associative with a basis B and L is a subsemigroup of K, then the (L,m)-subassociative (G;[ ]) generated by B is a free (L,m)-associative with a basis B.* □

Finally, we have the following

Theorem 5.5. *Every free (L,m)-associative is cancellative.*

Proof. If K is the set of all positive integers, then a (K,m)-associative is essentially the same as an (m+1,m)-semigroup. It is shown in [3] (T.6.9) that every free v.v. semigroup is cancellative, and therefore every free (L,m)-associative is cancellative as an (L,m)-subassociative of a cancellative (m+1,m)-semigroup. □

R E F E R E N C E S

[1] N. Celakoski; On semigroup associatives, Maced. Acad. Sci. and Arts, Contributions IX 2, Sect. Nat.-Math. Sci. (1977), 5-19.

[2] N. Celakoski, B. Janeva; Vector valued associatives, Proc. Conf. "Algebra and Logic", Cetinje 1986, 15-21.

[3] G. Čupona, N. Celakoski, S. Markovski, D. Dimovski; Vector valued groupoids, semigroups and groups (this volume).

[4] D. Dimovski; Free vector valued semigroups, Proc. Conf. "Algebra and Logic", Cetinje 1986, 55-62.

[5] Д. Димовски; Адитивни полугрупи на цели броеви, МАНУ, Прилози IX 2 (1977), 21-26.

[6] Ѓ. Чупона; Асоцијативи со кратење, Год. зб. ПМФ Скопје, 19 (1969), 5-14.