FREE (n+1,n)-GROUPS

Dončo Dimovski

Abstract. The goal of this paper is to give a combinatorial description of free (n+1,n)-groups. This description is also a proof that (n+1,n)-groups do exist.

§0. INTRODUCTION

Vector valued groups are defined in [1]. Here we focus on vector valued (n+1,n)-groups. An (n+1,n)-group is a set G together with a map $[\ ]:G^{n+1} \to G^n$, satisfying the following conditions:

$$[[x_1^{n+1}]x_{n+2}] = [x_1[x_2^{n+2}]] \quad \text{(associativity); and} \tag{0.1}$$

for given $a \in G$, $\underline{b} \in G^n$, there exist $\underline{x},\underline{y} \in G^n$, such that

$$[a\underline{x}] = \underline{b} = [\underline{y}a]. \text{ (solvability of equations).} \tag{0.2}$$

Above, x_1^m denotes $x_1x_2\ldots x_m$; $(x_1^m)=\underline{x}$ denotes the vector $(x_1,x_2,\ldots,x_m)$; $[x_1^m]$ denotes $[\ ](x_1^m)$. Because of (0.1), $[[\ldots[[x_1^{n+1}]y_1]\ldots]y_m]$ is denoted by $[x_1^{n+1}y_1^m]$. By $[\ ]_i$ we denote the i^{th} component of the map $[\ ]$, i.e. $[\underline{x}]_i=y_i$ if $[\underline{x}]=(y_1^n)$. The above definition for n=1 is a definition of ordinary (binary) group, thus, henceforth we assume that $n \geq 2$.

In [1], Theorem 4.3., it is stated that free (n+1,n)-groups are nontrivial, i.e. have more than one element. Although its statement is true, the proof is not correct. In professor Čupona's seminar at Skopje, we attempted to find a correct proof. Here we give a satisfactory combinatorial description of free (n+1,n)-groups, showing that they are nontrivial, in fact they are infinite.

The free (n+1,n)-groups are the only known examples of nontrivial (n+1,n)-groups. In [2] it was shown that some finite sets do not admit an (n+1,n)-group structure. Professor John Thompson provided me with an indirect proof that nontrivial finite (3,2)-groups do not exist. His proof can be generalized to a proof that finite nontrivial (n+1,n)-groups do not exist.

We need the following result proved in [2].

Proposition 0.1. *Let* $(G,[\ ])$ *be an* $(n+1,n)$*-group. Then* $(G^n,\circ)$*, where* $\underline{x}\circ\underline{y}=[\underline{x}\underline{y}]$*, is a group with identity element* $(e^n)=(e,\dots,e)$*,* $e\in G$*, such that:*

(i) for each $x\in G$*,* $[xe^n]=[e^n x]$*;*

(ii) if $[x_1^m]=(e^n)$*, then* $[x_2^m x_1]=(e^n)$ *(here* $m\geq n+1$*); and*

(iii) there exist a map $g:G\to G$*, called the* $[\ ]$*-inverse map, such that* $g^{n+1}=id_G$*, and for each* $x\in G$*,* $[xg(x)g^2(x)\dots g^n(x)]=(e^n)$*.* □

Henceforth, the number n shall be fixed, and due to technical reasons, it will be denoted by N.

§1. A COMBINATORIAL DESCRIPTION OF FREE (N+1,N)-GROUPS

Let X be a given (possibly empty) set. We are going to give a combinatorial description of a free (N+1,N)-group generated by X.

Definition 1.1. Let $X^{-i}=\{x^{-i}\mid x\in X\}$ for $i\in\{0,1,\dots,N\}$. Let $Y=\bigcup_{i=0}^{N}X^{-i}$. Define $h:Y\to Y$ by $h(x^{-i})=x^{-(i+1)}$, where + is the addition in the group Z_{N+1}.

Note that for $X=\emptyset$, $Y=\emptyset$. Moreover, for $X\neq\emptyset$, the map h is a bijection and $h^{N+1}=id_Y$.

Next we are going to define a sequence of sets A_k, $k\geq 0$, by induction on k.

Definition 1.2. Let $A_0=\{e\}\cup Y$, $e\notin Y$. Suppose that A_k is defined. Let

$A_{k+1}=A_k\cup(B_{k+1}\setminus C_{k+1})\times M_N$, where

$M_N=\{1,2,\dots,N\}$,

$B_{k+1} = \{x_1^n \mid x_j \in A_k,\ n \geq N+1\}$ and

$C_{k+1} = \{x_1^r e^N x_{r+1}^s \mid x_j \in A_k,\ x_{r+1} \neq e,\ r+1 \leq s < N\}$.

Let $A = \bigcup_{k=0}^{\infty} A_k$. (Compare with [3]).

Above, $x_1^j x_{j+1}^r$ denotes x_1^r if $j=0$, and $x_1^j x_{j+1}^r$ denotes x_1^j if $r \leq j$.

Next, let $S(A)$ be the free semigroup generated by A, $S^1(A)=S(A) \cup \{1\}$, where 1 is the empty word. From now on, the letters u,v,w,u_j,v_j,w_j will be used for elements from A.

Definition 1.3. We say that $a=u_1^n \in S(A)$ has dimension n, and write $\dim(a)=n$. Define the length $|a|$ of a, by induction, as follows: $|e|=1$; $|y|=1$ for $y \in Y$; $|u_1^r|=|u_1|+\ldots+|u_n|$; $|(u_1^n,i)|=|u_1^n|$. Let $\dim(1)=0=|1|$.

Similarly as in [3], we are going to define a map $\psi: S(Y) \to S(A)$ called reduction by induction on the length. This will lead us to the description of a free (N+1,N)-group generated by X.

Definition 1.4. (a) For $u \in A_0$, let $\psi(u)=u$.

(b) Suppose that ψ is defined on S_m, where $S_m=\{a \mid a \in S(A),\ |a| \leq m\}$, i.e. $\psi: S_m \to S(Y)$, and moreover, for $a \in S_m$:

$$|\psi(a)| \leq |a|; \qquad (1.1)$$

$\psi(a) \neq a$ and $|\psi(a)| = |a|$ if and only if
$$a=u_1^r e^N u_{r+1}^s,\quad u_{r+1} \neq e,\ r+1 \leq s < N, \text{ and } \psi(u_j)=u_j \qquad (1.2)$$
for each $j=1,2,\ldots,s$;

$$\psi\psi(a) = \psi(a); \qquad (1.3)$$

$$\dim(a)<N \text{ or } \dim(\psi(a))<N \text{ implies } \dim(a)=\dim(\psi(a)); \qquad (1.4)$$

$\dim(\psi(a)) \geq N+1$ implies that $(\psi(a),i) \in A$, for each $i \in M_N$; and (1.5)

If $a=u_1^N$ with $\psi(u_j)=u_j$ for each $j \in M_N$, then: $\psi(a) \neq a$ if and only if $u_j=(v_1^p,j)$ for each $j \in M_N$. (1.6)

Moreover, in this case, $\psi(a)=v_1^p$.

Remark: (1) (1.1) implies that $\psi(S_m)\subseteq S_m$; (2) Definition 1.2. and (1.2) imply that for $a\in A\cap S_m$, $\psi(a)\neq a$ if and only if $|\psi(a)|<|a|$; (3) (1.1) and (1.4) imply that $\psi(A\cap S_m)\subseteq A\cap S_m$.

(c) The extension of $\psi:S_m \to S(Y)$ to a map $\psi:S_{m+1} \to S(A)$ is given by the following algorithm, i.e. for $a\in S_{m+1}$, $\psi(a)$ is defined by the first possible application of one of the following steps:

(I) If $2\le \dim(a)=n<N$, then $\psi(a)=\psi_1(a)$, where $\psi_1(u_1^t)$ is only a notation for $\psi(u_1)\psi(u_2)\ldots\psi(u_t)$.

(II) If $\dim(a)=n\ge N$, then:

(II.1) If $|\psi_1(a)|<|a|$, then $\psi(a)=\psi(\psi_1(a))$;

(II.2) If $a=u_1^r e^N u_{r+1}^s$, $u_{r+1}\neq e$, $r+1\le s<N$, then $\psi(a)=u_1^s e^N$;

(II.3) If $a=bu_1^N c$, where $b,c\in S^1(A)$, $u_j=(v_1^p,j)$, $j\in M_N$ and b is of the smallest such dimension, then $\psi(a)=\psi(bv_1^p c)$;

(II.4) If $a=be^N c$, where $b,c\in S^1(A)$, $\dim(bc)\ge N$, and b is of the smallest such dimension, then $\psi(a)=\psi(bc)$;

(II.5) If $a=bu_1^{N+1}c$, where $b,c\in S^1(A)$, $u_1=y\in Y$, $u_{j+1}=h^j(y)$, $j\in M_N$, and b is of the smallest such dimension, then $\psi(a)=\psi(bce^N)$;

(II.6) If $a=bu_{r+1}^N cu_1^r d$, where $b,c,d\in S^1(A)$, $u_j=(v_1^p,j)$, $j\in M_N$, $\psi(v_1^p,c)=e^N$, b is of the smallest such dimension and c is of the smallest such dimension for this b, then $\psi(a)=\psi(bde^N)$;

(II.7) $\psi(a)=a$.

(III) If $\dim(a)=1$, i.e. $a=(u_1^p,j)\in A$, then $\psi(a)=(\psi(u_1^p),j)$. Here, if $\psi(u_1^p)=w_1^N$, then $(\psi(u_1^p),j)=(w_1^N,j)$ is only a notation for w_j.

Proposition 1.5. *(A) (I) to (III) from Definition 1.4. is indeed an algorithm, which extends the map* $\psi:S_m \to S(A)$ *to a map* $\psi:S_{m+1} \to S(A)$.

(B) For each $a\in S_{m+1}$, $\psi:S_{m+1} \to S(A)$ *satisfies (1.1) to (1.6).*

The proof will be given later.

Proposition 1.5. completes the inductive step in Definition 1.4, and so we have defined the map $\psi:S(A) \to S(A)$. Moreover, the construction of ψ together with Proposition 1.5 implies that the map ψ satisfies (1.1) to (1.6) for each $a \in S(A)$, and (1.1) and (1.4) imply that $\psi(A) \subseteq A$.

Next, we need the following theorem whose proof is very technical and long, and it will be given later. Its main point is that the order of the steps in the defining algorithm for ψ is not essentially important.

Theorem 1.6. *The map $\psi:S(A) \to S(A)$ satisfies the following conditions:*

For each $a \in A_0$, $\psi(a)=a$; (1.7)

$\psi(e^N) = e^N$; (1.8)

$\psi(abc) = \psi(a\psi(b)c)$ *for* $a,c \in S^1(A)$, $b \in S(A)$; (1.9)

$\psi(ae^N b) = \psi(abe^N)$ *for* $a,b \in S^1(A)$; (1.10)

$\psi(ae^N) = \psi(a)$ *for* $a \in S(A)$, $dim(a) \geq N$; (1.11)

$\psi(u_1^N) = \psi(v_1^p)$ *for* $u_j=(v_1^p,j)$, $j \in M_N$; (1.12)

$\psi(u_1^{N+1}) = e^N$ *for* $u_1=y \in Y$, $u_{j+1}=h^j(y)$, $j \in M_N$; (1.13)

$\psi(u_{r+1}^N a u_1^r) = e^N$ *for* $u_j=(v_1^p,j)$, $j \in M_N$ *and* $\psi(v_1^p\ a)=e^N$; (1.14)

$\psi(ab) = e^N$ *implies* $\psi(ba) = e^N$; *and* (1.15)

If $(a,j) \in A$, *then* $\psi(a,j)=(\psi(a),j)$. *(See remark in (III) of Definition 1.4.)* (1.16)

Let $Q=\psi(A)$. It is obvious that $Q \subseteq A \subseteq S(A)$. Let $R=\{a \mid a \in S(A),\ \dim(a) \geq N\}$. For each $i \in M_N$, and $a \in R$, let

$\theta_i(a) = (\psi(a),i)$, and $\theta(a) = \theta_1(a)\ldots\theta_N(a)$. (1.17)

Lemma 1.7. *$\theta_i: R \to Q$ are well defined maps. Moreover*

$\theta(a) = \psi(a)$ *and* $\theta_i\theta(a) = \theta_i(a)$. (1.18)

Proof. (1.4) implies that $\dim(\psi(a)) \geq N$. If $\dim(\psi(a))=N$, let $\psi(a)=w_1^N$. Then $\psi\psi(a)=\psi(a)$ by (1.3), i.e. $\psi(w_1^N)=w_1^N$. If $\psi(w_i) \neq w_i$ for some i, then (1.2) implies that $|\psi(w_i)| < |w_i|$ and so, $|\psi(\psi_1(w_1^N))| < |\psi(w_1^N)| = |w_1^N|$. On the other hand, (1.9) implies that $\psi(w_1^N) = \psi(\psi_1(w_1^N))$. So, $\psi(w_i) = w_i$ for each $i \in M_N$. Hence, $w_i = (\psi(a), i) \in Q$. If $\dim(\psi(a)) \geq N+1$, then $(\psi(a), i) \in Q$ because of (1.5) and (1.16). The fact that the maps θ_i and θ satisfy (1.18), follows from (1.12) and (1.3). It is obvious that θ is a map from R to R. □

Definition 1.8. Let $[\ ]: Q^{N+1} \to Q^N$ be defined by

$$[u^{N+1}]_i = \theta_i(u_1^{N+1}). \tag{1.19}$$

The proof of the following main theorem is going to be divided into several lemmas.

Theorem 1.9. *$(Q, [\ ])$ is a free $(N+1,N)$-group generated by X.*

Lemma 1.10. *$(Q, [\ ])$ is an $(N+1,N)$-semigroup, i.e. $(Q, [\ ])$ satisfies (0.1).*

Proof. $[[u_1^{N+1}]u_{N+2}]_i = \theta_i([u_1^{N+1}]_1 \ldots [u_1^{N+1}]_N u_{N+2})$
$= \theta_i(\theta(u_1^{N+1})u_{N+2}) = (\theta(\theta(u_1^{N+1})u_{N+2}), i)$
$= (\psi(\psi(\theta(u_1^{N+1}))u_{N+2}), i)$ (by (1.9))
$= (\psi(\psi(u_1^{N+2})u_{N+2}), i)$ (by (1.18)) $= (\psi(u_1^{N+2}), i)$ (by (1.9))
$= (\psi(u_1\psi(u_2^{N+2})), i)$ (by (1.9))
$= (\psi(u_1\psi\theta(u_2^{N+2})), i)$ (by (1.18))
$= (\psi(u_1\theta(u_1^{N+2})), i)$ (by (1.9))
$= \theta_i(u_1\theta(u_2^{N+2})) = \theta_i(u_1[u_2^{N+2}]) = [u_1[u_2^{N+2}]]_i$. □

Lemma 1.11. *For $a = u_1^n$, $n \geq N$, $u_j \in Q$, $[ae^N]_i = [e^N a]_i = \theta_i(a)$ for each $i \in M_N$.*

Proof. $[ae^N]_i = \theta_i(ae^N) = (\psi(ae^N), i) = (\psi(a), i)$ (by (1.11)) $= \theta_i(a)$. Similarly, $[e^N a]_i = \theta_i(a)$. □

Lemma 1.12. *There are maps $g_i: Q \to Q$, $i \in M_N$, such that for each $u \in Q$,*

$$[ug_1(u)g_2(u)\ldots g_N(u)] = (e^N). \tag{1.20}$$

Proof. For $y \in Y \subseteq Q$, define $g_i(y)=h^i(y)$. For e, define $g_i(e)=[e^{2N-1}]_i$. Lemma 1.10 implies that $[eg_1(e)\dots g_N(e)]=[e^{2N}]$, and (1.11) and (1.8) imply that $[e^{2N}]=(e^N)$. Now, suppose that the maps g_i are defined on $Q_m=\{u \mid u\in Q,\ |u| \le m\}$, and satisfy (1.20). Let $u=(a,j)=(u_1^n,j)\in Q_{m+1}$. Define $g_i(u)=\theta_i(b)$ where:

$b = (a,j+1)\dots(a,N)G(a)(a,1)\dots(a,j-1)$;

$G(a) = G(u_1^N) = G_1(u_n)G_1(u_{n-1})\dots G_1(u_1)$; and

$G_1(v) = g_1(v)g_2(v)\dots g_N(v)$.

Now, for each $i\in M_N$, $\theta_i(uG_1(u))=\theta_i(u\theta_1(b)\dots\theta_N(b))=$
$= \theta_i(u\theta(b)) = (\psi(u\theta(b)),i) = (\psi(u\psi\theta(b)),i)$
$= (\psi(u\psi(b)),i)$ (by (1.18)) $= (\psi(ub),i)$ (by (1.9))
$= (e^N,i)$ (by (1.14), because $\psi(aG(a))=\psi(u_1^{n-1}u_nG_1(u_n)G(u_1^{n-1}))$
$= \psi(u_1^{n-1}e^NG(u_1^{n-1}))$ (by (1.19) and (1.9)) $= \psi(u_1^{n-1}G(u_1^{n-1}))$
(by (1.10) and (1.11)) $= \psi(e^N)$ (by induction) $= (e^N)$ (by (1.8)).
Hence, $\theta_i(uG_1(u))=(e^N,i)=e$ for each $i\in M_N$, i.e. $[uG_1(u)]=(e^N)$. □

Lemma 1.13. *The equations* $[a\underline{x}]=\underline{b}=[\underline{y}a]$ *have solutions in* Q.

Proof. For given $a\in Q$ and $\underline{b}\in Q^N$, let $\underline{x}=[G_1(a)\underline{b}]$ and $\underline{y}=[\underline{b}G_1(a)]$. Then $[a\underline{x}]=[aG_1(a)\underline{b}]=[e^N\underline{b}]=\underline{b}$, by Lemmas 1.10, 1.11, and 1.12. Using (1.15), we have that $[G_1(a)a]=(e^N)$, and again by Lemmas 1.10, 1.11, and 1.12,

$$[\underline{y}a] = [\underline{b}G_1(a)a] = [\underline{b}e^N] = \underline{b}. \quad \square$$

Lemmas 1.10, 1.11, 1.12, and 1.13 imply that (Q,[ ]) is an (N+1,N)-group, with identity element e, and [ ]-inverse map $g=g_1$. This implies that $g_i=g^i$ for each $i\in M_N$, and $g^{N+1}=id_Q$.

The proof that (Q,[ ]) is generated as an (N+1,N)-group by X is the same as the similar proof in [3]. Here, we note that (Q,[ ]) as a semigroup is generated by $A_o=\{e\}\cup Y$, where X is identified by X^{-o}. Moreover, the map $\lambda:X \to Q$ defined by $\lambda(x)=x^{-o}$ is an injection.

In order to complete the proof of Theorem 1.9, we have to prove the following Lemma.

Lemma 1.14. *Let $(H,[\ ]')$ be an $(N+1,N)$-group with identity element e' and $[\ ]'$-inverse map $g':H \to H$. Let $f':X \to H$ be a given map. Then there exist a unique $(N+1,N)$-homomorphism $f:Q \to H$, such that $f\circ\lambda=f'$.*

Proof. Define a map $\bar{f}:A \to H$ by induction as follows: $\bar{f}(e)=e'$; $\bar{f}(x^{-o})=f'(x)$; $\bar{f}(x^{-i})=(g')^i(f'(x))$, $i\in M_i$, i.e. $\bar{f}(x^{-i})=\bar{f}(h^i(x^{-o}))=(g')^i(f'(x))$; and $\bar{f}(a,i)=[\bar{f}_1(a)]'_i$ for $(a,i)\in A$, where $\bar{f}_1(u_1^n)=\bar{f}(u_1)\dots\bar{f}(u_n)$. We have to show that $\bar{f}(\psi(a),i)=[\bar{f}_1(a)]'_i$ for each $a\in R$ and $i\in M_N$, where $[\bar{f}_1(u_1^N)]'_i$ stands for $\bar{f}(u_i)$. The proof is by induction on the length. Suppose that for each $a\in R_m=\{a \mid a\in R,\ |a|\le m\}$,

$$\bar{f}(\psi(a),i) = [\bar{f}_1(a)]'_i. \tag{1.21}$$

It follows directly from the definition of $\bar{f}$ that (1.21) holds for the initial value m=N. This assumption implies that for $(a,i)\in A\cap S_m$,

$$\bar{f}\psi(a,i) = [\bar{f}_1(a)]'_i \tag{1.22}$$

since $\psi(a,i)=(\psi(a),i)$.

Now, let $a\in R_{m+1}$, and $\psi(a)\ne a$.

(1) If $|\psi_1(a)| < |a|$, then $\bar{f}(\psi(a),i)=\bar{f}(\psi(\psi_1(a)),i)$ (by (1.9)) $= [\bar{f}_1\psi_1(a)]'_i$ (by (1.21)) $= [\bar{f}_1(a)]'_i$ (by (1.22)).

(2) If $\psi_1(a)=a$, and $a=be^Nc$, $\dim(bc)<N$, then $\bar{f}(\psi(a),i)= =\bar{f}(\psi(bce^N,i)$ (by (1.10)) $= \bar{f}(bce^N,i)$ (by (II.7)) since $\psi_1(a)=a) = [\bar{f}_1(b)(e')^N\bar{f}_1(c)]'_i$ (by the definition of $\bar{f}$) $= [\bar{f}_1(a)]'_i$.

(3) If $a=be^Nc$ with $\dim(bc)\ge N$, then $\bar{f}(\psi(a),i)=\bar{f}(\psi(bc),i)$ (by (1.10) and (1.11)) $= [\bar{f}_1(b)\bar{f}_1(c)]'_i$ (by (1.21)) $= [\bar{f}_1(b)(e')^N\bar{f}_1(c)]'_i = [\bar{f}_1(a)]'_i$.

(4) If $a=bu_1^Nc$ with $u_j=(v_1^p,j)$ for each $j\in M_N$, then $\bar{f}(\psi(a),i) = \bar{f}(\psi(bv_1^pc),i)$ (by (1.9) and (1.12)) $= [\bar{f}_1(b)\bar{f}_1(v_1^p)\bar{f}_1(c)]'_i$ (by (1.21)) $= [f(b)[\bar{f}_1(v_1^p)]'_1\dots[f_1(v_1^p)]'_N\bar{f}_1(c)]'_i = [\bar{f}_1(a)]'_i$ (by the definition of f).

(5) Let $a=byh(y)\dots h^N(y)c$ for some $y=x^{-i}\in Y$. Then $\overline{f}(\psi(a),i)=\overline{f}(\psi(bce^N),i)$ (by (1.9), (1.10) and (1.13)) $=$
$= [\overline{f}_1(b)(e')^N\overline{f}_1(c)]'_i$ (by (1.21)) $=$
$= [\overline{f}_1(b)(g')^i(f'(x))\dots(g')^N(f'(x))f'(x)g'(f'(x))\dots(g')^{i-1}$
$(f'(x)\overline{f}_1(i)]'_i$ (because $(g')^{N+1}=id_M$ and $h^j(x^{-i})=x^{-(i+j)}$) $=$
$= [\overline{f}_1(a)]'_i$.

(6) Let $a=bu^N_{r+1}cu^r_1d$, where $\psi_1(a)=a$, $u_j=(v^p_1,j)$ and $\psi(v^p_1c)=e^N$. Then $\overline{f}(e^N,i)=e'$ for each $i\in M_N$, and so $\overline{f}(\psi(v^p_1c),i)=[\overline{f}_1(v^p_1)\overline{f}_1(c)]'_i=$ $=e'$, i.e. $[\overline{f}_1(v^p_1)\overline{f}_1(c)]'=((e')^N)$. Now, Proposition 0.1 (ii) implies that

$$[[\overline{f}_1(v^p_1)]'_{r+1}\dots[\overline{f}_1(v^p_1)]'_N\overline{f}_1(c)[f_1(v^p_1)]'_1\dots[\overline{f}_1(v^p_1)]'_r] = ((e')^N),$$

i.e. $[\overline{f}_1(u^N_{r+1}cu^r_1)]'_i=e'$ for each $i\in M_N$. So, $\overline{f}(\psi(a),i)=\overline{f}(\psi(be^Nd),i)$ (by (1.9) and (1.14)) $= [\overline{f}_1(b)(e')^N\overline{f}_1(d)]'_i$ (by (1.21)) $=$
$= [\overline{f}_1(b)\overline{f}_1(u^N_{r+1}cu^r_1)\overline{f}_1(d)]'_i = [\overline{f}_1(a)]'_i$.

Hence for $\psi(a)\neq a$, $\overline{f}(\psi(a),i) = [\overline{f}_1(a)]'_i$.

Now let $\psi(a)=a$. If $\dim(a)\geq N+1$, then $(a,i)\in A$, and so $\overline{f}(\psi(a),i)=[\overline{f}_1(a)]'_i$ by the definition of f. If $\dim(a)=N$, then, since $\psi(a)=a$, (a,i) is only a notation for u_i, where $a=u^N_1$. Then $\overline{f}(\psi(a),i)=\overline{f}(u_i)$, and $[\overline{f}_1(a)]'_i$ is only a notation for $\overline{f}(u_i)$. Thus, we have completed the inductive step, i.e. we have proved that for each $a\in R$, $\overline{f}(\psi(a),i)=[\overline{f}\ (a)]'_i$. Let $f:Q\to H$ be defined by $f(u)=\overline{f}(u)$. Now, let $(u^{N+1}_1)\in Q^{N+1}$. Then $f([u^{N+1}_1]_i)=\overline{f}(\theta_i(u^{N+1}_1))=$ $=\overline{f}(\psi(u^{N+1}_1),i)=[\overline{f}_1(u^{N+1}_1)]'_i=[f_1(u^{N+1}_1)]'_i$, implies that the map f is an (N+1,N)-homomorphism. Moreover, from the definition of f it follows that $f\circ\lambda=f'$.

The fact that f is a unique (N+1,N)-homomorphism with this property, follows from the fact that Q as an (N+1,N)-group is generated by X. (End of Proof of Theorem 1.9.) □

<u>Corollary 1.15</u>. *The (N+1,N)-group (Q,[ ]) is nontrivial moreover the set Q is infinite.*

<u>Proof</u>. For example, $u_n=(u_{n-1}e^N,1)$, $n>1$, $u_1=(e^{N+1},1)$ is a sequence of distinct elements in Q . □

Corollary 1.16. *If the set X is empty, then (Q,[ ]) is an initial object in the category of (N+1,N)-groups. Moreover, Q is an infinite (countable) set.* □

§2. PROOF OF PROPOSITION 1.5.

(1) The right hand side in (I) is well defined, because $|u_j| < |a|$. Hence $\psi(a)$ is well defined. Moreover: $\psi(a)=a$ or $|\psi(a)| < |a|$, by (1.1); $\psi(\psi(a))=\psi(a)$ by (1.3); and $\dim(\psi(a))=\dim(a)$. Hence ψ satisfies (1.1) to (1.6) for $a \in S_{m+1}$ with $2 \le \dim(a) < N$.

(2) The right hand side in (II.2) is uniquely determined by the form of a.

(3) In (II.1) and (II.3) to (II.6) the right hand side is of the form $\psi(a')$ where $|a'| < |a|$, and so, it is well defined. Moreover:

(3.1) $|\psi(a)| = |\psi(a')| \le |a'| < |a|$ by (1.1).

Because $\dim(a') \ge N$, (1.4) implies that

(3.2) $\dim(\psi(a)) \ge N$.

(4) The above shows that $\psi(a)$ is well defined for each $a \in S_{m+1}$ with $\dim(a) \ge N$. Moreover, for such a's ψ satisfies: (1.1) - because of (2) and (3.1); (1.2) - because of (3.1) and (2); (1.3) - because of (2) and the fact that no step of (II.1) to (II.6) is applicable on $v_1^s e^N$ for $1 \le s < N$ and $\psi(v_j)=v_j$; (1.4) - because of (3.2) and (2), i.e. $\dim(a) \ge N$ and $\dim(\psi(a)) \ge N$; (1.5) - because of (1.3) i.e. for $\psi(a)=v_1^n$, $n \ge N+1$ and $\psi(v_1^n)=v_1^n$ it follows that v_1^n is not of the form $u_1^r e^N u_{r+1}^s$ with $u_{r+1} \ne e$; (1.6) - because no step of (II.1), (II.2) and (II.4) to (II.6) is applicable on u_1^N with $\psi(u_j)=u_j$.

(5) Now let $(a,i) \in A \cap S_{m+1}$. Then (3.2) implies that $\dim(\psi(a)) \ge N+1$ or $\dim(\psi(a))=N$. If $\dim(\psi(a)) \ge N+1$, then (1.5) from (4) implies that $(\psi(a),i) \in A$. If $\dim(\psi(a))=N$, then $\psi(a)=v_1^N$ is well defined, and so $(\psi(a),i)=v_i \in A$. For $(a,i) \in A$, $\psi(a) \ne a$ implies that $|\psi(a)| < |a|$ by (1.2) from (4). Hence $|\psi(a,i)| \le |(a,i)|$, i.e. ψ satisfies (1.1), and moreover, ψ satisfies (1.2).

Using (4) it follows that ψ satisfies (1.3) on $A \cap S_{m+1}$. The definition of ψ shows that ψ satisfies (1.4). Because $\dim((a,i))=1$, it is trivial that ψ satisfies (1.5) and (1.6) on $A \cap S_{m+1}$.

The proof of (A) and (B) from Proposition 1.5. follows, in fact is, (1) to (5). □

§3. PROOF OF THEOREM 1.6.

The fact that ψ satisfies (1.7) follows from Definition 1.4. (a). The only step applicable on e^N is (II.7). Hence ψ satisfies (1.8).

The rest of the proof is divided into several lemmas.

Lemma 3.1. *For $a \in S(A)$, $\psi(a)=\psi(\psi_1(a))$.*

Proof. If $\dim(a)=1$, then $\psi(\psi_1(a))=\psi(\psi(a))=\psi(a)$ by (1.3). If $2 \le \dim(a) < N$, then $\psi(a)=\psi_1(a)$ by (I), and again by (1.3), $\psi(a)=\psi(\psi_1(a))$. If $\dim(a) \ge N$, then: $\psi(a)=\psi(\psi_1(a))$ if $\psi_1(a)\neq a$ by (II.1); and $\psi(a)=\psi(\psi_1(a))$ if $\psi_1(a)=a$. □

Lemma 3.2. *For each $a = buc$, $b,c \in S^1(A)$, $u \in A$, $\psi(a) = \psi(b\psi(u)c)$.*

Proof. It follows from Lemma 3.1 and (1.3), because $\psi(b\psi(u)c)=\psi(\psi_1(b\psi(u)c))=\psi(\psi_1(buc))=\psi(buc)$. □

Lemma 3.3. *$\psi(ae^Nb) = \psi(abe^N)$ for $a,b \in S^1(A)$, and $1 \le \dim(ab) < N$.*

Proof. By induction on the length. If $\psi_1(ab)\neq ab$, then $\psi(ae^Nb)=\psi(\psi_1(a)e^N\psi_1(b))=\psi(\psi_1(ab)e^N)=\psi(abe^N)$. If $\psi_1(ab)=ab$ and $b=e^tv_1^p$, where $v_1\neq e$, then $\psi(ae^Nb)=ae^tv_1^pe^N=\psi(abe^N)$ by (II.2). □

Lemma 3.4. *Let $a=bu_1^Nc$, where $u_j=(z,j)$ for each $j \in M_N$, and $\psi(z)=z$. Then $\psi(a)=\psi(bzc)$.*

Proof. By induction on the length. If $\psi_1(bc)\neq bc$, then $\psi(bu_1^Nc)=\psi(\psi_1(b)u_1^N\psi_1(c))=\psi(\psi_1(b)z\psi_1(c))=\psi(bzc)$ by Lemma 3.2. If $\psi_1(bc)=bc$, then $\psi_1(a)=a$, and so, (II.1) is not applicable on a. Because $u_1^N\neq e^N$, (II.2) is not applicable on a. Since $u_j=(z,j)$, (II.3) is applicable on a. If b is of smallest such dimension, then $\psi(a)=\psi(bzc)$ by (II.3). If not, then $b=b'w_1^N d$, where $w_j=(x,j)\in A$, with b' of the smallest such dimension. So, $\psi(a)=\psi(b'xdu_1^N c)=\psi(b'xdzc)=\psi(bzc)$, because $|b'xdu_1^N c|<|a|$ and $|bzc|<|a|$. ||

Lemma 3.5. *Let* $a=be^N c$, $dim(bc)\geq N$. *Then* $\psi(a)=\psi(bc)$.

Proof. By induction on the lenght. If $\psi_1(bc)\neq bc$, then $\psi(a)=\psi(\psi_1(b)e^N\psi_1(c))=\psi(\psi_1(bc))=\psi(bc)$, because $|\psi_1(b)e^N\psi_1(c)|<|a|$. If $\psi_1(bc)=bc$, then (II.1) is not applicable on a, and since $\dim(a)\geq N$, (II.2) is not applicable on a. If (II.3) is applicable on a, then the conclusion follows from Lemma 3.4 and the inductive hypothesis. If (II.3) is not applicable on a, then (II.4) is applicable on a. If b is of the smallest such dimension, then $\psi(a)=\psi(bc)$ by (II.4). If not, then $b=b'e^N d$, for some $d\in S^1(A)$ or $b=b'e^t$ for some $0<t\leq N$, and with b' of the smallest such dimension. In both of these cases, the conclusion follows from (II.4) and the inductive hypothesis. ||

Lemma 3.6. $\psi(ae^N b)=\psi(abe^N)$ *for* $a,b\in S^1(A)$.

Proof. If $1\leq\dim(ab)<N$, then $\psi(ae^N b)=\psi(abe^N)$ by Lemma 3.3. If $\dim(ab)\geq N$, then $\psi(ae^N b)=\psi(ab)=\psi(abe^N)$ by Lemma 3.5. ||

Lemma 3.7. $\psi(au_1^{N+1}b)=\psi(abe^N)$ *for* $u_1=y\in Y$ *and* $u_{j+1}=h^j(y)$.

Proof. By induction on the lenght. If $\psi_1(ab)\neq ab$, the conclusion follows from Lemma 3.2 and the inductive hypothesis. Let $\psi_1(ab)=ab$. Then (II.1) and (II.2) are not applicable on $au_1^{N+1}b$. If (II.3) is applicable on $au_1^{N+1}b$, then $a=a'v_1^N a''$, or $b=b'v_1^N b''$ where $v_j=(z,j)\in A$. Then the conclusion follows from the inductive hypothesis and Lemma 3.4. If (II.4) is applicable on $au_1^{N+1}b$, then $a=a'e^N a''$ or $b=b'e^N b''$, and the inductive hypothesis and Lemma 3.5 imply that $\psi(au_1^{N+1}b)=\psi(abe^N)$. If (II.1) to (II.4) are not applicable on $au_1^{N+1}b$, then (II.5) is. If a is of the smallest such dimension, then $\psi(au_1^{N+1}b)=\psi(abe^N)$ by (II.5). If not, then $a=a'v_1^{N+1}a''$ or $au_1^t=a'v_1^{N+1}$ where $v_1\in Y$ and $v_{j+1}=h^j(v_1)$, with a' of smallest such dimension. In the first case the conclusion follows from the inductive hypothesis and (II.5); in the second one the conclusion follows from (II.5) and the fact that in this case $v_1=h^t(u_1)$. □

Lemma 3.8. *(a)* $\psi(abc) = \psi(a\psi(b)c)$;
(b) $\psi(ab) = e^N$ *implies* $\psi(ba) = e^N$;
(c) $\psi(zc) = e^N$ *implies* $\psi(au^N_{r+1}cu^r_1d) = \psi(ade^N)$ *for* $u_j = (z,j)\in A$ *for each* $j\in M_N$.

Proof. By induction on the lenght, for (a) of abc, for (b) of ab, and for (c) of $au^N_{r+1}cu^r_1d$. The proof is divided into several steps.

Step 1. $\psi(au) = \psi(\psi(a)u)$ *for* $a\in S(A)$, $u\in A$.

Proof. (1) If $\psi_1(au)\neq au$ then Lemma 3.2 and the inductive hypothesis imply that $\psi(au)=\psi(\psi_1(a)\psi_1(u))=\psi(\psi(\psi_1(a))\psi_1(u))= =\psi(\psi(a)\psi_1(u))=\psi(\psi(a)u)$.

(2) If $\psi(a)=a$, then $\psi(au)=\psi(\psi(a)u)$.

(3) Let $\psi_1(au)=au$, $\dim(a)\geq 2$, and $\psi(a)\neq a$. Then (I), (II.1) and (III) are not applicable on a.

(3.i) Let $a=v^r_1e^Nv^s_{r+1}$, $r+1\leq s<N$. Then $\psi(a)=v^s_1e^N$, and $\psi(\psi(a)u)=\psi(v^s_1e^Nu)$. If $s<N-1$, then by Lemma 3.3 $\psi(v^s_1e^Nu)= =\psi(v^s_1ue^N)=\psi(v^r_1e^Nv^N_{r+1}u)=\psi(au)$. If $s=N-1$ then $\psi(v^s_1e^Nu)=\psi(v^s_1u)$ by Lemma 3.5, and $\psi(au)=\psi(v^s_1u)$ again by Lemma 3.5.

(3.ii) Let $a=bu^N_1c$, $u_j=(z,j)$. Then Lemma 3.4 and the inductive hypothesis imply that $\psi(\psi(a)u)=\psi(\psi(bzc)u)=\psi(bzcu)=\psi(au)$.

(3.iii) Let $a=be^Nc$, $\dim(bc)\geq N$. Then by Lemma 3.5 and the inductive hypothesis $\psi(\psi(a)u)=\psi(\psi(bc)u)=\psi(bcu)=\psi(be^Ncu)=\psi(au)$.

(3.iv) Let $a=bu^{N+1}_1c$, $u_1=y\in Y$, $u_{j+1}=h^j(y)$. Then by Lemma 3.7 and the inductive hypothesis, $\psi(\psi(a)u)=\psi(\psi(bce^N)u)=\psi(bce^Nu)= =\psi(bcue^N)=\psi(bu^{N+1}_1cu)=\psi(au)$. Here we have used inductively (a) from Lemma 3.8 for bce^Nu, because $|bce^Nu|<|au|$.

(3.v) Let $a=bu^N_{r+1}cu^r_1d$ with $u_j=(z,j)$, $\psi(zc)=e^N$, b of the smallest such dimension, c of the smallest such dimension for this b, and (II.1) to (II.5) are not applicable on a. Then by (II.6) $\psi(a)=\psi(bde^N)$, and

$$\psi(\psi(a)u) = \psi(\psi(bde^N)u) = \psi(bde^Nu) = \psi(bdue^N)$$

by the inductive hypothesis and Lemma 3.3.

Now, we examine ua. It is obvious that (I), (II.1), (II.2), (II.7) and (III) are not applicable on au.

(3.v.1) Let (II.3) be applicable on au. Since (II.3) is not applicable on $a=bu_{r+1}^{N}cu_1^r d$, it follows that (II.3) is applicable on du or on $u_1^r du$. In the first case, let $du=d'w_1^N$, with $w_j=(x,j)\in A$. Then by Lemma 3.4 and the inductive hypothesis, $\psi(au)= =\psi(bu_{r+1}^N cu_1^r d'x)=\psi(bd'xe^N)=\psi(bd'w_1^N e^N)=\psi(bdue^N)=\psi(\psi(a)u)$. In the second case, let $u_1^r du=u_1^N$. Hence, in this case $u=u_N$. Then $\psi(au)= =\psi(bu_{r+1}^N cz)$ (by Lemma 3.4) $=\psi(bu_{r+1}^N e)$ (by Lemma 3.8 (a) and (b), since $\psi(zc)=e^N$, and $|bu_{r+1}^N cz|<|au|) = \psi(bdue^N)$.

(3.v.2) Let (II.4) be applicable on au. Again, since (II.4) is not applicable on a, it follows that $du=d'e^N$. Then $\psi(bu_{r+1}^N cu_1^r du)=\psi(bu_{r+1}^N cu_1^r d')=\psi(bd'e^N)=\psi(bd'e^N e^N)=\psi(bdue^N)$, by Lemma 3.8 (c) inductively, and Lemma 3.5.

(3.v.3) Let (II.5) be applicable on au. Again, since (II.5) is not applicable on a, it follows that $du=d'yh(y)\ldots h^N(y)$. Then $\psi(bu_{r+1}^N cu_1^r du)=\psi(bu_{r+1}^N cu_1^r d'e^N)=\psi(bd'e^N e^N)=\psi(bd'e^N)=\psi(bdu)=\psi(bdue^N)$ by Lemma 3.8 (c) inductively, Lemma 3.7 and Lemma 3.5.

(3.v.4) If (II.3), (II.4) and (II.5) are not applicable on au, then (II.6) is applicable on au, and the assumptions in (3.v) imply that $\psi(au)=\psi(bdue^N)$ by (II.6).

The above discussion shows that $\psi(au)=\psi(\psi(a)u)$, i.e. we have completed the inductive step for Step 1.

<u>Step 2</u>. $\psi(ua) = \psi(u\psi(a))$.

<u>Proof</u>. (i) If $\psi_1(ua)\neq ua$, then the conclusion follows from the inductive hypothesis and Lemma 3.2.

(ii) If $\psi(a)=a$ then $\psi(ua)=\psi(u\psi(a))$. If $\dim(a)=1$, then $\psi(ua)=\psi(u\psi(a))$ by Lemma 3.2.

(iii) Let $\psi_1(ua)=ua$, $\dim(a)\geq 2$, and $\psi(a)\neq a$. Then (I), (II.1), (II.7) and (III) are not applicable on a.

(iii.1) Let (II.2) be applicable on a, i.e. $a=v_1^r e^N v_{r+1}^s$, $r+1\leq s<N$, $v_{r+1}\neq e$. Then $\psi(a)=v_1^s e^N$, and $\psi(u\psi(a))=\psi(uv_1^s e^N)$. The

conclusion follows from Lemma 3.3. in the case $s < N-1$, and from Lemma 3.5 in the case $s=N-1$.

(iii.2) Let (II.3) be applicable on a, i.e. $a=bu_1^N c$, $u_j=(z,j)$, $j\in M_N$. Then by Lemma 3.4 and the inductive hypothesis $\psi(u\psi(a))=\psi(u\psi(bzc))=\psi(ubzc)=\psi(ua)$.

(iii.3) Let (II.4) be applicable on a, i.e. $a=be^N c$, $\dim(bc)\geq N$. Then by Lemma 3.5 and the inductive hypothesis $\psi(u\psi(a))=\psi(u\psi(bc))=\psi(ubc)=\psi(ube^N c)$.

(iii.4) Let (II.5) be applicable on a, i.e. $a=byh(y)\dots h^N(y)c$, $y\in Y$. Then by Lemma 3.7, and the inductive hypothesis $\psi(u\psi(a))=$ $=\psi(u\psi(bce^N)=\psi(ubce^N)=\psi(ua)$.

(iii.5) Let (II.6) be applicable on a, i.e. $a=bu_{r+1}^N cu_1^r d$, with $u_j=(z,j)\in A$, $\psi(zc)=e^N$ and b,c are of the smallest such dimensions. If $\dim(d)\geq 1$, then by Step 1, $\psi(ua)=\psi(\psi(ubu_{r+1}^N cu_1^r d')v)$ (where $d=d'v$, $v\in A)=\psi(\psi(ubd'e^N)v)=\psi(u\psi(bu_{r+1}^N cu_1^r d))=\psi(u\psi(a))$. So, let $a=bu_{r+1}^N cu_1^r$. It is obvious that (I), (II.1), (II.7) and (III) are not applicable on ua, since we have already assumed that (II.1) to (II.5) are not applicable on a, and $\psi_1(ua)=ua$.

(iii.5.1) Let (II.2) be applicable on ua. Then $ub=e^N b'$, and so, by Lemma 3.4 and the inductive hypothesis $\psi(ua)=$ $=\psi(b'u_{r+1}^N cu_1^r)=\psi(b'e^N)=\psi(e^N b'e^N)=\psi(ube^N)=\psi(u\psi(a))$.

(iii.5.2) Let (II.3) be applicable on ua. Then $ub=w_1^N b'$ with $w_j=(x,j)\in A$ or $ub=u_1^r$. In the first case Lemma 3.4 and the inductive hypothesis imply that $\psi(ua)=\psi(xb'u_{r+1}^N cu_1^r)=$ $=\psi(xb'e^N)=\psi(w_1^N b'e^N)=\psi(ube^N)=\psi(u\psi(be^N))=\psi(u\psi(a))$. In the second case, Lemma 3.4 and the inductive hypothesis imply that $\psi(ua)=\psi(zcu_1^r)=\psi(\psi(zc)u_1^r)=\psi(e^N u_1^r)=\psi(u_1^r e^N)=\psi(ube^N)=\psi(u\psi(be^N)=$ $=\psi(u\psi(a))$. (In this case b=1.)

(iii.5.3) Let (II.4) be applicable on ua. Then the only possibility (since (II.4) is not applicable on a) is $ub=e^N b'$. So, by Lemma 3.5 and the inductive hypothesis $\psi(ua)=\psi(b'u_{r+1}^N cu_1^r)=$ $=\psi(b'e^N)=\psi(e^N b'e^N)=\psi(ube^N)=\psi(u\psi(be^N))=\psi(u\psi(a))$.

(iii.5.4) Let (II.5) be applicable on ua. Then $ub=yh(y)\dots h^N(y)b'$, for $y\in Y$. So, by Lemma 3.7 and the inductive hypothesis $\psi(ua)=\psi(b'u^N_{r+1}cu^r_1e^N)=\psi(b'e^Ne^N)=\psi(e^Nb'e^N)=\psi(ub)= =\psi(ube^N)=\psi(u\psi(be^N))=\psi(u\psi(a))$.

(iii.5.5) If (II.1) to (II.5) are not applicable on ua, then (II.6) is applicable on ua. If ub is of the smallest such dimension, then by (II.6) $\psi(ua)=\psi(ube^N)=\psi(u\psi(be^N))=\psi(u\psi(a))$. If not, then $ua=v^N_{t+1}gv^t_1a'$, where $v_j=(x,j)\in A$, $\psi(xg)=e^N$, and g is of the smallest such dimension. Now, there are only two possibilities for v^N_{t+1}.

(a) $v^N_{t+1} = ub'$ where $b = b'b"$; and

(b) $v^N_{t+1} = ubu^N_{r+1}$. In this case $u_j = v_j$, and $b = 1$.

In the case (a) there are four possibilities for b'.

(a.1) $b = b'gv^t_1b'''$, i.e. $b"=gv^t_1b'''$;

(a.2) $b" = gv^r_1$, $v_{r+1} = u_{r+1}$, $t > r$;

(a.3) $g = b"u^N_{r+1}c'$, where $c = c'v^t_1c"$; and

(a.4) $g = b"u^N_{r+1}c$, $t \le r$.

In the case (b) there are two possibilities:

(b.1) $c = gv^t_1c"$; and

(b.2) $c = g$, $t < r$.

In the case (a.1) the inductive hypothesis and (II.6) imply that $\psi(ua)=\psi(v^N_{t+1}gv^t_1b'''u^N_{r+1}cu^r_1)=\psi(b'''u^N_{r+1}cu^r_1e^N)=\psi(b'''e^Ne^N)= =\psi(v^N_{t+1}gv^t_1b'''e^N)=\psi(u\psi(a))$.

In the case (a.2) $v_j=u_j$ for each j, and so (II.3) is applicable on a which contradicts our assumption.

In the case (a.3) $ua=v^N_{t+1}b"u^N_{r+1}c'v^t_1c"u^r_1$. Then $\psi(ua)= =\psi(c"u^r_1e^N)$ (by (II.6)) $= \psi(c"u^r_1\psi(u^N_{r+1}c'xb")$ (because $\psi(xb"u^N_{r+1}c')=e^N$ and using inductively Lemma 3.8 (a) and (b)) $=$ $\psi(c"u^r_1u^N_{r+1}c'xb")$ (by the inductive hypothesis) $= \psi(c"zc'xb")$ (by Lemma 3.4) $= \psi(c"zc'v^N_1b")$ (by Lemma 3.4) $=$ $\psi(\psi(c"zc'v^t_1)v^N_{t+1}b")$ (by the inductive hypothesis) $= \psi(e^Nv^N_{t+1}b")$ (by Lemma 3.8 (b) inductively since $|c"zc'v^t_1| < |ua|$) $=\psi(v^N_{t+1}b"e^N)= =\psi(v_{t+1}\psi(v^N_{t+2}b"e^N))=\psi(v_{t+1}\psi(a))=\psi(u\psi(a))$.

In the case (a.4) $u_j=v_j$, and $ua=u^N_{t+1}b"u^N_{r+1}cu^r_1$. Then $\psi(ua)=\psi(u^r_{t+1}e^N)=\psi(u^r_{t+1}\psi(u^N_{r+1}b"e^N))=\psi(u^N_{t+1}b"e^N)=\psi(u_{t+1}be^N)=$

$=\psi(u_{t+1}\psi(a))=\psi(u\psi(a))$, because $e^N=\psi(zb"u^N_{r+1}c)=\psi(u^N_{r+1}czb")=$ $=\psi(u^N_{r+1}b"e^N)=\psi(b"e^Nu^N_{r+1})=\psi(b"u^N_{r+1}e^N)$, using the fact that $|zb"u^N_{r+1}c|<|a|$.

In the case (b.1) $ua=u^N_{t+1}c'u^t_1c"u^{t+1}_1$. Then $\psi(zc')=e^N=$ $=\psi(zc'u^t_1c")$ implies that $\psi(u^t_1c"e^N)=e^N$, and by Lemma 3.8 (b) indictively, $\psi(c"u^t_1e^N)=e^N$. Then $\psi(ua)=\psi(c"u^{t+1}_1e^N)=\psi(u_{t+1}e^N)=$ $=\psi(u\psi(a))$.

In the case (b.2) $ua=u^N_{t+1}cu^{t+1}_1$. Then $\psi(ua)=\psi(u_{t+1}e^N)=$ $=\psi(u_{t+1}\psi(a))=\psi(u\psi(a))$ by (II.6).

Thus we have proved Step 2.

Step 3. $\psi(abc)=\psi(a\psi(b)c)$.

Proof. If $c=c'u$, $u\in A$, then by Step 1 $\psi(abc)=\psi(\psi(abc')u)=$ $=\psi(\psi(a\psi(b)c')u)=\psi(a\psi(b)c'u)=\psi(a\psi(b)c)$. If $a=ua'$, $u\in A$, then by Step 2 $\psi(abc)=\psi(u\psi(a'bc))=\psi(u\psi(a'\psi(b)c))=\psi(ua'\psi(b)c)=\psi(a\psi(b)c)$. If $a=1=c$, then $\psi(b)=\psi(\psi(b))$ by (1.3).

With Steps 1 to 3 we have just completed the inductive step for (a) from Lemma 3.8.

Step 4. Proof of the inductive step for Lemma 3.8 (b).

Let $\psi(ub)=e^N$, where $u\in A$. We are going to prove that $\psi(bu)=e^N$. Several applications of this fact will imply the inductive step for Lemma 3.8 (b).

If $|\psi(b)|<|b|$, then $e^N=\psi(ub)=\psi(u\psi(b))=\psi(\psi(b)u)=\psi(bu)$. If $|\psi(u)|<|u|$, then $e^N=\psi(ub)=\psi(\psi(u)b)=\psi(b\psi(u))=\psi(bu)$. Let $|\psi(b)|=|b|$ and $\psi(b)\neq b$. Then $b=v^r_1e^Nv^s_{r+1}$, $r+1\le s<N$, and $\psi(b)=v^s_1e^N$. So, $e^N=\psi(ub)=\psi(u\psi(b))=\psi(uv^s_1e^N)$. This implies that $s=N-1$ and $\psi(uv^s_1)=e^N$. But then $e^N=\psi(v^s_1u)$ by the inductive hypothesis, and so, $e^N=\psi(v^s_1u)=\psi(v^s_1e^Nu)$ (by Lemma 3.5) = $=\psi(\psi(v^s_1e^N)u)=\psi(\psi(b)u)=\psi(bu)$.

Now let $\psi(b)=b=b'v$, $v\in A$, and $\psi(u)=u$.

If $|\psi(ub')|=|ub'|$ and $\psi(ub')\neq ub'$, then $ub'=u^r_1e^Nu^s_{r+1}$, $r+1\le s<N$, and $e^N=\psi(ub)=\psi(u^s_1e^Nv)$ implies that $s=N-1$ and $\psi(u^s_1v)=e^N=\psi(u^s_2vu_1)=\psi(u^s_2e^Nvu_1)=\psi(b'vu)=\psi(bu)$.

Next, we assume that $\psi(b)=b=b'v$, $|\psi(ub')|<|ub'|$ and $\psi(u)=u$. Let b=cd, where c is of the smallest dimension with $|\psi(uc)|<|uc|$. It follows that $1\leq \dim(c)\leq \dim(b')$. From $\psi(u)=u$, $\psi(b)=b$, it follows that $\psi_1(uc)=uc$, and so (I), (II.1), (II.2), (II.7) and (III) are not applicavle on uc.

(1) Let (II.3) be applicable on uc. Then $uc=u_1^N$ where $u_j=(z,j)$. So, $e^N=\psi(ucd)=\psi(zd)$. Now we examine $cdu=u_2^N du_1$. The assumptions imply that (I), (II.1) and (III) are not applicable on cdu. Furthermore, (II.2) is not applicable on $u_2^N du_1$, because $\dim(u_2^N du_1)\geq N$. If (II.3) is applicable on $u_2^N du_1$, then it must be applicable on d, but then $\psi(b)\neq b$. If (II.4) is applicable on $u_2^N du_1$, then it must be $d=e^N$, i.e. $e^N=\psi(ze^N)=\psi(z)$, but then $\psi(u)\neq u$. If (II.5) is applicable on $u_2^N du_1$, then $d=d'yh(y)\dots h^N(y)d''$, but then $\psi(b)\neq b$. Because $\psi(zd)=e^N$, it follows that (II.6) is applicable on $u_2^N du_1$. For $u_2^N du_1$, the b of the smallest such dimension in (II.6) is the empty word. Now, let $u_2^N du_1=u_2^N d'u_1d''u_1$, where d' is of the smallest such dimension. Then $\psi(b)=\psi(u_2^N d)=$ $=\psi(u_2^N d'ud'')\neq u_2^N d=b$. Hence d is of the smallest such dimension, and $\psi(u_2^N du_1)=e^N$ by (II.6).

(2) Let (II.4) be applicable on uc. Then it must be: (a) $uc=uc'e^N$, $\dim(c')=N-1$; or (b) $uc=e^{2N}$, d=1; or (c) $uc=e^N c'$, $\dim(c')\geq N$. In the case (a) $e^N=\psi(ucd)=\psi(uc'e^N d)=\psi(uc'd)=\psi(c'du)=$ $=\psi(c'e^N du)=\psi(bu)$. In the case (b) $uc=ub=e^N$, and so $\psi(bu)=e^N$. In the case (c) $e^N=\psi(ucd)=\psi(e^N c'd)=\psi(c'd)$ implies by Step 3 that $\psi(cdu)=\psi(e^{N-1}\psi(c'd)e)=\psi(e^{2N})=e^N$.

(3) Let (II.5) be applicable on uc. Then $u=y\in Y$ and $c=h(y)\dots h^N(y)$. Then $\psi(ucd)=\psi(yh(y)\dots h^N(y)d)=\psi(e^N d)=\psi(de^N)=e^N$. If d=1 then $\psi(bu)=\psi(cu)=\psi(h(y)\dots h^N(y)y)=e^N$. If $d\neq 1$ then $\psi(d)=e^N$, and so $\psi(bu)=\psi(h(y)\dots h^N(y)dy)=\psi(h(y)\dots h^N(y)\psi(d)y)=$ $=\psi(h(y)\dots h^N(y)e^N y)=\psi(h(y)\dots h^N(y)y)=e^N$.

(4) Let (II.6) be applicable on uc. Then $uc=u_{r+1}^N c'u_1^r$ with c' of the smallest such dimension, $u_j=(z,j)$ and $\psi(zc')=e^N$. Moreover, since (II.3) is not applicable on uc it follows that $r\geq 1$. Then $\psi(uc)=e^N$ by (II.6). Now, $e^N=\psi(ucd)=\psi(e^N d)=\psi(de^N)$ implies that $\dim(d)\geq N$ and $\psi(d)=e^N$, or d=1. Let $\dim(d)\geq N$. If $d\neq e^N$, then

$\psi(b)\neq b$. Hence $d=e^N$, and so $\psi(bu)=\psi(cdu)=\psi(ce^Nu)=\psi(cu)=e^N$. Now, let d=1. Then $ub=u_{r+1}^N c'u_1^r$ and $bu=u_{r+2}^N c'u_1^{r+1}$. If $r+1\neq N$, then $\psi(bu)=e^N$ by (II.6), because, of our assumptions, (II.1) to (II.5) are not applicable on bu. If r+1=N, then by Lemma 3.4 and the inductive hypothesis ($\psi(zc')=e^N=\psi(c'z)$ because $|c'z|<|ub|$), $\psi(bu)=\psi(c'u_1^N)=\psi(c'z)=e^N$.

Hence for $|\psi(ub'|<|ub'|$ we have proved that $\psi(bu)=e^N$.

Now let $\psi(ub')=ub'$. If $ub=e^{2N}$, then $\psi(bu)=e^N$. So let $ub\neq e^{2N}$. Then, the assumptions imply that (I), (II.1), (II.7) and (III) are not applicable on ub. Since $\psi(ub)=e^N$, (II.2) is not applicable on ub. If (II.3) is applicable on ub, then $ub=u_1^N$ for $u_j=(z,j)$ and $\psi(z)=z$; hence $\psi(ub)\neq e^N$. If (II.4) is applicable on ub, then $\psi(ub')\neq ub'$ or $\psi(b)\neq b$. If (II.5) is applicable on ub, then $ub=yh(y)\dots h^N(y)$, and then $\psi(bu)=\psi(h(y)\dots h^N(y)y)=$ $=e^N$. If (II.6) is applicable on ub, then $ub=u_{r+1}^N c'u_1^r$ with $u_j=(z,j)$ and $\psi(zc)=e^N$. Then, if $r\neq N-1$, no step of (II.1) to (II.5) is applicable on $u_{r+2}^N c'u_1^{r+1}$, and by (II.6) $\psi(u_{r+2}^N c'u_n^{r+1})=$ $=\psi(e^N)=e^N$, i.e. $\psi(bu)=e^N$, since c' is of the smallest such dimension. If r=N-1, then by Lemma 3.4 and the inductive hypothesis $\psi(bu)=\psi(c'u_1^N)=\psi(c'z)=\psi(zc')=e^N$.

Thus we have completed Step 4, i.e. $\psi(ub)=e^N$ implies $\psi(bu)=e$, which in turns implies that if $\psi(ab)=e^N$ then $\psi(ba)=e^N$.

<u>Step 5</u>. *$\psi(zc)=e^N$ implies that $\psi(au_{r+1}^N cu_1^r d)=\psi(ade^N)$ for $u_j=(z,j)$, $r\geq 1$.*

<u>Proof</u>. If $ad\neq 1$, then using Lemma 3.8 and the above steps inductively, we have $\psi(au_{r+1}^N cu_1^r d)=\psi(a\psi(u_{r+1}^N cu_1^r)d)=\psi(ae^N d)=\psi(ade^N)$. If a=1=d, then $e^N=\psi(zc)$. If $\psi(z)=z$, then by Lemma 3.4 and Step 5, $e^N=\psi(zc)=\psi(u_1^N c)=\psi(u_{r+1}^N cu_1^r)$. Now, let $\psi(z)=w_1^N$. Then $\psi(u_j)=w_j$, and $e^N=\psi(zc)=\psi(\psi(z)c)=\psi(w_1^N c)=\psi(\psi_1(u_1^N)c)=\psi(u_1^N c)=\psi(u_{r+1}^N cu_1^r)$. The last case is when $\psi(z)=w_1^t$, $t\geq N+1$. Then $\psi(u_j)=(w_1^t,j)$, and $e^N=\psi(zc)=\psi(\psi(z)c)=\psi(w_1^t c)=\psi(\psi(\psi_1(u_1^N))c)=\psi(u_1^N c)=\psi(u_{r+1}^N cu_1^r)$.

In the above steps the show that if ψ satisfies Lemma 3.8 for words with lenght less than or equal to m, then ψ satisfies

Lemma 3.8 for words with lenght equal to m+1. Hence, by induction, ψ satisfies Lemma 3.8, i.e. we have proved Lemma 3.8. ||

Lemma 3.9. $\psi(au_1^N b) = \psi(azb)$ *for* $u_j=(z,j)$.

Proof. $\psi(au_1^N b)=\psi(a\psi(u_1^N)b)$ (by Lemma 3.7) = $= \psi(a\psi(\psi_1(u_1^N))b)$ (by Lemma 3.1) = $\psi(a\psi_1(u_1^N)b)$ = $= \psi(aw_1^t b)$ (where $\psi(z)=w_1^t$)$=\psi(a\psi(z)b)=\psi(azb)$. ||

Proof of Theorem 1.6. (1.9) is Lemma 3.8 (a); (1.10) is Lemma 3.6; (1.11) is Lemma 3.5; (1.12) is Lemma 3.9 for a=b=1; (1.13) is Lemma 3.7 for a=b=1; (1.14) is Lemma 3.8 (c); (1.15) is Lemma 3.8 (b); and (1.16) follows from the definition of ψ. ||

R E F E R E N C E S

[1] G.Čupona; Vector Valued semigroups, Semigroup Forum, Vol. 26 (1983), 65-74.

[2] D.Dimovski; Some existence conditions for vector valued groups, God. Zbor. na Mat. fak., 33-34 (1982-83), 99-103.

[3] D.Dimovski; On (3,2)-groups, Proc. of the Conf. Algebra and Logic, Cetinje 1986, 63-71.