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SOME CHARACTERIZATIONS OF n-BANDS

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Abstract. An idempotent n-semigroup is called an n-band. Some well-known properties of bands (i.e. of 2-bands) are generalized in this paper.

0. First we will state the necessary preliminary definitions and results.

If $(x_1,\dots,x_n) \mapsto x_1x_2\dots x_n$ is an associative n-ary operation on a set S, then we say that S is an n-semigroup. A subset A of S is called an i-ideal of S iff $S^{i-1}AS^{n-i} \subseteq A$ (as usual, $S^0AS^{n-1}=AS^n$, $S^{n-1}AS^0=S^{n-1}A$); if $i=n$, then A is called a left ideal, and if $i=1$ - a right ideal; A is a two-sided ideal iff it is a left and right ideal; A is an ideal iff it is an i-ideal for every $i\in\{1,2,\dots,n\}$. An n-semigroup S is said to be two sided simple (left-simple) if it has no proper two sided ideal (left ideal). An ideal A is said to be completely prime iff: $x_1x_2\dots x_n\in A \iff (\exists_i)x_i\in A$. A filter of S is any subset $B\subseteq S$ which complement in S is a completely prime ideal. If x is a given element of S, then $N(x)$ will denote the intersection of all filters of S which contain x.

An n-semigroup S is called an n-band iff it is idempotent, i.e. iff the identity $x^n=x$ holds in S. If, in addition, S is commutative and for every $i_\nu, j_\nu > 0$ such that

$$i_1 + i_2 +\dots+ i_k = j_1 + j_2 +\dots+ j_k = n,$$

the identity

$$x_1^{i_1} x_2^{i_2} \dots x_k^{i_k} = x_1^{j_1} x_2^{j_2} \dots x_k^{j_k}$$

holds in S, then we say that S is an n-semilattice. A congruence α in S is called a semilattice congruence iff S/α is an n-semilattice.

We will formulate some results proved in [5] and [6]. Throughout the paper, S will denote a given n-semigroup if it is not said otherwise.

This paper is in final-form and no version of it will be submited for publication elsewher.

0.1. ([5] 3.2) The relation η defined in S by

$$x\eta y \iff N(x) = N(y)$$

is the minimal semilattice congruence on S.

(If $x\in S$, then the η-class which contains x will be denoted by N_x.)

0.2. ([6] 2.2) If $x\in S^{n-1}xS^{n-1}$ for every $x\in S$, then

$$N_x = \{y\in S \mid x\in S^{n-1}yS^{n-1},\ y\in S^{n-1}xS^{n-1}\},$$

also for every $x\in S$.

0.3. ([5] 2.1) The following conditions on an n-semigroup S are equivalent.

(i) Every η-class of S is a left simple n-semigroup.

(ii) For every $x\in S$, $x\in S^{n-1}x^n$ and $xS^{n-1}\subseteq S^{n-1}x$.

(iii) For every $x\in S$, $N_x=\{y\in S \mid x\in S^{n-1}y,\ y\in S^{n-1}x\}$.

1. Now we will give a characterization of n-bands by means of the η-classes.

Proposition 1. An n-semigroup S is an n-band iff for every $x\in S$ the following equality holds:

$$N_x = \{y\in S \mid x=(xy^{n-1})^{n-1}x,\quad y=(yx^{n-1})^{n-1}y\}. \qquad (1)$$

Proof. Let S be an n-band. Then

$$x = x^n = x^{n-1}x^n x^{n-1}\in S^{n-1}x^n S^{n-1},$$

and so, by 0.2., we have

$$N_x = \{y\in S \mid x\in S^{n-1}yS^{n-1},\ y\in S^{n-1}xS^{n-1}\}.$$

By this it follows that, if $x\in S$ and $y\in N_x$, then there exist $a_1,\dots,a_{n-1}$, $b_1,\dots,b_{n-1}$ such that

$$x = a_1\dots a_{n-1}yb_1\dots b_{n-1}.$$

Therefore:

$$x = xx^{n-1} = a_1 \dots a_{n-1} y b_1 \dots b_{n-1} x^{n-1}$$
$$= a_1 \dots a_{n-1} (y b_1 b_2 \dots b_{n-1} x^{n-1}) =$$
$$= a_1 \dots a_{n-1} (y b_1 b_2 \dots b_{n-1} x^{n-1})^n =$$
$$= (a_1 a_2 \dots a_{n-1} y b_1 b_2 \dots b_{n-1}) x^{n-1} (y\, b_1 b_2 \dots b_{n-1} x^{n-1}) \dots (y b_1 b_2 \dots b_{n-1} x^{n\cdot}$$
$$= x y^{n-1} (y b_1 b_2 \dots b_{n-1} x^{n-1})^{n-1} =$$
$$= (x y^{n-1})^n (y b_1 b_2 \dots b_{n-1} x^{n-1})^{n-1} =$$
$$= (x y^{n-1})^{n-1} x y^{n-1} (y b_1 b_2 \dots b_{n-1} x^{n-1})^{n-1} =$$
$$= (x y^{n-1})^{n-1} x.$$

By symmetry: $y=(yx^{n-1})^{n-1}y$ *and so (1) is proved.*

Conversely, suppose that S is an n-semigroup in which (1) holds. Then, if $x \in S$, *we have* x, $x^n \in N_x$ *and:*

$$x = (xx^{n-1})^{n-1} x = x^{n(n-1)+1}$$
$$x^n = (x^n x^{n-1})^{n-1} x^n = x^{(2n-1)(n-1)+n} =$$
$$= x^{n(n-1)+1} x^{n(n-1)} = x,$$

i.e. S is an n-band. ||

2. It is well known (see for example ([3] p. 45) that a band is a two-sided simple iff it is a rectangular band, i.e. iff it satisfies the identity $xyx=x$. *We give a description of two-sided simple n-bands, for any n, in the following proposition.*

Proposition 2. An n-semigroup S is a two-sided simple n-band iff it satisfies the identity

$$(xy^{n-1})^{n-1} x = x. \qquad (2)$$

Proof. Let S be a two-sided simple n-band and let $x,y \in S$. *Then* $S^{n-1} y S^{n-1} = S$, *by which it follows that there exist* $a_1, \dots, a_{n-1}$, $b_1, \dots, b_{n-1} \in S$ *such that*

$$x = a_1 \dots a_{n-1} y b_1 \dots b_{n-1}.$$

Using this, we have that:

$$x = a_1 \dots a_{n-1} y b_1 \dots b_{n-1} x^{n-1}$$
$$= a_1 \dots a_{n-1} (y b_1 \dots b_{n-1} x^{n-1})^n$$
$$= a_1 \dots a_{n-1} y b_1 \dots b_{n-1} x^{n-1} (y b_1 \dots b_{n-1} x^{n-1})^{n-1}$$
$$= x x^{n-1} (y^n b_1 \dots b_{n-1} x^{n-1})^{n-1}$$
$$= x y^{n-1} (y b_1 \dots b_{n-1} x^{n-1})^{n-1}$$
$$= (x y^{n-1})^{n-1} x y^{n-1} (y b_1 \dots b_{n-1} x^{n-1})^{n-1}$$
$$= (x y^{n-1})^{n-1} x,$$

i.e. that the identity (2) is true.

Conversely, if an n-semigroup S satisfies the identity (2), then by Proposition 1, S is an n-band. Beside this, for all $x,y \in S$ we have

$$y = (yx^{n-1})^{n-1} y \in S^{n-1} x S^{n-1},$$

i.e. $S=S^{n-1}xS^{n-1}$, by which it follows that S is two-sided simple. ||

3. We will give here a description of one more class of n-bands.

Proposition 3. *If S is an n-band, then the following statements are equivalent*:

(i) *The identity $xy^{n-1}=x$ holds in every η-class of S.*

(ii) *For every $x \in S$, $xS^{n-1} \subseteq S^{n-1}x$.*

(iii) *For every $x \in S$, $N(x)=\{y \in S \mid xy^{n-1}=x\}$.*

(iv) *S satisfies the identity*

$$xy^{n-1} = (xy^{n-1})^{n-1} x.$$

Proof. (i) ⟹ (ii). It is clear that if an n-semigroup satisfies the identity $xy^{n-1}=x$ then it is left-simple. Thus, by 0.3., one obtains that (i) implies (ii).

(ii) ⟹ (iii). Let $xS^{n-1} \subseteq S^{n-1}x$ for every $x \in S$. Since $x=x^n=x^{n-1}x=x^{n-1}x^n \in S^{n-1}x^n$, by 0.3 we obtain that

$$N_x = \{y \in S \mid x \in S^{n-1}y,\ y \in S^{n-1}x\}.$$

By this, it follows that if $y\in N_x$, *then there exist* $a_1,\dots,a_{n-1}\in S$, *such that* $x=a_1a_2\dots a_{n-1}y$. *Therefore*

$$x = x^{n-1}a_1\dots a_{n-1}y=x^{n-1}a_1\dots a_{n-1}y^n=xy^{n-1}.$$

We will prove now that the set $T=\{y\in S \mid x\in S^{n-1}y\}$ *is a filter.*

Let $u_1u_2\dots u_n\in T$. *Then*

$$x\in S^{n-1}u_1u_2\dots u_n = S^{n-1}u_1u_2\dots u_i^{n-i+1}u_i^{i-1}\dots u_n\subseteq S^{n-1}u_1u_2\dots u_{i-1}u_i^{n-i+1}S^{n-1}$$

$$\subseteq S^{n-1}u_1u_2\dots u_i^{n-i+1}\subseteq S^{n-1}u_i.$$

Therefore $u_i\in T$ *for* $i=1,2,\dots,n$.

Conversely, let $u_1,u_2,\dots,u_n\in T$. *Then* $x\in S^{n-1}u_1$, $x\in S^{n-1}u_2,\dots,x\in S^{n-1}u_n$ *and so, using (ii),*

$$x\in S^{n-1}x^n\subseteq S^{n-1}S^{n-1}u_1S^{n-1}u_2\dots S^{n-1}u_n\subseteq$$

$$\subseteq S^{n-1}u_1S^{n-1}u_2\dots S^{n-1}u_{n-1}u_n\subseteq\dots\subseteq$$

$$\subseteq S^{n-1}u_1u_2\dots u_n.$$

Therefore $u_1u_2\dots u_n\in T$, *and since* $x=x^n\in S^{n-1}x$, *it follows that* $N(x)\subseteq T$.

Let $y\in T$. *Then* $x=a_1a_2\dots a_{n-1}y\in N(x)$, *and since* $N(x)$ *is a filter, it follows that* $y\in N(x)$ *which means that* $T\subseteq N(x)$. *Hence* $T=N(x)$, *i.e.* $N(x)=\{y\in S \mid x\in S^{n-1}y\}$.

But, $x\in S^{n-1}y$ *iff* $x=xy^{n-1}$, *and therefore we have* $N(x)=\{y\in S \mid x=xy^{n-1}\}$.

(iii) $\Longrightarrow$ *(iv). The set* $N(xy^{n-1})$ *is a filter of S. By the assumption,*

$$N(xy^{n-1}) = \{z\in S \mid xy^{n-1} = xy^{n-1}z^{n-1}\}.$$

Since $x,y\in N(xy^{n-1})$, *it follows that* $xy^{n-1}=xy^{n-1}x^{n-1}$ *and* $y^{n-1}x=y^{n-1}xy^{n-1}$ *which implies that*

$$xy^{n-1} = xy^{n-1}x^{n-1} = xy^{n-1}xx^{n-2} = xy^{n-1}xy^{n-1}x^{n-2} = \dots =$$

$$= (xy^{n-1})^{n-1}x.$$

(iv) $\Longrightarrow$ *(i). If S is an n-band, then for every* $x\in N_x$, *by Proposition 1, we have*

$$x = (xy^{n-1})^{n-1}x = xy^{n-1}.$$

4. We will show now that the n-semilattices are, in fact, usual binary semilattices (i.e. commutative idempotent semigroups).

Proposition 4. An n-semigroup S is an n-semilattice iff there exists a binary operation "." on S such that $(S;\cdot)$ is a semilattice and the following identity is satisfied:

$$x_1x_2\ldots x_n = x_1\cdot x_2\cdot\ldots\cdot x_n. \tag{4.1}$$

Proof. Suppose first that S is an n-semilattice and define a binary operation "." on S by

$$x\cdot y = xy^{n-1}. \tag{4.2}$$

Then, $(S;\cdot)$ is a commutative groupoid and

$$(x\cdot y)\cdot z = xy^{n-1}z^{n-1},$$

$$x\cdot(y\cdot z) = x(yz^{n-1})^{n-1} = xy^{n-1}(z^{n-1})^{n-1}$$
$$= xy^{n-1}z^{n-2}z^{1+(n-2)(n-1)} =$$
$$= xy^{n-1}z^{n-1};$$

thus $(S;\cdot)$ is a semilattice. Also we have that:

$$x_1\cdot x_2\cdot\ldots\cdot x_n = x_1x_2^{n-1}x_3^{n-1}\ldots x_n^{n-1} =$$
$$= x_1x_2^{n-1}\ldots x_{n-2}^{n-1}(x_{n-1}^{n-1}x_n)x_n^{n-2} =$$
$$= x_1x_2^{n-1}\ldots x_{n-2}^{n-1}x_{n-1}x_n^{n-1}x_n^{n-2} = \ldots =$$
$$= x_1x_2x_3\ldots x_{n-2}x_{n-1}x_n^{1+(n-2)(n-1)} =$$
$$= x_1x_2\ldots x_n,$$

i.e. (4.1) is true.

If (S,o) is a semilattice such that

$$x_1x_2\ldots x_n = x_1ox_2o\ldots ox_n,$$

then

$$xoy = xoyoyo\ldots oy = xy^{n-1},$$

i.e. $xoy = x\cdot y$. ||

5. The following problem is considered in the paper [4; pp. 138-139]: given a class $\mathcal{C}$ of semigroups, find a "reasonable" definition of the corresponding class $\mathcal{C}(n)$ of n-semigroups. One of the possible solutions is to say that:

"An n-semigroup S belongs to $\mathcal{C}(n)$ iff there exists a semigroup $(S;\cdot)\in\mathcal{C}$ such that the identity (4.1) is satisfied".

The proposition 4 gives the possibility to characterize in such a way the class of n-semilattices, but we should note that this kind of defining classes of n-semigroups is not suitable.

In order to illustrate this assertion we will consider the symmetric group of permutations $G=\{(1),(12),(13),(23),(123),(132)\}$. The set of transpositions, $S=\{(12),(13),(23)\}$ is a ternary semigroup with respect to superposition of mappings; moreover, this 3-semigroup is a 3-band and a 3-group. But, there is no binary semigroup $(S;\cdot)$ in which (4.1) would hold.

We mention that the Propositions 1, 2 and 3 are well known for n=2. (See, for example, [3].) This propositions suggest to call S a rectangular n-band iff S satisfies the identity (2). For example, the 3-semigroup $S=\{(12),(13),(23)\}$ satisfies this identity, because $(xy^2)^2x=x^2x=x$, and so we can consider S as a rectangular 3-band.

Note that if we want to carry over the property of anticommutativity of rectangular bands to the n-ary case, then we come to another class of n-semigroups as it is shown in [1]. Namely, one obtains that an n-semigroup S satisfies the quasiidentity

$$xz_1\ldots z_{n-1}y = yz_1\ldots z_{n-1}x \Longrightarrow x=y$$

iff there exists a binary rectangular band $(S;\cdot)$ such that $x_1\ldots x_n = x_1\cdot x_n$.

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