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ON (3,2) - GROUPS

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Abstract. The goal of this paper is to put together some known facts about (3,2)-groups. Some equivalent definition for (3,2)-groups are given. It is mentioned that finite (3,2)-groups do not exist. An elementary proof that finite (3,2)-groups with less than 12 elements do not exist is given. At the end, it is shown that (3,2)-groups do exist, by giving a combinatorial description of a free (3,2)-group without generators. Such a group is countable-infinite.

0. Introduction. Vector valued groups are defined in [1]. Here we focus on vector valued (3,2)-groups. A (3,2)-group is a set G together with a map $[\ ]: G^3 \to G^2$ satisfying the following conditions:

(1) $[[xyz]t] = [x[yzt]]$, (associativity); and

(2) For given $a,b,c \in G$, there exist $x,y,z,t \in G$ such that $[axy] = (b,c) = [zta]$ (solvability of equations).

Because of (1) $[[\dots[[xyz]t]\dots]u]$ is denoted by $[xyzt\dots u]$.

In [1], Theorem 4.3 it is stated that a free (3,2)-group is nontrivial, i.e. has more than one element. Although the statement is true, the proof has some gaps. In professor Čupona's seminar at Skopje, we tried to fill up these gaps. In this paper, a combinatorial description of free (3,2)-groups without generators is given, showing that they are nontrivial. Besides this, we give some basic facts, equivalent definitions, and nonexistence conditions for (3,2)-groups.

1. Basic facts. Let $(G,[\ ])$ be a (3,2)-group.

Proposition 1. $(G^2,\cdot)$ where $(x,y)\cdot(z,t) = [xyzt]$, is a group with identity element a pair (e,e). Moreover:

1) $(x,e)\cdot(e,y) = [xeey] = (x,y)$;
2) $[xyz] = [xab] \Leftrightarrow (y,z) = (a,b) \Leftrightarrow [yzx] = [abx]$;
3) For each $x \in G$, there exist unique $y,z \in G$, such that $[xyz] = [yzx] = [zxy] = (e,e)$, and if $x = y$ or $y = z$ or $z = x$, then $x = y = z$; and
4) For each $x \in G$, $[xee] = [eex]$.

The proof of this Proposition is given in [2]. ∎

We denote the pair $[xee]$ by $(\alpha(x),\beta(x))$. This defines maps $\alpha,\beta: G \to G$. In this notation, for each $x,y,z \in G$ $[xyz] = [xy\alpha(z)\beta(z)] = [x\alpha(y)\beta(y)z] = [\alpha(x)\beta(x)yz]$. Moreover $[e\alpha(e)\beta(e)] = (e,e) = (\alpha(e),\beta(e))\cdot(\alpha(e),\beta(e))$.

Proposition 2. Let $f: G^2 \to G^2$ be the involution $f(x,y) = (y,x)$. Then $(G,[\ ]')$, where $[xyz]' = f([zyx])$, is a (3,2)-group. In the group $(G^2,\circ)$ associated to $(G,[\ ]')$, the pair (e,e) is still the identity element, and $f((x,y)\circ(z,t)) = f(z,t)\cdot f(x,y)$. Moreover, $\alpha' = \beta$ and $\beta' = \alpha$ i.e. $(\alpha'(x),\beta'(x)) = [xee]' = (\beta(x),\alpha(x))$.

Proof. (i) Associativity: $[[xyz]'t]' = [f([zyx])t]' =$
$= f([tf(f([zyx]))]) = f([t[zyx]]) = f([[tzy]x]) =$
$= [xf([tzy])]' = [x[yzt]']'$.

(ii) The solvability of the equations for $[\ ]'$ follows from the solvability of the equations for $[\ ]$ and the fact that f is a bijection.

(iii) The operation $\circ$ on G^2 is defined by
$(x,y)\circ(z,t) = [xyzt]' = f(f(z,t)\cdot f(x,y))$. Hence,
$(x,y)\circ(e,e) = f(f(e,e)\cdot f(x,y)) = f((e,e)\cdot f(x,y)) = ff(x,y) =$
$= (x,y)$, and
$[xee]' = f([eex]) = f(\alpha(x),\beta(x)) = (\beta(x),\alpha(x))$. ∎

Proposition 3. If for some $x \in G$ (i) $\alpha(x) = x$, or (ii) $\beta(x) = x$, or (iii) $\alpha(G) \cap \beta(G) \neq \emptyset$, then $|G| = 1$.

Proof. Because of Proposition 2. it is enough to consider only (i) and (iii).

(i) Let $\alpha(x) = x$. Then $[xee] = (\alpha(x),\beta(x)) =$

$= [x\beta(x)ee]$ implies that $[\beta(x)ee] = (e,e)$. Now, for each $z \in G$, $(z,\beta(x)) = [z\beta(x)ee] = [zee] = [eez] = [\beta(x)eez] =$
$= (\beta(x),z)$, implies that $z = \beta(x)$, i.e. $|G|=1$.

(iii) Let $\alpha(x) = \beta(y)$. Then from $[xee] = (\alpha(x),\beta(x))$ and $[yee] = (\alpha(y),\beta(y)) = (\alpha(y),\alpha(x))$ we have that $(\alpha(y),x) = [\alpha(y)xee] = [\alpha(y)\alpha(x)\beta(x)] = [yee\beta(x)] = (y,\beta(x))$, i.e. $y = \alpha(y)$. Now, (i) implies that $|G| = 1$. ∎

<u>Proposition 4</u>. (1) If $e \in \alpha(\alpha(G))$, then $|G| = 1$.
(2) If for some $x \in G$, $x = \alpha(\alpha(x))$, then $e \in \beta(G)$.
(3) If for some $x \in G$, $x = \beta(\beta(x))$, then $e \in \alpha(G)$.
(4) If $e \in \beta(\beta(G))$, then $|G| = 1$.
(5) If for some $x,y \in G$, $x = \alpha(\alpha(x))$ and $y = \beta(\beta(y))$, then $|G| = 1$.

<u>Proof</u>. (4) follows from (1) and Proposition 2. (3) follows from (2) and Proposition 2. (5) follows from (2), (3) and Proposition 3.(iii).

(1) Let $e = \alpha(\alpha(x))$, i.e. $[\alpha(x)ee] = (e,\beta(\alpha(x))$. Then, $[\beta(\alpha(x))ee] = (e,\alpha(x)) = (\alpha(\beta(\alpha(x))),\beta(\beta(\alpha(x))))$, i.e. $\alpha(G) \cap \beta(G) \neq \phi$. Now apply Proposition 3.(iii).

(2) Let $x = \alpha(\alpha(x))$. Then $[xee] = (\alpha(x),\beta(x)) =$
$= [\alpha(x)ee\beta(x)] = [\alpha(\alpha(x))\beta(\alpha(x))\beta(x)] = [x\beta(\alpha(x))\beta(x)]$ implies that $\beta(\alpha(x)) = \beta(x) = e$, i.e. $e \in \beta(G)$. ∎

<u>2. Equivalent definitions of (3,2)-groups</u>. The next Proposition gives an equivalent definition for (3,2)-groups, analogous to the definition of (ordinary) groups via a binary, unary and nullary operations.

<u>Proposition 5</u>. The existence of a (3,2)-group structure on a set G is equivalent to the existence of: maps $[\ ]:G^3 \to G^2$ and $g:G \to G$, and an element $e \in G$ satisfying the following conditions: (i) The map $[\ ]$ is associative, i.e. $[[xyz]t] = [x[yzt]]$; (ii) $[x[yee]] = (x,y)$; (iii) $g^3 = id_G$; and (iv) $[xg(x)g^2(x)] = (e,e)$. We say that g is the $[\ ]$-<u>inverse map</u>.

<u>Proof</u>. Let $(G,[\ ])$ be a (3,2)-group. Then $[\ ]$ and $e \in G$ (Proposition 1.) satisfy (i) and (ii). For $x \in G$, let $g(x) = y$, where $[xyz] = (e,e)$. Then Proposition 1. implies that (iii) and (iv) are satisfied.

Conversely, let $[\ ]$, g, and $e \in G$ be given, satisfying (i) to (iv). The condition (i) is the associativity for $[\ ]$. Let $a,b,c \in G$. Then $(b,c) = [bcee] = [bcg(a)g^2(a)a] = [eebc] = [ag(a)g^2(a)bc]$ implies that $[axy] = (b,c) = [uva]$ for $(x,y) = [g(a)g^2(a)bc]$ and $(u,v) = [bcg(a)g^2(a)]$. ∎

Proposition 6. The existence of a (3,2)-group structure on a set G is equivalent to the existence of a group structure on G^2 and elements $e, e', \alpha, \beta \in G$ satisfying the following conditions: (1) (e',e) is the identity element in the group G^2; (2) $(x,e)(e',y) = (x,y)$; and (3) $(\alpha,\beta)(x,e) = (e',x)(\alpha,\beta)$.

Proof. Let $(G,[\ ])$ be a (3,2)-group. Then $(G^2,\cdot)$ is a group (Proposition 1.) and $e = e'$, $\alpha = \alpha(e)$, $\beta = \beta(e)$ satisfy (1), (2) and (3).

Conversely, let G^2 be a group and $e, e', \alpha, \beta \in G$ satisfy (1), (2) and (3). Define a map $[\ ]: G^3 \to G^2$ by:

$$[xyz] = (x,e)(\alpha,\beta)(y,z).$$

Then,

(i) $[xyz] = (x,e)(\alpha,\beta)(y,e)(e',z) = (x,e)(e',y)(\alpha,\beta)(e',z) = (x,y)(\alpha,\beta)(e',z)$.

(ii) $[[xyz]t] = [xyz](\alpha,\beta)(e',t) = (x,y)(\alpha,\beta)(e',z)(\alpha,\beta)(e',t) = (x,e)(e',y)(\alpha,\beta)(\alpha,\beta)(z,e)(e',t) = (x,e)(\alpha,\beta)(y,e)(\alpha,\beta)(z,t) = (x,e)(\alpha,\beta)[yzt] = [x[yzt]]$.

(iii) For given $a,b,c \in G$, $[axy] = (b,c) = [uva]$ where $(x,y) = (\alpha,\beta)^{-1}(a,e)^{-1}(b,c)$ and

$$(u,v) = (b,c)(e',a)^{-1}(\alpha,\beta)^{-1}. \ ∎$$

Proposition 7. The existence of a (3,2)-group structure on a set G is equivalent to the existence of a group structure on a set H and $X \subseteq H$, such that $|X| = |G|$ and each element from H is a unique product of two elements from X.

Proof. Let $(G,[\ ])$ be a (3,2)-group. Define

$$X = \{(\alpha(x),\beta(x)) \mid x \in G\}.$$

Then Proposition 1. implies that each element from the group $(G^2,\cdot)$ is a unique product of two elements from X, and $|X| = |G|$.

Conversely, let H be a group and $X \subseteq H$ satisfies the condition from the Proposition. Then the map $f: X^2 \to H$ defined by $f(x,y) = xy$, is a bijection. Define $[\ ]$ on X by $[xyz] = f^{-1}(xyz)$. Then $[xyz] = f^{-1}(f(x,y)z) = f^{-1}(xf(y,z))$, and so: $[[xyz]t] = f^{-1}(f([xyz])t) = f^{-1}(xyzt) = [x[yzt]]$. Moreover, for given $a,b,c \in X$, $f^{-1}(a^{-1}bc)$, $f^{-1}(bca^{-1}) \in X^2$, and so $[af^{-1}(a^{-1}bc)] = (b,c) = [f^{-1}(bca^{-1})a]$. Hence, $(X,[\ ])$ is a (3,2)-group. Since there is a bijection from X to G, it follows that there is a (3,2)-group structure on G. ∎

<u>Proposition 8</u>. The existence of a (3,2)-group structure on a set G is equivalent to the existence of inclusions $\varphi, \psi: G \to \mathrm{Perm}(G^2)$ satisfying the following conditions:

(1) $\varphi(x)(a,y) = \psi(y)(x,a)$ for each $x,y,a \in G$; and

(2) $\varphi(x) \circ \psi(y) = \psi(y) \circ \varphi(x)$ for each $x,y \in G$, where $\circ$ is the composition of permutations.

<u>Proof</u>. Let $(G,[\ ])$ be a (3,2)-group. Define φ and ψ by $\varphi(x)(a,b) = [xab]$ and $\psi(x)(a,b) = [abx]$. The (3,2)-group structure on G implies that φ and ψ are inclusions from G to $\mathrm{Perm}(G^2)$ and satisfy the conditions (1) and (2).

Conversely, let φ and ψ are given and satisfy (1) and (2). Define $[\ ]: G^3 \to G^2$ by $[xyz] = \varphi(x)(y,z)$. Then (1) implies that $[xyz] = \psi(z)(x,y)$, and (2) implies that $[\ ]$ is associative. For given $a,b,c \in G$, $[axy] = (b,c) = [uva]$, where $(x,y) = (\varphi(a))^{-1}(b,c)$ and $(u,v) = (\psi(a))^{-1}(b,c)$. ∎

3. (<u>Non</u>) <u>Existence conditions for</u> (3,2)-<u>groups</u>. In [2] it is shown that the existence of a (3,2)-group structure on a finite set G, implies that 6 is a divisor of $|G|$ or $|G| = 1$. (Here $|G|$ is the number of elements in G.) The next Proposition gives an elementary proof that on a set with 6 elements there does not exist a (3,2)-group structure. Professor John Thompson provided me with a proof that finite (3,2)-groups do not exist. He proved that if G is a finite group and $X \subseteq G$ such that $XX = \{xy \mid x,y \in X\} = G$ and $|X|^2 = |G|$, then $|G| = 1$. Then we apply Proposition 7. His proof uses group algebras over the field of complex numbers, Wedderburn's theorem, and representations and characters of finite groups.

<u>Proposition 9</u>. A set with 6 elements does not admit a (3,2)-group structure.

<u>Proof</u>. Let $|G| = 6$, and let (e,e) be the identity element in the group $(G^2,\cdot)$. Propositions 3. and 4. imply that $|\{e,\alpha(e),\alpha(\alpha(e)),\beta(e),\beta(\beta(e))\}| = 5$. Proposition 4. implies that $\alpha(\alpha(\alpha(e))) \neq e \neq \beta(\beta(\beta(e)))$, $\alpha(\alpha(\alpha(e)))\neq\alpha(\alpha(e))$, and $\beta(\beta(\beta(e))) \neq \beta(\beta(e))$. Since $\alpha(\alpha(\alpha(e))) \neq \beta(\beta(\beta(e)))$ and $|G| = 6$, it follows that $\alpha(\alpha(\alpha(e))) = \alpha(e)$ or $\beta(\beta(\beta(e)))=\beta(e)$. By Proposition 2. we can assume that $\alpha(\alpha(\alpha(e))) = \alpha(e)$. Then Proposition 4. implies that $\beta(\beta(\beta(e))) \neq \beta(e)$, and so:
$G = \{e,\alpha(e),\alpha(\alpha(e)),\beta(e),\beta(\beta(e)),\beta(\beta(\beta(e)))\}$.
Now, $\alpha(\alpha(\alpha(e))) = \alpha(e)$ implies that $\beta(\alpha(e)) = \beta(\alpha(\alpha(\alpha(e)))) = e$. If $\beta(\beta(\beta(\beta(e)))) = \beta(\beta(e))$, then Proposition 3. implies that $|G|= 1$. Hence, $\beta(\beta(\beta(\beta(e)))) = \beta(e)$. Then $[eee] = (\alpha(e),\beta(e))$ and $[eee] \neq [ee\beta(\beta(\beta(e)))]$ imply that $\alpha(\beta(\beta(\beta(e))))=\alpha(\alpha(e))$. Now, $[ee\beta(\beta(\beta(e)))]=[\alpha(\alpha(e))\alpha(\beta(e))\alpha(\beta(\beta(e)))\beta(\beta(\beta(e)))]$ implies that $[\alpha(\alpha(e))\alpha(\beta(e))\alpha(\beta(\beta(e)))] = (e,e)$. This, together with $[e\alpha(e)\beta(e)] = (e,e)$ and $\alpha(\alpha(e))\neq e,\alpha(e),\beta(e)$ implies that $\alpha(\beta(e)) \neq e,\alpha(e),\beta(e)$, and since $\alpha(\beta(e))\neq\beta(\beta(e))$, $\alpha(\beta(e)) \neq \beta(\beta(\beta(e)))$, it follows that $\alpha(\beta(e)) = \alpha(\alpha(e))$, and by Proposition 1., $\alpha(\beta(\beta(e))) = \alpha(\alpha(e))$. Now,
$(e,e) = [\alpha(\alpha(e))\alpha(\alpha(e))\alpha(\alpha(e))] = [\alpha(e)e\alpha(e)e\alpha(e)e] =$
$= [e\alpha(e)e\alpha(e)e\alpha(e)]$ implies that $(\beta(e),\beta(e))=(e,\alpha(e))$. ∎

4. <u>Free</u> (3,2)-<u>groups</u> <u>without</u> <u>generators</u>. In this section we give a combinatorial description of a free (3,2)-group without generators, i.e. generated by the empty set $\emptyset$. This group is an initial object in the category of (3,2)-groups, and is countable-infinite.

Let $A_0 = \{e\}$. If A_k, $k \geq 0$ is defined, let

$$A_{k+1} = \{(x_1^n,i)\mid x_j \in A_k,\ n \geq 3,\ i = 1,2\} \cup A_k \setminus \{(eex,i)\mid i = 1,2,\ x \in A_k,\ x \neq e\}.$$

Above, x_1^n stands for $x_1x_2\ldots x_{n-1}x_n$.

Let $A = \bigcup_{k=0}^{\infty} A_k$, and let $S(A)$ be the free semigroup generated by A. We say that $x_1^n = a \in S(A)$, $x_j \in A$ has <u>dimension</u> n and write $\dim(a) = n$. Define a <u>length</u> $|a|$ of $a = x_1^n \in S(A)$, $x_j \in A$, by induction, as follows:

$|e| = 1$; $|a| = |x_1^n| = |x_1| + |x_2| + \ldots + |x_n|$; and

$|(x_1^n,i)| = |x_1^n|$. Roughly put, a length of $x \in S(A)$ is the number of appearances of e in x.

Let $S_k = \{x | x \in S(A),\ |x| \leq k\}$.

Define a map $\varphi : S(A) \to S(A)$ by induction on the length, as follows:

(1) $\varphi(e) = e$;

(2) Suppose that φ is defined on S_{k-1}. Let

$M_k = \{x | x \in S(A),\ |x| = k,\ \dim(x) = 1\}$;
$N_k = \{eeu \,|\, |u| = k-2,\ \dim(u) = 1,\ \varphi(u) = u \neq e\}$;
$R_k = \{u_1^n \,|\, u_j \in A,\ n \geq 2,\ |u_1^n| = k,\ \varphi(u_j) = u_j\} \setminus N_k$; and
$T_k = S_k \setminus (S_{k-1} \cup M_k \cup N_k \cup R_k)$.

Then $N_k \cap R_k = M_k \cap (N_k \cup R_k) = T_k \cap (N_k \cup R_k \cup M_k) = \emptyset$ and $S_k = S_{k-1} \cup M_k \cup N_k \cup R_k \cup T_k$.

The extension of φ on $S_{k-1} \cup N_k$ is given by

(A) $\varphi(eeu) = uee$.

Next, the extension of φ on $S_{k-1} \cup N_k \cup R_k$ is defined as follows: Let $u_1^n \in R_k$. Define $\varphi(u_1^n)$ to be:

(B.1) $\varphi(\varphi(u_1^{n-1})u_n)$ if $|\varphi(u_1^{n-1})| < |u_1^{n-1}|$;

(B.2) $\varphi(u_1\varphi(u_2^n))$ if $|\varphi(u_2^n)| < |u_2^n|$;

(B.3) v_1^t $(= \varphi(v_1^t))$ if $n = 2$, $u_i = (v_1^t, i)$, $i=1,2$, $t \geq 3$;

(B.4) $\varphi(uv)$ if $n = 4$, $u_1^4 = uvee$, or $u_1^4 = ueev$, or $u_1^4 = eeuv$;

(B.5) ee if $u_1 = (v_1^t,2)$, $u_n = (v_1^t,1)$, $\varphi(v_1^t u_2^{n-1}) = ee$ and $t \geq 3$;

(B.6) u_1^n otherwise.

The next extension of φ on $S_{k-1} \cup N_k \cup R_k \cup T_k$ is given by

(C) $\varphi(u_1u_2\ldots u_n) = \varphi(\varphi(u_1)\varphi(u_2)\ldots\varphi(u_n))$.

At the end, the extension of φ on S_k is given by

(D) $\varphi((u_1^n,i)) = (\varphi(u_1^n),i)$, $n \geq 3$, $i = 1,2$. (Here we use the notation $(u_1u_2,i) = u_i$, $i = 1,2$. It is necessary, because $\dim(\varphi(u_1^n)) \geq 2$ for $n \geq 2$.)

The essential parts in the definition of φ are (A), (B.3), (B.4) and (B.5). The part (A) implies the fact $[xee] = [eex]$ (Proposition 1. 4)), the part (B.3) implies the associativity, the part (B.4) implies that (e,e) is the identity element in the associated group (Proposition 1.) and the part (B.5) implies that each x has unique y,z such that $[xyz] = (e,e)$ (Proposition 1. 3)). The other parts are given only because of the technical difficulties in the proof of the following:

<u>Lemma 10</u>. The map φ is well defined and satisfies the following conditions:

1. $\varphi(e) = e$;
2. $\dim(\varphi(x)) = 1$ if and only if $\dim(x) = 1$;
3. $\dim(\varphi(u_1u_2))=2$ if and only if $\varphi(u_1u_2) = \varphi(u_1)\varphi(u_2)$;
4. $\varphi(eeu) = \varphi(u)ee$;
5. $\varphi(uee) = \varphi(u)ee$;
6. $|\varphi(u_1^n)| \leq |u_1^n|$;
7. If $u_1^n \neq eeu$ and $\varphi(u_j) = u_j$, then $|\varphi(u_1^n)| = |u_1^n|$ if and only if $\varphi(u_1^n) = u_1^n$;
8. $\varphi(u_1^n) = \varphi(u_1^{r-1}\varphi(u_r^s)u_{s+1}^n)$ for each $1 \leq r \leq s \leq n$;
9. $\varphi((u_1^n,i)) = (\varphi(u_1^n),i)$ for $n \geqslant 3$, $i = 1,2$; and
10. $\varphi(u_1^n) = (e,e)$ if and only if $\varphi(u_2^nu_1) = (e,e)$. ∎

The proof of this Lemma, although straightforward by induction on the length, is very long, so we do not give it here. It will appear in a paper about free (n+1,n)-groups.

Let $G = \varphi(A)$. Define maps $g_1,g_2:G \longrightarrow G$ by induction on the length as follows:

(a) $g_i(e) = (eee,i)$, $i = 1,2$;

(b) $g_i((u_1^n,1))=\varphi((u_1^n,2)g_1(u_n)g_2(u_n)\ldots g_1(u_1)g_2(u_1),i)$; and

(c) $g_i((u_1^n,2)) = \varphi(g_1(u_n)g_2(u_n)\ldots g_1(u_1)g_2(u_1)(u_1^n,1),i)$.

<u>Lemma 11</u>. For each $x \in G$, $\varphi(xg_1(x)g_2(x)) = ee$.

<u>Proof</u>. The following equations, using induction on the length and Lemma 10. imply Lemma 11.

$$\varphi(e(eee,1)(eee,2)) = \varphi(eeee) = ee;$$

$$\begin{aligned}\varphi((u_1^n,1)g_1((u_1^n,1))g_2((u_1^n,1))) &= \varphi(u_1^ng_1(u_n)g_2(u_n)\ldots g_1(u_1)g_2(u_1)) \\ &= ee = \varphi(g_1(u_n)g_2(u_n)\ldots g_1(u_1)g_2(u_1)u_1^n) = \\ &= \varphi(g_1((u_1^n,2))g_2((u_1^n,2))(u_1^n,2)) = \\ &= \varphi((u_1^n,2)g_1((u_1^n,2))g_2((u_1^n,2))). \ ∎\end{aligned}$$

Define a map $[\;]:G^3 \longrightarrow G^2$ by

$[uvw] = (\varphi(uvw,1),\varphi(uvw,2))$.

Theorem 12. $(G,[\,])$ is a free (3,2)-group generated by the empty set Φ.

Proof. (i) $[[uvw]z] = [\varphi(uvw,1)\varphi(uvw,2)\ z] =$
$= (\varphi(uvwz,1),\varphi(uvwz,2)) = [u\varphi(vwz,1)\varphi(vwz,2)] = [u[vwz]]$.

(ii) Let $a,b,c \in G$. Then $[ax_1x_2] = (b,c) = [y_1y_2a]$, where $(x_1,x_2) = [g_1(a)g_2(a)bc]$ and $(y_1,y_2) = [bcg_1(a)g_2(a)]$.

(iii) Let $(G',[\,]')$ be a (3,2)-group, (e',e') the identity element in the associated group, and $g':G' \to G'$ the $[\,]'$-inverse map. Define $f:G \to G'$ by induction on the length as follows: $f(e) = e'$; $f((u_1^n,i)) = a_i$, $i = 1,2$, $n \geq 3$ and $(a_1,a_2) = [f(u_1)f(u_2)\ldots f(u_n)]'$. The map f is well defined, and the proof that f is a (3,2)-homomorphism, i.e. $(f,f)([xyz]) = [f(x)f(y)f(z)]'$, is by induction on the length and using Lemma 10. Moreover, f is a unique (3,2)-homomorphism from $(G,[\,])$ to $(G',[\,]')$.

Remark 13. Since $(G,[\,])$ is a (3,2)-group and $[xg_1(x)g_2(x)] = (e,e)$, it follows that $g = g_1$ is the $[\,]$-inverse map.

Remark 14. The set G is infinite (countable). For example, $u_n = (u_{n-1}ee,1)$, $u_1 = e$, $n > 1$, is a sequence of distinct elements in G.

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