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EXAMPLES OF VECTOR VALUED GROUPS

Dedicated to Academician Petar Serafimov on the occasion of his 70-th anniversary

Abstract: The main purpose of this paper is to give examples of $(2m+s, m)$ vector valued groups with more than one element. First, some sufficient existence conditions for such groups are given and then concrete examples are constructed. Using the structure of $(2m+s, m)$-groups, some consequences about congruences of certain sums of binomial coefficients are obtained.

1. In [1] vector valued groups are defined. $(G, [\ ])$ is an (m, n)-group, $m - n = k > 0$, iff $[\ ]: G^m \to G^n$ is a map satisfying:

associativity, i.e. for each $1 \leqslant i \leqslant k$

$$[[x_l^m]\, x_{m+1}^{m+k}] = [x_1^i\, [x_{i+1}^{i+m}]\, x_{i+m+1}^{m+k}] \quad ; \tag{1.1}$$

and solvability of the equations

$$[\mathbf{a}\ \mathbf{x}] = \mathbf{b} = [\mathbf{y}\ \mathbf{a}]. \tag{1.2}$$

Above, $\mathbf{a} \in G^t$ denotes a vector (a_1^t) i.e. $(a_1, a_2, \ldots, a_t)$ and $[x_1^t]$ denotes $[\ ](x_1^t)$ i.e. $[x_1\, x_2 \ldots x_t]$.

$(2m, m)$-groups are discussed in [3]. In [2] it is shown that the existence of nontrivial (with more than one element) finite $(n+1, n)$-groups requires certain properties on the cardinality of these groups. In particular, it is shown that some sets do not admit $(n+1, n)$-group structure. The existence question of nontrivial finite $(n+1, n)$-groups, and more generally $(n+k, n)$-groups for $1 \leqslant k \leqslant n$, is still open. The aim of this paper is to give examples of a wide class of vector valued groups, finite and infinite that include, in a natural way, the "trivial" $(2m, m)$-groups from [3]. At the end of the paper, as a consequence of the definition of some of these examples, certain congruences of sums of binomial coefficients are obtained.

2. Let $m, s \in \mathbf{N}, m \geqslant 1$ and G and G' be groups with identity elements e and e'. Supose that $f:G^{m+s} \to G'$ is a homomorphism satisfying the following conditions:

$$f(x_1^{m+s}) = e' \Rightarrow f(x_2^{m+s}, x_1) = e'; \tag{2.1}$$

$$f(e^s, x_1^m) = e' \Leftrightarrow (x_1^m) = (e^m) \text{ ; and} \tag{2.2}$$

$$(\forall \mathbf{x} \in G^{m-s}) (\exists \mathbf{y} \in G^m) \;\; f(\mathbf{x}) = f(e^s, \mathbf{y}). \tag{2.3}$$

Clearly (2.1) is equivalent to:

$$f(x_1^{m+s}) = e' \Leftrightarrow f(x_i^{m+s}, x_1^i) = e' \text{ for each } 1 \leqslant i \leqslant m-2s. \tag{2.1'}$$

Above, G^t denotes the product group, and (e^t) denotes the vector $\underbrace{(e, e, \ldots, e)}_{t}$.

We are going to show that the above assumptions give a $(2m+s, m)$-group structure on the set G. First we state the following facts whose proofs follow directly from the assumptions.

Fact 2.1. The restriction of f on $(e^s) \times G^m \subseteq G^{m+s}$ is a monomorphism, whose image coincides with the image of f. ∎

Fact 2.2. $f(x_1^{m+s}) \; f(y_1^{m+s}) \ldots f(z_1^{m+s}) = f(u_1^{m+s}) f(v_1^{m+s}) \ldots$
$\ldots f(w_1^{m+s}) \Rightarrow f(x_2^{m+s}, x_1) \; f(y_2^{m+s}, y_1) \ldots f(z_2^{m+s}, z) =$
$= f(u_2^{m+s}, u_1) \; f(v_2^{m+s}, v_1) \ldots f(w_2^{m+s}, w_1)$. ∎

Fact 2.3. Let $0 \leqslant p \leqslant m-s$, $t, r \in \mathbf{N}$, $|\mathbf{x}| = p$, $|\mathbf{x}_{2i-1}| = m+s-p$, $|\mathbf{x}_{2i}| = p$, $|\mathbf{y}_j| = p$, $|\mathbf{y}_{2j-1}| = m+s-p$ and $|\mathbf{y}_{2j}| = p$, for each $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant r$, where $|\mathbf{x}|$ denotes the length of the vector $\mathbf{x}$. Then:

(a) $f(e^{m+s-p}, \mathbf{x}) f(\mathbf{x}_1, \mathbf{x}_2) \ldots f(\mathbf{x}_{2t-1}, \mathbf{x}_{2t}) =$
$= f(\mathbf{x}_1, \mathbf{x}) f(\mathbf{x}_3, \mathbf{x}_2) \ldots f(e^{m+s-p}, \mathbf{x}_{2t})$; and

(b) $[f(e^{m+s-p}, \mathbf{x}) f(\mathbf{x}_1, \mathbf{x}_2) \ldots f(\mathbf{x}_{2t-1}, \mathbf{x}_{2t}) =$
$= f(e^{m+s-p}, \mathbf{y}) f(\mathbf{y}_1, \mathbf{y}_2) \ldots f(\mathbf{y}_{2r-1}, \mathbf{y}_{2r})] \Rightarrow$
$\Rightarrow [f(\mathbf{x}, \mathbf{x}_1) f(\mathbf{x}_2, \mathbf{x}_3) \ldots f(\mathbf{x}_{2t}, e^{m-s-p}) =$
$= f(\mathbf{y}, \mathbf{y}_1) f(\mathbf{y}_2, \mathbf{y}_3) \ldots f(\mathbf{y}_{2r}, e^{m-s-p})]$. ∎

Now, let $\overline{G} = \cup_{i=0}^{\infty} G^{m+i}$. The homomorphism f, induces a homomorphism $f:G^{t(m+s)} \to G'$ for each $t \geqslant 1$. (We use the same notation f.) If $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \overline{G}$ so that $|\mathbf{x}_1| = p$ and $|\mathbf{x}_2| = t(m+s)$ for some $t \geqslant 0$

and $0 < p < m + s$, we define $f(\mathbf{x}) = f(e^{m+s-p}, \mathbf{x}_1) f(\mathbf{x}_2)$, again abusing the notation f. Using $f: \bar{G} \to G'$, we define a relation $\sim$ on $\bar{G}$ by:

$$\mathbf{x} \sim \mathbf{y} \Leftrightarrow f(\mathbf{x}) = f(\mathbf{y}) \text{ and } |\mathbf{x}| \equiv |\mathbf{y}| \ (\text{mod } m+s). \tag{2.4}$$

By **Fact 2.3.** (b), this definition is equivalent to:

$$\begin{aligned} \mathbf{x} \sim \mathbf{y} \Leftrightarrow |\mathbf{x}| \equiv |\mathbf{y}| \equiv p \ (\text{mod } m+s),\ 0 < p \leqslant m+s \\ \text{and } f(\mathbf{x}, e^{m+s-p}) = f(\mathbf{y}, e^{m+s-p}). \end{aligned} \tag{2.4'}$$

Proposition 2.4. The following are equivalent:

(i) $\mathbf{x} \sim \mathbf{y}$;
(ii) There are $\alpha, \beta \in \mathbf{N}^\circ$, $(e^\alpha, \mathbf{x}, e^\beta) \sim (e^\alpha, \mathbf{y}, e^\beta)$;
(iii) For any $\alpha, \beta \in \mathbf{N}^\circ$, $(e^\alpha, \mathbf{x}, e^\beta) \sim (e^\alpha, \mathbf{y}, e^\beta)$.

Proof. It follows directly from the definition of $f: \bar{G} \to G'$ that $f(e^\alpha, \mathbf{x}) = f(\mathbf{x})$ for any $\alpha \in N^\circ$, $\mathbf{x} \in \bar{G}$. This, together with (2.4) and (2.4′) implies that (i), (ii), and (iii) are equivalent. ∎

Proposition 2.5. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \bar{G}$,
$$\mathbf{x} \sim \mathbf{y} \Rightarrow (\mathbf{x}, \mathbf{z}) \sim (\mathbf{y}, \mathbf{z}) \text{ and } (\mathbf{z}, \mathbf{x}) \sim (\mathbf{z}, \mathbf{y}).$$

Proof. It is enough to show this for $z \in G$. Let $\mathbf{x} \sim \mathbf{y}$, i.e. $|\mathbf{x}| \equiv |\mathbf{y}| \equiv p$ (mod $m+s$), $f(e^{m+s-p}, \mathbf{x}) = f(e^{m+s-p}, \mathbf{y})$ and $f(\mathbf{x}, e^{m+s-p}) = f(\mathbf{y}, e^{m+s-p})$. Then,

$$f(e^{m+s-1}, z) f(\mathbf{x}, e^{m+s-p}) = f(e^{m+s-1}, z) f(\mathbf{y}, e^{m+s-p})$$

and

$$f(e^{m+s-p}, \mathbf{x}) f(z, e^{m+s-1}) = f(e^{m+s-p}, \mathbf{y}) f(z, e^{m+s-1}),$$

i.e.

$$(e^{m+s-1}, z, \mathbf{x}, e^{m+s-p}) \sim (e^{m+s-1}, z, \mathbf{y}, e^{m+s-p})$$

and $(e^{m+s-p}, \mathbf{x}, z, e^{m+s-1}) \sim (e^{m+s-p}, \mathbf{y}, z, e^{m+s-1})$, which together with **Proposition 2.4.** implies that $(z, \mathbf{x}) \sim (z, \mathbf{y})$ and $(\mathbf{x}, z) \sim (\mathbf{y}, z)$. ∎

Now, let $[\ ] : G^{2m+s} \to G^m$ be defined by:

$$[x_1^{2m+s}] = (y_1^m) \Leftrightarrow (x_1^{2m+s}) \sim (y_1^m), \tag{2.5}$$

or equivalently by:

$$\begin{aligned} [x_1^{2m+s}] = (y_1^m) &\Leftrightarrow f(e^s, x_1^m) f(x_{m+1}^{2m+s}) = f(e^s, y_1^m) \\ &\Leftrightarrow f(x_1^{m+s}) f(x_{m+s+1}^{2m+s}) = f(y_1^m, e^s). \end{aligned} \tag{2.5'}$$

The conditions (2.2) and (2.3) imply that [ ] is well defined.

Theorem 2.6. $(G, [\ ])$ is a $(2m+s, m)$-group.

Proof. Let $0 \leqslant i \leqslant m+s$, $[x_1^i [x_{i+1}^{2m+s+i}] x_{2m+s+i+1}^{3m+2s}] = (y_1^m)$, and $[x_{i+1}^{2m+s+i}] = (z_1^m)$. Then $(x_{i+1}^{2m+s+i}) \sim (z_1^m)$ and $(x_1^i, z_1^m, x_{2m+s+i+1}^{3m+2s}) \sim (y_1^m)$, which together with **Proposition 2.5.** implies that (y_1^m) does not depend on i. Hence the associativity i.e. the condition (1.1) holds for [ ]. The proof that [ ] satisfies (1.2) is as follows. Let $\mathbf{a} \in G^{m+s}$, $\mathbf{b} \in G^m$ and let $\mathbf{c} \in G^{m+s}$ be the inverse for $\mathbf{a}$ in the product group G^{m+s}. Then, the condition (2.3), **Fact 2.1.** and **Fact 2.3.** imply that there are $\mathbf{x}, \mathbf{y} \in G^m$ such that $f(\mathbf{c})\, f(\mathbf{b}, e^s) = f(\mathbf{c} \cdot (\mathbf{b}, e^s)) = f(\mathbf{x}, e^s)$ and $f(e^s, \mathbf{b}) f(\mathbf{c}) = f((e^s, \mathbf{b}) \cdot \mathbf{c}) = f(e^s, \mathbf{y})$. But this is equivalent to $f(\mathbf{a}) f(\mathbf{x}, e^s) = f(\mathbf{b}, e^s)$ and $f(e^s, \mathbf{y}) f(\mathbf{a}) = f(e^s, \mathbf{b})$, i.e. $[\mathbf{a}\ \mathbf{x}] = \mathbf{b} = [\mathbf{y}\ \mathbf{a}]$. ∎

3. In order to be more convenient for constructing concrete examples of $(2m+s, m)$-groups, we restate the assumptions from **2.** as follows.

Proposition 3.1. Let G be a group with identity element e, and for $k \in \mathbf{N}$ let G^k be the product group. For given $m, s \in \mathbf{N}$, the following are equivalent:

(A) There is a group G' and a homomorphism f from G^{m+s} to G' satisfying the conditions (2.1), (2.2) and (2.3).

(B) There is a homomorphism $g : G^s \to G^m$ satisfying:

Image (g) is contained in the centre $Z(G^m) = Z(G)^m$ of G^m; and (3.1)

$$g(x_1^s) = (z_1^m) \Leftrightarrow g(x_2^s, z_1) = (z_2^m, x). \tag{3.2}$$

(C) There is a homomorphism $h : G^{m+s} \to G^m$ satisfying:

$$h(e^s, x_1^m) = (x_1^m); \quad \text{and} \tag{3.3}$$

$$h(x_1^{m+s}) = (e^m) \Rightarrow h(x_2^{m+s}, x_1) = (e^m). \tag{3.4}$$

Proof: $(A) \Rightarrow (B)$: Let $g : G^s \to G^m$ be defined by: $g(x_1^s) = (z_1^m) \Leftrightarrow \Leftrightarrow f(x_1^s, z_1^m) = e'$. Because $f(x_1^s, z_1^m) = e'$ is equivalent to $f(x_1^s, e^m) = f(e^s, u_1^m)$ for (u_1^m) the inverse of (z_1^m) in the group G^m, the conditions (2.2) and (2.3) imply that g is a well defined homomorphism. If $f(x_1^s, e^m) = f(e^s, u_1^m)$ then $f(e^s, (u_1^m) \cdot (y_1^m)) = f(e^s, u_1^m) f(e^s, y_1^m) = f(x_1^s, e^m) f(e^s, y_1^m) = f(x_1^s, y_1^m) = f(e^s, y_1^m) f(x_1^s, e^m) = f(e^s, y_1^m) f(e^s, u_1^m) = f(e^s, (y_1^m)(u_1^m))$, which together with (2.2) implies (3.1). Using the definition of g and (2.1) we have: $g(x_1^s) = (z_1^m) \Leftrightarrow f(x_1^s, z_1^m) = e' \Leftrightarrow f(x_2^s, z_1, z_2^m, x_1) = e' \Leftrightarrow g(x_2^s, z_1) = (z_2^m, x_1)$.

$(B) \Rightarrow (C)$: Let $h : G^{m+s} \to G^m$ be defined by:

$h(x_1^{m+s}) = (g(x_1^s))^{-1} \cdot (x_{s+1}^{s+m})$, i.e. by $g(x_1^s) \cdot h(x_1^{m+s}) = (x_{s+1}^{m+s})$.

Then (3.1) implies that h is a homomorphism. Because g is a homomorphism, it follows that $h(e^s, x_1^m) = (x_1^m)$, i.e. (3.3). If $h(x_1^{m+s}) = (e^m)$, then $g(x_1^s) = (x_{s+1}^{m+s})$, and by (3.2) $g(x_2^s, x_{s+1}) = (x_{s+2}^{m+s}, x_1)$, i.e. $g(x_2^{s+1}) = (x_{s+2}^{m+s}, x_1)$. So, $h(x_2^{m+s}, x_1) = (e^m)$.

$(C) \Rightarrow (A)$: Let $G' = G^m$ and $f = h$. Then (2.1) follows from (3.4), (2.2) follows from (3.3), and (2.3) follows from (3.3) and the fact that h is a homomorphism from G^{m+s} to G^m. ∎

Using **Proposition 3.1.**, **Theorem 2.6.** and the fact that in abelian groups the condition (3.1) is trivially satisfied, we get the following:

Corollary 3.2. Let $(G, +)$ be an abelian group with zero O, and for given $m, s \in \mathbf{N}$ let $g: G^s \to G^m$ be a homomorphism, so that $h(x_1^{m+s}) = (O^m)$ implies $h(x_2^{m+s}, x_1) = (O^m)$. Then $[\,]: G^{2m+s} \to G^m$ defined by $[x_1^{2m+s}] = (x_1^m) - g(x_{m+1}^{m+s}) + (x_{m+s+1}^{2m+s})$ furnishes G with a $(2m+s, m)$-group structure. ∎

Now we are going to show how a solution of a certain matrix equation can give us a homomorphism satisfying (B) from **Proposition 3.1**. Let F be a field or a commutative ring with 1. We use the following notations for certain types of matrices over F, for given $k, t \in \mathbf{N}$: I_t — denotes the identity $t \times t$ matrix;

$$J_{k\times t} = [a_{ij}]_{k\times t} \quad \text{where} \quad a_{ij} = \begin{cases} 1 & i = 1, \ j = t \\ 0 & \text{otherwise} \end{cases},$$

$$\text{i.e. with block matrices } J_{k\times t} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right]_{k\times t};$$

$$E_t = [a_{ij}]_{t\times t} \quad \text{where} \quad a_{ij} = \begin{cases} 1 & i = j+1 \\ 0 & \text{otherwise} \end{cases},$$

$$\text{i.e. with block matrices } E_t = \left[\begin{array}{c|c} 0 & 0 \\ \hline I_{t-1} & 0 \end{array}\right]_{t\times t}.$$

For given $m, s \in \mathbf{N}$ we consider the following matrix equation over F:

$$E_s \cdot X_{s\times m} + X_{s\times m} \cdot J_{m\times s} \cdot X_{s\times m} = J_{s\times m} + X_{s\times m} E_m. \tag{3.5}$$

Proposition 3.3. If (3.5) has a solution $A = A_{s\times m}$ over F, then for each F-module G the map $g: G^s \to G^m$ defined by $g(x_1^s) = (x_1^s) \cdot A$, satisfies (B) from **Proposition 3.1.**

Proof. It is clear that g is a module homomorphism. Because G is an abelian group it follows that g satisfies (3.1). Let

$$g(x_1^s) = y_1^m \qquad \text{i.e. } (x_1^s)\, A = (y_1^m)\,.$$

Then,

$$g(x_2^s, y_1) = (x_2^s, y_1) \cdot A = ((x_1^s) \cdot E_s + (y_1^m) \cdot J_{m\times s}) \cdot A =$$
$$= (x_1^s)(E_s + A \cdot J_{m\times s}) \cdot A = (x_1^s)(E_s \cdot A + A \cdot J_{m\times s} \cdot A) =$$
$$= (x_1^s)(J_{s\times m} + A \cdot E_m) = (x_1^s) \cdot J_{s\times m} + (y_1^m) \cdot E_m = (y_2^m, x_1),$$

which shows that g satisfies (3.2). ∎

Remark 3.4. It is easy to check that the equation (3.5) is equivalent to the following system of sm equations with sm unknowns over F:

$$x_{11} \cdot x_{sm} = 1; \tag{3.6.i}$$

$$x_{(i-1)m} + x_{i1} \cdot x_{sm} = 0, \qquad i = 1, 3, \ldots, s; \tag{3.6.ii}$$

$$x_{11} \cdot x_{sj} = x_{1\,(j+1)}, \qquad j = 1, 2, \ldots, m-2, m-1; \tag{3.6.iii}$$

$$x_{(i-1)j} + x_{i1} \cdot x_{sj} = x_{i(j+1)}, \qquad i \neq 1 \text{ and } j \neq m. \tag{3.6.iv}$$

4. In this part we give concrete examples of $(2m+s, m)$-groups. For $m, s, j \in N$ and $1 \leqslant i \leqslant s$, let:

$$a_{ij} = (-1)^{s-i} \binom{s+j-i-1}{j-1} \binom{s+j-1}{i-1}\,. \tag{4.1}$$

Suppose that there is an integer $T \in \mathbf{N}$, such that:

$$a_{i(m+1)} \equiv \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases} \qquad (\text{mod } T). \tag{4.2}$$

Then we have:

Proposition 4.1. The matrix $A = A_{s\times m} = [a_{ij}]$ over $F = \mathbf{Z}_T = \mathbf{Z}/_{T\mathbf{Z}}$, for a_{ij} defined by (4.1) and satisfying (4.2), satisfies the equation (3.5). So, for $T \neq 1$, any nontrivial $\mathbf{Z}_T$-module G can be given a $(2m+s, m)$-group structure by

$$[x_1^m\, y_1^s\, z_1^m] = (x_1^m) + (z_1^m) - (y_1^s) \cdot A. \tag{4.3}$$

Proof. First we show that for each $j \in \mathbf{Z}$:

$$a_{11} \cdot a_{sj} = a_{1(j+1)}\,; \quad \text{and} \tag{4.4}$$

$$a_{ij} + a_{(i+1)1} \cdot a_{sj} = a_{(i+1)\,(j+1)} \quad \text{for } 1 \leqslant i \leqslant s-1. \tag{4.5}$$

Proof of (4.4).

$$a_{11}\cdot a_{sj}=(-1)^{s-1}\binom{s+1-1-1}{1-1}\binom{s+1-1}{1-1}(-1)^{s-s}\binom{s+j-s-1}{j-1}\binom{s+j-1}{s-1}=$$

$$=(-1)^{s-1}\binom{s+j-1}{s-1}=(-1)^{s-1}\binom{s+(j+1)-1-1}{(j+1)-1}\binom{s+(j+1)-1}{1-1}=$$

$$=a_{1(j+1)}\cdot\ \square$$

Proof of (4.5).

$$a_{ij}+a_{(i+1)1}\cdot a_{sj}=$$

$$=(-1)^{s-i}\binom{s+j-i-1}{j-1}\binom{s+j-1}{i-1}+(-1)^{s-i-1}\binom{s+1-i-1-1}{1-1}\binom{s+1-1}{i+1-1}\cdot$$

$$\cdot(-1)^{s-s}\binom{s+j-s-1}{j-1}\binom{s+j-1}{s-1}=(-1)^{s-i}\cdot\frac{(s+j-1)!}{(s-i)!\cdot j!\cdot i!\cdot(s-j-i)}\cdot$$

$$\cdot[ij-s(s-j-i)]=(-1)^{s-i-1}\cdot\frac{(s+j-1)!}{(s-i)!\cdot j!\cdot i!\cdot(s+j-i)}\cdot(s+j)\cdot(s-i)=$$

$$=(-1)^{s-i-1}\cdot\frac{(s+j)!}{(s+j-i)!\cdot i!}\cdot\frac{(s+j-i-1)!}{j!(s-i-1)!}=a_{(i+1)(j+1)}\quad\cdot\ \square$$

Now, (4.4) and (4.2) imply (3.6.i); (4.4) implies (3.6.iii); (4.5) and (4.2) imply (3.6.ii); and (4.5) implies (3.6.iv). ■

Example 4.2. For $s=1$, $m\in\mathbf{N}$, $a_{1(m+1)}=1$. So, any abelian group $(G,+)$ is a $(2m+1, m)$-group with $[x_1^m\ u\ y_1^m]=(x_1^m)+(y_1^m)-(u^m)$.

Example 4.3. For $s=2$, $m\in\mathbf{N}$, (4.2) holds for any divisor T of $m+2$. So, any $\mathbf{Z}_T$ module is a $(2m+2, m)$-group with $[x_1^m\ u\ v\ y_1^m]=x_i+y_i-ua_{1i}-va_{2i}$, where $[x_1^m\ u\ v\ y_1^m]_i$ is the i-th component of $[x_1^m\ u\ v\ y_1^m]$.

Example 4.4. For $s=3$, $m\in\mathbf{N}$, (4.2) holds for any divisor T of $m+3$ if $m+3$ is odd, and any divisor T of $\frac{m+3}{2}$ if $m+3$ is even.

Example 4.5. For $s=5$ and $m=7$, (4.2) holds only for $T=1$. So, the discussed procedure gives only trivial ($|G|=1$) (19, 7)-groups, and does not give answer to the existence question for nontrivial finite (19, 7)-groups.

The following fact is useful for showing existence of (m, n)-groups.

Fact 4.6. The existence of nontrivial (finite), (km, kn)-groups is equivalent to the existence of nontrivial (finite) (m, n)-groups.

Proof. If $(G, [\,])$ is a (km, kn)-group, then $(G^k, [\,]')$ is an (m, n)-group, where

$$[x_1^k\, x_{k+1}^{2k} \ldots x_{(m+1)k+1}^{mk}]'_i = ([x_1^{mk}]^{ki-k+1}, \ldots, [x_1^{mk}]_{ki}).$$

If $(G\, [\,])$ is an (m, n)-group, then $(G, [\,]')$ is a (km, kn)-group, where $[x_1^m\, y_1^m \ldots z_1^m]'_{ki-k+j} = [x_j\, y_j \ldots z_j]_i$. ∎

Example 4.7. For $s = 6$ and $m = 4$, (4.2) holds only for $T = 1$, and similarly as in **Example 4.5.**, **Proposition 4.1.** does not give direct construction of (14, 4)-groups, But (14, 4)-groups can be constructed by **Fact 4.6.** ($14 = 2 \cdot 7$, $6 = 2 \cdot 3$) and **Example 4.4.** ($s = 3$, $m = 2$).

Proposition 4.8. Let $m + s$ be prime number, Then (4.2) holds for $T = m + s$. So, any vector space over the field $\mathbf{Z}_{m+s}$ can be given a $(2m + s, m)$-group structure.

Proof. $$a_{1(m+1)} = (-1)^{s-1} \binom{s+m+1-1-1}{m+1-1} \cdot \binom{s+m+1-1}{1-1} =$$

$$= (-1)^{s-1} \binom{s+m-1}{m} = (-1)^{s-1} \frac{(s+m-1)!}{m!\ (s-1)!} =$$

$$= (-1)^{s-1} \frac{(s+m-1) \ldots (s+m-(s-1))}{(s-1)!} \equiv$$

$$\equiv (-1)^{s-1} \frac{(-1)(-2) \ldots (-(s-1))}{(s-1)!} \pmod{(m+s)} = 1.$$

For $i \neq 1$, $$a_{i(m+1)} = (-1)^{s-i} \binom{s+m+1-i-1}{m+1-1} \binom{s+m+1-1}{i-1} =$$

$$= (-1)^{s-i} \frac{(s+m-i)!}{m!\ (s-i)!} \frac{(s+m)!}{(s+m-i+1)!\ (i-1)!} \equiv 0 \pmod{(m+s)}. \ ∎$$

Example 4.9. For $s = 1$, $m = 2$, the system of equations (3.6) is: $(x_{11})^2 = x_{12}$, $x_{11} \cdot x_{12} = 1$. This system has three solutions over the field of complex numbers **C**, one of which is $x_{11} = -\dfrac{1+i\sqrt{3}}{2}$, $x_{12} = -\dfrac{1-i\sqrt{3}}{2}$.

So, any vector space G over **C** becomes a (5, 2)-group by

$$[u_1^5] = \left(u_1 + u_4 + u_3 \cdot \frac{1+i\sqrt{3}}{2},\ u_2 + u_5 + u_3 \frac{1-i\sqrt{3}}{2}\right).$$

5. Now we state some consequences about congruences of sums of binomial coefficient, obtained from the definition (4.3) of $[\ ]:G^{2m+s} \to G^m$ and its associativity. Let m, s, T be as in **Proposition 4.1.**, and let $G = \mathbf{Z}_T$. First, (4.3) can be stated as:

$$[x_1^{2m+s}]_i = x_i + x_{m+s+i} - \sum_{j=1}^{s} x_{m+j} \cdot a_{ji} .$$

Then:

$$[[x_1^{2m+s}]\, x_{2m+s+1}^{3m+2s}]_i = \tag{5.1}$$

$$= x_i + x_{m+s+i} + x_{2m+2s+i} - \sum_{j=1}^{s} (x_{m+j} + x_{2m+s+j})\, a_{ji} .$$

The associativity of [ ], tells us that for $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant m+s$

$$[x_1^t\, [x_{t+1}^{t+2m+s}]\, x_{t+2m+s+1}^{3m+2s}]_i = [[x_1^{2m+s}]\, x_{2m+s+1}^{3m+2s}]_i .$$

There are several casses to be discussed. Here we consider only the case: $s \leqslant m$, $t \leqslant s$ and $i \leqslant t$. The rest of the cases are left as an exercise for the reader. For $s \leqslant m$, $t \leqslant s$, $i \leqslant t$ we have:

$$[x_1^t\, [x_{t+1}^{t+2m+s}]\, x_{t+2m+s+1}^{3m+2s}]_i = x_i + x_{2m+2s+i} +$$

$$+ x_{m+s+i} \cdot \left(\sum_{k=1}^{t} a_{(s-t+i)\,(m-t+k)} \cdot a_{ki} \right) - \sum_{j=1}^{t} x_{m+j}\, a_{ji} - \tag{5.2}$$

$$- \sum_{j=1}^{s} x_{2m+s+j} \cdot a_{ji} + \sum_{r=1}^{s} x_{m+t+r} \left(\sum_{k=1}^{t} a_{r(m-t+k)} \cdot a_{ki} \right) .$$

Now, (5.1), (5.2), associativity of [ ], and the fact that $G = \mathbf{Z}_T$, imply that:

$$\sum_{k=1}^{t} a_{(s-t+i)(m-t+k)} \cdot a_{ki} \equiv 1 \pmod{T}; \tag{5.3}$$

$$\sum_{k=1}^{t} a_{r(m-t+k)} \cdot a_{ki} \equiv a_{(t+r)i} \pmod{T}, \text{ for } 1 \leqslant r \leqslant s-t; \tag{5.4}$$

$$\sum_{k=1}^{t} a_{r(m-t+k)} \cdot a_{ki} \equiv 0 \pmod{T}, \text{ for } s-t < r \leqslant s. \tag{5.5}$$

For example, when $m + s$ is a prime number, (5.3) can be stated as:

$$\sum_{k=1}^{t} (-1)^{s+t-i-k} \binom{m+k-i-i}{m-t+k-1} \binom{s+m-t+k-1}{s-t+i-1} \binom{s+i-k-1}{i-1} \binom{s+i-1}{k-1} \equiv$$

$$\equiv 1 \pmod{(m+s)}.$$

6. We finish this paper with the following questions. We point out that examples of nontrivial $(2m+s, m)$-groups are not found for all $m, s \in \mathbf{N}$.

Question 6.1. Do there exist nontrivial finite $(2m+s, m)$-groups, for any $m, s \in \mathbf{N}$? Specially, do there exist nontrivial finite (19, 7)-group?

Question 6.2. Do there exist F and a solution for the matrix equation (3.5) for any $m, s \in \mathbf{N}$?

Question 6.3. For given $m, s \in \mathbf{N}$, is there an algorithm for producing a ring F and a solution for the matrix equation (3.5)?

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ПРИМЕРИ НА ВЕКТОРСКО ВРЕДНОСНИ ГРУПИ

(Резиме)

Основна цел на оваа работа е да се дадат примери на $(2m+s, m)$ векторско вредносни групи со повеќе од еден елемент. На почетокот се дадени неколку доволни услови за постоење на такви групи, а потоа се конструирани конкретни примери. Користејќи ја структурата на $(2m+s, m)$-групи, добиени се неколку конгруенции помеѓу одреден вид од суми од биномни коефициенти.

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