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SUBALGEBRAS OF CANCELLATIVE SEMIGROUPS

Ġ.Čupona, S.Markovski, S.Crvenković, G.Vojvodić

Abstract. *A universal algebra (i.e. an Ω-algebra)* $\mathfrak{A}=(A;\Omega)$ *is called weakly cancellative if there exist a cancellative monoid* $\underline{M}$ *and a mapping* $\omega \longmapsto \hat{\omega}$ *from* Ω *into* $\bigcup\{M^{n+1} \mid n \geq 1\}$ *such that* $A \subseteq M$ *and*

$$\hat{\omega}=(\omega_0,\omega_1,\ldots,\omega_n),\ \omega(a_1,\ldots,a_n)=\omega_0 a_1 \omega_1 \ldots \omega_{n-1} a_n \omega_n,$$

for any n-ary operator $\omega \in \Omega$ *and* $a_1,\ldots,a_n \in A$. *If in addition: (i)* $\omega_k=1$ *(1 is the identity of the monoid) for any* $\omega \in \Omega$ *and* $k \geq 1$, *(ii)* $\omega_k=1$ *for any* $\omega \in \Omega$ *and* $k \geq 0$, *(iii) all operators are binary and* $\omega_0=\omega_2=1$ *for any* $\omega \in \Omega$ *– then* $(A;\Omega)$ *is termed: (i) a cancellative Ω-algebra, (ii) a cancellative Ω-associative, (iii) a cancellative polysemigroup-respectively. Convenient axiom systems for each of the quasivarieties of cancellative Ω-algebras, cancellative Ω-associatives and cancellative Ω-polysemigroups are given in this paper. A description of weakly cancellative Ω-algebras is given in* [10] *, and such a description is found in this paper as well, but both of them are not suitable enough.*

1. Cancellative universal algebras.

Let $\mathfrak{A}=(A;\Omega)$ be an Ω-algebra[1] with a carrier A. A semigroup $\underline{S}=(S;\cdot)$ is said to be a CR-*semigroup* (Cohn-Rebane semigroup) for $\mathfrak{A}$ if $A \subseteq S$ and there is a mapping $\omega \longmapsto \bar{\omega}$ from Ω into S such that

$$\omega(a_1,\ldots,a_n)=\bar{\omega}a_1\ldots a_n \qquad (1.1)$$

for any $\omega \in \Omega(n)$, $a_1,\ldots,a_n \in A$, $n \geq 1$. It is well known that any Ω-algebra admits a CR-semigroup ([2], [10]).

We say that an Ω-algebra $\mathfrak{A}$ is *cancellative* iff for any Ω-terms[1] $\xi'\eta_1\xi''$, $\xi'\eta_2\xi''$, $\zeta'\eta_1\zeta''$, $\zeta'\eta_2\zeta''$ the following quasiidentity[1] is satisfied in $\mathfrak{A}$:

$$\xi'\eta_1\xi''=\xi'\eta_2\xi'' \Longrightarrow \zeta'\eta_1\zeta''=\zeta'\eta_2\zeta''. \qquad (1.2)$$

Theorem 1.1. An Ω-algebra $\mathfrak{A}=(A;\Omega)$ is cancellative iff it admits a cancellative CR-semigroup.

Proof. Clearly, if some CR-semigroup of $\mathfrak{A}$ is cancellative, then $\mathfrak{A}$ is cancellative as well.

Let $\mathfrak{A}$ be a cancellative Ω-algebra and let $\phi \in \Omega(k)$, where $k \geq 2$. Consider the free semigroup $(A \cup \Omega)^+$ [2] on $A \cup \Omega$. An element $u \in (A \cup \Omega)^+$ is called Ω-*word* if $u \in A$ or $u=\omega u_1 u_2 \ldots u_n$, where $\omega \in \Omega(n)$ and $u_1,\ldots,u_n$ are Ω-words. The $\mathfrak{A}$-value $[u]$ of an Ω-word is defined in the usual way. Namely, if $u \in A$, then $[u]=u$, and $[u]=\omega(a_1,\ldots,a_n)$ if $u=\omega u_1 \ldots u_n$ and

$[u_i] = a_i$ for $i=1,\dots,n$. Ω-words can be characterized in another way. Define first the notion of the potency $p(u)$ of an element $u \in (A\cup\Omega)^+$. Namely, p is the homomorphism from $(A\cup\Omega)^+$ into the additive semigroup of integers which extends the mapping $p_0: A\cup\Omega \longrightarrow Z$ defined by $a\in A \Rightarrow p_0(a)=1$, $\omega \in \Omega(n) \Rightarrow p_0(\omega)=1-n$. Therefore, if $u,v\in (A\cup\Omega)^+$, then $p(uv)=p(u)+p(v)$. It is easy to show that an element $u\in(A\cup\Omega)^+$ is an Ω-word iff it satisfies the following two conditions:

$$p(u)=1 \qquad (1.3)$$

$$u=vw \Rightarrow p(v)\leqslant 0. \qquad (1.4)$$

The condition of cancellativity of $\mathfrak{a}$ is now equivalent with the following statement. If $u'uu''$, $u'vu''$, $v'uv''$, $v'vv''$ are Ω-words, then:

$$[u'uu''] = [u'vu''] \Longrightarrow [v'uv''] = [v'vv'']. \qquad (1.2')$$

Now, we are going to find a cancellative CR-semigroup of $\mathfrak{a}$.

Define a relation $\approx$ on $(A\cup\Omega)^+$ in the following way:

(*) $u\approx v$ iff there exist $u',u''\in(A\cup\Omega)^*$ [2] such that $u'uu''$, $u'vu''$ are Ω-words and $[u'uu''] = [u'vu'']$.

Clearly,

$$u\approx v \Longrightarrow p(u)=p(v). \qquad (1.5)$$

The relation $\approx$ is reflexive, for if $u\in(A\cup\Omega)^+$ and $a\in A$, then there exist $i,j\geqslant 0$ such that $\phi^i ua^j$ is an Ω-word. Clearly, $\approx$ is symmetric. Assume that $u\approx v$ and $v\approx w$, i.e. there exist $u',u'',v',v''\in(A\cup\Omega)^*$ such that $u'uu''$, $u'vu''$, $v'vv''$, $v'wv''$ are Ω-words and $[u'uu''] = [u'vu'']$, $[v'vv''] = [v'wv'']$. By (1.5), we have $p(u)=p(v)=p(w)$, and therefore if $a\in A$, then there exist $i,j\geqslant 0$ such that $\phi^i ua^j, \phi^i va^j, \phi^i wa^j$ are Ω-words. Now, by (1.2′) we obtain $[\phi^i ua^j]=[\phi^i va^j]=[\phi^i wa^j]$, i.e. $u\approx w$. Thus, $\approx$ is transitive.

Let $u\approx v$ and $w\in(A\cup\Omega)^+$. Therefore, there exist $u',u''\in(A\cup\Omega)^*$ such that $u'uu''$, $u'vu''$ are Ω-words with the same value. Let $a\in A$, and, by (1.5), we can choose $i,j\geqslant 0$ such that $\phi^i uwa^j, \phi^i vwa^j$ are Ω-words. By (1.2′) we have that $[\phi^i uwa^j]=[\phi^i vwa^j]$, and therefore $uw\approx vw$. Similarly, it can be shown that $wu\approx wv$. So, $\approx$ is a congruence on $(A\cup\Omega)^+$.

Let $uw\approx vw$. Then, there exist u', $u''\in(A\cup\Omega)^*$ such that $u'uwu''$, $u'vwu''$ are Ω-words with the same value. If $a\in A$, then there exist $i,j\geqslant 0$ such that $\phi^i ua^j$, $\phi^i va^j$ are Ω-words (by (1.5)), and by (1.2′) it follows $[\phi^i ua^j]=[\phi^i va^j]$, which implies $u\approx v$. Similarly, $wu\approx wv \Rightarrow u\approx v$.

Thus, we have shown that $(A\cup\Omega)^+/\approx$ is a cancellative semigroup.

If $a,b \in A$ are such that $a \approx b$, then there exist Ω-words $u'au''$, $u'bu''$ with the same value, and therefore $a=[a]=[b]=b$. So, we can assume that $A \subseteq (A \cup \Omega)^{+}/\approx$. Finally, if $a=\omega(a_1,\ldots,a_n)$ in $\mathfrak{A}$, then $a \approx \omega a_1 \ldots a_n$ in $(A \cup \Omega)^{+}$. Therefore, $\hat{\mathfrak{A}} = (A \cup \Omega)^{+}/\approx$ is a CR-semigroup for $\mathfrak{A}$.

Further on, if $\omega \in \Omega$, then we will denote by $\hat{\omega}$ the $\approx$-equivalence class containing ω. Thus, if $a=\omega(a_1,\ldots,a_n)$ in $\mathfrak{A}$, then $a=\hat{\omega}a_1\ldots a_n$ in $\hat{\mathfrak{A}}$.

There remains the case when $\Omega=\Omega(1)$ consists of unary operators only. The cancellativity of the (unary) Ω-algebra $\mathfrak{A}=(A;\Omega)$ means that the following quasiidentities are satisfied in $\mathfrak{A}$:

$$\begin{aligned}&\omega(x)=\omega(y) \Longrightarrow x=y,\\ &\omega_1\ldots\omega_p(x)=\tau_1\ldots\tau_q(x) \Longrightarrow \omega_1\ldots\omega_p(y)=\tau_1\ldots\tau_q(y)\end{aligned} \qquad (1.2'')$$

where $p,q \geq 0$, $\omega_\nu,\tau_\lambda \in \Omega$.

Certainly, it is possible that two different elements $\omega',\omega'' \in \Omega$ define the same transformation of A. We can, however, reduce Ω to a set Ω' in such a way that different elements of Ω' induce different transformations of A. Namely, we can define a relation $\equiv$ on Ω by $\omega \equiv \tau$ iff $(\exists x \in A)\,\omega(x)=\tau(x)$; by (1.2) we have $\omega \equiv \tau$ iff $(\forall x \in A)\,\omega(x)=\tau(x)$. Thus, we obtain that $\equiv$ is an equivalence on Ω, and we can define Ω' to be a subset of Ω which contains exactly one element from each $\equiv$-equivalence class. Therefore, we can assume that Ω' is a set of transformations of A such that (1.2) holds. Denote by Γ the monoid of transformations of A generated by Ω'. Now, the conditions (1.2'') reduce to the following ones:

$$\begin{aligned}&\omega(x)=\omega(y) \Longrightarrow x=y,\\ &\omega(x)=\tau(x) \Longrightarrow \omega=\tau.\end{aligned} \qquad (1.2''')$$

We notice that the unit of Γ is the identity transformation $I=1_A$: $a \mapsto a$. Denote by $\hat{A}^1$ the set $A^* \times \Gamma$ and define an operation $\bullet$ on $\hat{A}^1$ by

$$\begin{aligned}&(\underline{a},\omega)\bullet(b\underline{c},\tau)=(\underline{a}\omega(b)\underline{c},\tau)\\ &(\underline{a},\omega)\bullet(1,\tau)=(\underline{a},\omega\tau)\end{aligned} \qquad (1.6)$$

for any $\underline{a},\underline{c} \in A^*$, $b \in A$, $\omega,\tau \in \Gamma$. It is easy to check that $\hat{\mathfrak{A}}^1=(\hat{A}^1;\bullet)$ is a cancellative semigroup with an identity $(1,1)$ and that the mapping $\omega \mapsto (1,\omega)$ is an injective homomorphism from Γ into $\hat{\mathfrak{A}}^1$. Moreover, if $b=\omega(a)$, then we have $(1,\omega)\bullet(a,1)=(\omega(a),1)=(b,1)$. Thus, by identifying ω with (1,ω) and a with (a,1), we see that $A\cup\Gamma\subseteq\hat{A}^1$, and ω(a)=b in $\mathfrak{A}$ implies ωa=b. If $\omega\in\Omega$, then there is an $\omega'\in\Omega'$ such that $(\forall x\in A)\,\omega(x)=\omega'(x)$, and therefore we have a mapping $\omega\mapsto\omega'$ from Ω into $\hat{A}^1$ such that $(\forall x\in A)\,\omega(x)=\omega' x$. In such a way we obtain a CR-monoid[3)] $\hat{\mathfrak{A}}^1$ for $\mathfrak{A}$. Clearly, if $u,v\in\hat{A}^1$ and uv=(1,1), then u=v=(1,1), and thus the set $\hat{A}$ of nonidentity elements of $\hat{A}^1$ is a subsemigroup of $\hat{\mathfrak{A}}^1$. The semigroup $\hat{\mathfrak{A}}=(\hat{A};\cdot)$ is a CR-semigroup for $\mathfrak{A}$.

This completes the proof of Theorem 1.1.

Proposition 1.2. Let $\mathfrak{A}=(A;\Omega)$ be a cancellative Ω-algebra and $\hat{\mathfrak{A}}$ be the cancellative CR-semigroup for $\mathfrak{A}$ obtained in the previous proof. If $\underline{S}=(S;\cdot)$ is an arbitrary cancellative CR-semigroup for $\mathfrak{A}$, then there exists a (unique) homomorphism $\phi:\hat{\mathfrak{A}}\mapsto\underline{S}$ such that $\phi(a)=a$, $\phi(\hat{\omega})=\bar{\omega}$ for every $a\in A$, $\omega\in\Omega$.

Proof. Define a mapping $\phi_0: A\cup\Omega\mapsto S$ by $\phi_0(a)=a$, $\phi_0(\omega)=\bar{\omega}$ for every $a\in A$, $\omega\in\Omega$, and extend ϕ_0 to the unique homomorphism $\phi_1:(A\cup\Omega)^+\mapsto\underline{S}$. Then φ is the unique homomorphism from $\hat{\mathfrak{A}}$ into $\underline{S}$ such that $\phi\,\mathrm{nat}\approx\,=\phi_1$. We wish to show that φ is well defined, i.e. if $u,v\in(A\cup\Omega)^+$ and $u\approx v$, then $\phi_1(u)=\phi_1(v)$. Now, $u\approx v$ implies that there exist $u',u''\in(A\cup\Omega)^*$ such that $[u'uu'']=[u'vu'']$ in $\mathfrak{A}$. Thus we have

$$\phi_1(u')\phi_1(u)\phi_1(u'')=\phi_1(u'uu'')=\phi_0([u'uu''])=\phi_0([u'vu''])=$$
$$=\phi_1(u'vu'')=\phi_1(u')\phi_1(v)\phi_1(u''),$$

i.e. $\phi_1(u)=\phi_1(v)$.

Proposition 1.2 sugests that $\hat{\mathfrak{A}}$ is the *universal CR-semigroup* for $\mathfrak{A}$.

Proposition 1.3. Assume that $\Omega\neq\Omega(1)$. If $\mathfrak{A}$ is an Ω-algebra, then there is a CR-group[3)] for $\mathfrak{A}$ iff $\hat{\mathfrak{A}}$ is a subsemigroup of a group.

Proof. Certainly, if $\hat{\mathfrak{A}}$ is a subsemigroup of a group $\underline{G}$, then $\underline{G}$ is a CR-group for $\mathfrak{A}$.

Assume now that there is a CR-group for $\mathfrak{A}$, and consider the group $\underline{G}$ with the following presentation $\langle A\cup\Omega;\{a=\omega a_1\ldots a_n \mid a=\omega(a_1,\ldots,a_n)$ in $\mathfrak{A}\}\rangle$ in the variety of groups. Then $\underline{G}$ is a CR-group for $\mathfrak{A}$.

Let $F=A\cup A^{-1}\cup\Omega\cup\Omega^{-1}$, and define a relation ≡ in F^+ by $u\equiv v$ iff v can be obtained from u by a finite number of applications of the following types:

$u'cc^{-1}u''\text{—}uu''$, $uc^{-1}cu''\text{—}uu''$, $u'\omega a_1\ldots a_n u''\text{—}u'au''$

where $a=\omega(a_1\ldots a_n)$ in $\mathfrak{A}$, $c\in A\cup\Omega$, $u',u''\in F^+$. Clearly, $\underline{G}=F^+/\equiv$.

Extend the notion of potency. Namely, let $p:F^+\longrightarrow Z$ be the homomorphism from F^+ into additive semigroup of integers generated by $p(a)=1$, $p(\omega)=1-n$, $p(a^{-1})=-1$, $p(\omega^{-1})=n-1$, for any $a\in A,\omega\in\Omega(n)$. It is clear that $u\equiv v$ implies $p(u)=p(v)$. If $u,v\in(A\cup\Omega)^+$ and $u\approx v$, then $u\equiv v$, for $\underline{G}$ is a cancellative $\underline{C}\underline{R}$-semigroup for $\mathfrak{A}$.

Assume now that $u,v\in(A\cup\Omega)^+$ and $u\equiv v$. Then $p(u)=p(v)$, and therefore there exist $i,j\geqslant 0$ such that $\phi^i ua^j$, $\phi^i va^j$ are Ω-words, where $\phi\in\Omega(n)$ for some $n\geqslant 2$, $a\in A$. Also, from $u\equiv v$ it follows $\phi^i ua^j\equiv\phi^i va^j$, i.e. $[\phi^i ua^j]=[\phi^i va^j]$, which implies $u\approx v$.

Therefore, the mapping $a\longmapsto a,\omega\longmapsto\omega$ induces an injective homomorphism from $\hat{\mathfrak{A}}$ into $\underline{G}$, i.e. we can assume that $\hat{\mathfrak{A}}$ is a subsemigroup of the group $\underline{G}$.

This completes the proof.

We notice that it is not known whether the same result holds in the case when $\Omega=\Omega(1)$ consists of unary operators only.

The set $\mathcal{O}(A)=\{\mathcal{O}_n(A)\ |n\geqslant 1\}$ of (non-nullary) finitary operations on a set A is a semigroup under the usual operation $\circ$ of superposition, i.e.

$$\omega\circ\tau(x_1,\ldots,x_{m+n-1})=\omega(\tau(x_1,\ldots,x_m),x_{m+1},\ldots,x_{m+n-1}) \qquad (1.7)$$

where $\omega\in\mathcal{O}_n(A)$, $\tau\in\mathcal{O}_m(A)$. Any subsemigroup Γ of the semigroup $(\mathcal{O}(A),\circ)$ is called a *semigroup of operations* on A. In such a way we get a universal algebra (A,Γ). We say that Γ is a *cancellative semigroup of operations* if Γ is cancellative and (A,Γ) is a cancellative Γ-algebra. The following result is stated in[5].

Proposition 1.4. A semigroup Γ of operations on a set A is a cancellative semigroup of operations iff there is a semigroup $\underline{S}$ and an injective homomorphism $\Gamma\longrightarrow\bar{\Gamma}$ from Γ into $\underline{S}$ such that $A\subseteq S$ and (1.1) is satisfied for any $a_1,\ldots,a_n\in A$ and any n-ary operation $\omega\in\Gamma$.

Proof. We can assume that $1_A\in\Gamma$, i.e. that Γ is a monoid. The case when Γ is a semigroup of transformations on A is considered in the second part of the proof of Theorem 1.1.

If there is a $k\geqslant 2$ and a k-ary operation $\phi\in\Gamma$, then we consider first the monoid $\underline{M}=A^*\sqcup\Gamma$ which is the free product of A^* and Γ in the class of monoids. Then we define the notion of Γ-words similarly as in the proof of the first part of Theorem 1.1. Finally, we define in the same manner the corresponding congruence $\approx$ and we obtain the desired cancellative semigroup $\underline{S}=\underline{M}/\approx$.

2. Cancellative associatives.

An Ω-algebra $\mathfrak{A}=(A;\Omega)$ is said to be an Ω-associative if it satisfies any identity $\xi=\eta$ for any Ω-terms ξ,η which admit the same sequences of variables, i.e. $V(\xi)=V(\eta)$. An Ω-algebra $\mathfrak{A}=(A;\Omega)$ is called an *Ω-subassociative of a semigroup* $\underline{S}$ if $A\subseteq S$ and

$$\omega(a_1,\ldots,a_n)=a_1\ldots a_n \tag{2.1}$$

for any $\omega\in\Omega(n)$, $n\geqslant 1$, $a_i\in A$. We notice that the class of Ω-subassociatives of semigroups is a quasivariety, which is a subquasivariety of the variety of Ω-associatives. (Different problems concerning Ω-associatives are treated in several papers, and a review of these papers can be found in [1]). The main result of this part of the paper is the following

Theorem 2.1. An Ω-associative $\mathfrak{A}$ is an Ω-subassociative of a cancellative semigroup iff $\mathfrak{A}$ is a cancellative Ω-associative.

Proof. Clearly, if $\mathfrak{A}$ is an Ω-subassociative of a cancellative semigroup , then $\mathfrak{A}$ is a cancellative Ω-associative.

1.) Let $\mathfrak{A}=(A;\Omega)$ be an Ω-associative. If $n\geqslant 1$ and $\omega,\tau\in\Omega(n)$, then we have

$$\omega(a_1,\ldots,a_n)=\tau(a_1,\ldots,a_n)$$

for every $a_1,\ldots,a_n\in A$, and thus we can assume that $|\Omega(n)|\leqslant 1$. Specially, if $\omega\in\Omega(1)$, then $\omega(a)=a$ for every $a\in A$, i.e. $\omega=1_A$.

If $\Omega=\Omega(1)$, then $\mathfrak{A}$ is an Ω-subassociative of any cancellative semigroup $\underline{S}$ generated by A. For this reason we can assume that $\Omega\neq\Omega(1)$

Denote by J_Ω the subset of positive integers

$$J_\Omega=\{n-1 \mid \Omega(n)\neq\emptyset,\ n\geqslant 1\},$$

and let d_Ω be the greatest common divisor of J_Ω. Further, let $K=K_\Omega$ be the additive semigroup generated by J_Ω and let $K^0=K_\Omega\cup\{0\}$.

We can now define m+1-ary operation $[\quad]_m$ on A for any $m\in K^0$ in the following inductive way:

$$[a]_0=a,$$

$$m=m_1+m_2, m_1,m_2\in K^0 \Rightarrow [a_0\ldots a_m]_m=[[a_0\ldots a_{m_1}]_{m_1} a_{m_1+1}\ldots a_{m_2}]_{m_2}.$$

The fact that $\mathfrak{A}$ is an Ω-associative implies that $[\quad]^2_m$ is well defined and moreover if $m=m_0+m_1+\ldots+m_p\in K^0$, $m_\nu\in K^0$, $p\in K^0$, then

$$[a_0 \ldots a_m]_m = [[a_0 \ldots a_{m_0}]_{m_0} \ldots [\ldots a_m]_{m_p}]_p \, .$$

This implies that we can write $[\ \ ]$ instead of $[\ \ ]_m$. Also, instead of "an Ω-associative", further on, we shall use " a K-associative".

Now, the condition of cancellativity has following form:

$$[\underline{x}'\underline{z}_1\underline{x}''] = [\underline{x}'\underline{z}_2\underline{x}''] \Longrightarrow [\underline{y}'\underline{z}_1\underline{y}''] = [\underline{y}'\underline{z}_2\underline{y}''] \qquad (2.2)$$

for any $\underline{x}', \underline{x}'', \underline{y}', \underline{y}'', \underline{z}_1, \underline{z}_2 \in A^*$ such that

$|\underline{x}'| + |\underline{z}_1| + |\underline{x}''|$, $|\underline{x}'| + |\underline{z}_2| + |\underline{x}''|$, $|\underline{y}'| + |\underline{z}_1| + |\underline{y}''|$,
$|\underline{y}'| + |\underline{z}_2| + |\underline{y}''| \in K^0+1$.

We note that there is an $s_0 \geqslant 1$ such that $s \geqslant s_0 \Longrightarrow sd_\Omega \in K$,([8] ,[9])

2.) Assume now that $\mathfrak{a} = (A;\Omega)$ is a cancellative Ω-associative and $K = K_\Omega$, $d = d_\Omega$ are defined as above, i.e. $\mathfrak{a}$ is an cancellative K-assocative.

Consider the free semigroup A^+ on A. If $u \in A^+$ and $|u| \in K^0+1$, then $[u]$ is a uniquely determined element of A, i.e. $[\ \ ]$ is a partial mapping of A^+ in A. Define a relation $\approx$ on A^+ in the following way:

$$u \approx v \Longleftrightarrow (\exists i \geqslant 0, a \in A)\,(|u|+i, |v|+i \in K^0+1, [a^i u] = [a^i v]). \qquad (2.3)$$

By (2.2) we have

$$u \approx v \Longleftrightarrow (\forall i \geqslant 0, a \in A)\,(|u|+i, |v|+i \in K^0+1, 0 \leqslant j \leqslant i \Longrightarrow$$
$$\Longrightarrow [a^j u a^{i-j}] = [a^j v a^{i-j}]). \qquad (2.3')$$

Clearly, $\approx$ is reflexive and symmetric. Also, notice that if $u \approx v$, then $|u| \equiv |v| \pmod d$. Now, let $u \approx v$, $v \approx w$, and let $|u| \equiv |v| \equiv |w| \equiv i \pmod d$ for $0 \leqslant i \leqslant d$. Then there is some $s \geqslant 1$, such that $|u| + d - i + sd + 1$, $|v| + d - i + sd + 1$, $|w| + d - i + sd + 1 \in K^0+1$, and for any $a \in A$ we have

$$[a^{d-i+sd+1}u] = [a^{d-i+sd+1}v] = [a^{d-i+sd+1}w],$$

i.e. $u \approx w$. Thus, $\approx$ is transitive.

In the same manner it can be easily seen that $\approx$ is a congruence on A^+ and that $A^+/\approx\, = \hat{\mathfrak{a}}$ is a cancellative semigroup.

Let $b, c \in A$ and $b \approx c$. Then, for $a \in A$, $[a^{sd}b] = [a^{sd}c]$, which implies, by (2.2), that $b = c$. Thus, we can assume that $A \subseteq \hat{\mathfrak{a}}$.

Finally, if $a = [a_0 \ldots a_m]$ in $\mathfrak{a}$, then $a \approx a_0 \ldots a_m$, i.e. $a = a_0 \ldots a_m$ in $\hat{\mathfrak{a}}$.

This completes the proof of Theorem 2.[illegible].

We notice that in the definition of cancellative Ω-associative we have an infinite set of defining quasiidentities. But, in fact, all those quasiidentities are consequences of only one (or two) quasiidentity, as the following proposition shows.

Proposition 2.2. Let $\mathfrak{A}=(A;\Omega)$ be a K-associative, and let $n\in K$, $n\geqslant 2$. The following statements are equivalent:

(i) $\mathfrak{A}$ is cancellative.

(ii) The quasiidentities

$$[xz_1\ldots z_n]=[yz_1\ldots z_n]\Rightarrow x=y$$
$$[z_1\ldots z_nx]=[z_1\ \ldots z_ny]\Rightarrow x=y$$

are true in $\mathfrak{A}$.

(iii) There is an $r\in\{2,\ldots,n\}$ such that the quasiidentity

$$[z_1\ldots z_{r-1}xz_r\ldots z_n]=[z_1\ldots z_{r-1}yz_r\ldots z_n]\Rightarrow x=y$$

is true in $\mathfrak{A}$.

The cancellative semigroup $\hat{\mathfrak{A}}$ obtained in the proof of Theorem 2.1 is called *the universal cancellative covering semigroup* for the K-associative $\mathfrak{A}$. Namely, the following proposition can be proved in the same way as Proposition 1.2.

Proposition 2.3. If $\mathfrak{A}$ is an Ω-subassociative of a cancellative semigroup $S=(S;\cdot)$, then the identity mapping $a\mapsto a$ can be extended in a unique way to a homomorphism $\phi:\hat{\mathfrak{A}}\longmapsto S$.

The following analogy of Proposition 1.3 can be also proved in the same way.

Proposition 2.4. If $\mathfrak{A}$ is a cancellative Ω-associative, then it is an Ω-subassociative of a group iff $\hat{\mathfrak{A}}$ is a subsemigroup of a group.

3. Cancellative polysemigroups

An Ω-algebra $\mathfrak{A}=(A;\Omega)$ with binary operators only is said to be a *polysemigroup* iff the following equation,

$$\omega(a,\tau(b,c))=\tau(\omega(a,b),c), \qquad (3.1)$$

is satisfied for any $\omega,\tau\in\Omega$, $a,b,c\in A$. Then we will write $x\omega y$ instead of $\omega(x,y)$. It can be easily seen, by a usual induction, that if $a_0,a_1,\ldots,a_n\in A,\omega_1,\ldots,\omega_n\in\Omega$, then the "continued product" $a_0\omega_1a_1\omega_2\ldots a_{n-1}\omega_na_n$ is uniquely defined. A polysemigroup $\mathfrak{A}=(A;\Omega)$ is said to be *cancellative* iff the following quasiidentities are satisfied in $\mathfrak{A}$ for any $\omega,\tau\in\Omega$:

$$x\omega y=x\omega z\Rightarrow y=z,$$
$$y\omega x=z\omega x\Rightarrow y=z,$$
$$x\omega z=y\tau z\Rightarrow x\omega z'=y\tau z', \qquad (3.2)$$
$$x\omega y=x\tau z\Rightarrow x'\omega y=x'\tau z.$$

We say that a polysemigroup $\mathfrak{A}=(A;\Omega)$ is a *polysubsemigroup* *of a semigroup* $\underline{S}=(S;\cdot)$ if $A\subseteq S$ and there is a mapping $\omega\mapsto\bar{\omega}$ of Ω into S such that

$$a\omega b=a\cdot\bar{\omega}\cdot b \tag{3.3}$$

for any $a,b\in A,\omega\in\Omega$. It is known ([3], [6]) that any polysemigroup is a polysubsemigroup of a semigroup.

<u>Theorem 3.1. A polysemigroup is cancellative iff it is a polysubsemigroup of a cancellative semigroup.</u>

<u>Proof.</u> It is clear that if $\mathfrak{A}=(A;\Omega)$ is a polysubsemigroup of a cancellative semigroup, then $\mathfrak{A}$ is a cancellative polysemigroup.

Assume now that $(A;\Omega)$ is a cancellative polysemigroup and let $\omega\mapsto\bar{\omega}$ be a bijection from Ω into $\bar{\Omega}=\{\bar{\omega}|\omega\in\Omega\}$, where $A\cap\bar{\Omega}=\emptyset=\Omega\cap\bar{\Omega}$. Consider the free semigroup $(A\cup\bar{\Omega})^+$ on $A\cup\bar{\Omega}$. Every element $u=a_0\bar{\omega}_1a_1\ldots\bar{\omega}_na_n\in(a\cup\bar{\Omega})^+$, where $\omega_\nu\in\Omega$, $a_\nu\in A$, is called an Ω-word. Then, $[u]=a_0\omega_1a_1\ldots\omega_na_n\in A$ is uniquely determined value of u in $\mathfrak{A}$. Clearly, an element $v=c_1c_2\ldots c_p\in(A\cup\bar{\Omega})^+$ is a "subword" of an Ω-word iff for any $i\in\{2,3,\ldots,p-1\}$, the following implications hold:

$c_1\in A\Rightarrow c_2\in\bar{\Omega}$; $c_1\in\bar{\Omega}\Rightarrow c_2\in A$; $(p\geqslant 2)$

$c_i\in A\Rightarrow c_{i-1},c_{i+1}\in\bar{\Omega}$; $c_i\in\bar{\Omega}\Rightarrow c_{i-1},c_{i+1}\in A$.

It is also clear that if $u\in(A\cup\bar{\Omega})^+$, then there exists a unique sequence $u_1,u_2,\ldots,u_p$ such that $u=u_1u_2\ldots u_p$, and $u_1,\ldots,u_p$ are maximal subwords of u where each of them is a subword of an Ω-word. Then we say that $u_1\ldots u_p$ is the canonical decomposition of u.

Define a relation $\approx$ on $(A\cup\bar{\Omega})^+$ in the following way.

Let $u=u_1\ldots u_m$, $v=v_1\ldots v_m$ be the canonical decompositions of u,v respectively. Then:

$u\approx v$ iff m=n and there exist u'_ν,u''_ν such that $u'_\nu u_\nu u''_\nu$, $u'_\nu v_\nu u''_\nu$ are Ω-words and $[u'_\nu u_\nu u''_\nu]=[u'_\nu v_\nu u''_\nu]$, is satisfied for $\nu\in\{1,2,\ldots,m\}$.

It can be shown that $\approx$ is a congruence on $(A\cup\bar{\Omega})^+$ such that $\hat{\mathfrak{A}}=(A\cup\bar{\Omega})^+/\approx$ is a cancellative semigroup. Moreover, if $a,b\in A$ and $a\approx b$, then it follows from (3.2) that a=b, and $a=b\omega c$ in $\mathfrak{A}$ implies $a\approx b\bar{\omega}c$, which means that $\mathfrak{A}$ is a polysubsemigroup of $\hat{\mathfrak{A}}$.

It is clear that the semigroup $\hat{\mathfrak{A}}$ has the corresponding property of "universality".

The answer to the question "whether it is true that any cancellative polysemigroup is a cancellative universal algebra" is negative. Namely, if a polysemigroup $\mathfrak{A}=(A;\Omega)$ is cancellative as a universal algebra, then it has to satisfy, for example, the following quasiidentity:

$$\omega(x_1,x_2)=\omega(y_1,y_2)\Longrightarrow\tau(x_1,x_2)=\tau(y_1,y_2),$$

i.e.

$$x_1\omega x_2=y_1\omega y_2\Longrightarrow x_1\tau x_2=y_1\tau y_2,$$

for any $\omega,\tau\in\Omega$. Consider the symmetric group $\mathcal{S}_3$ on $\{1,2,3\}$, and put $\omega=(1)$, $\tau=(12)$. Then we have

$$(12)(1)(12)=(13)(1)(13)\text{ but }(12)(12)(12)=(12),$$
$$(13)(12)(13)=(23).$$

Now, if $\mathfrak{A}=\mathcal{S}_3$, $\Omega=\{\omega,\tau\}$ and $\omega(x,y)=xy$, $\tau(x,y)=x(12)y$, then the algebra $(A;\omega,\tau)$ is a cancellative universal algebra.

4. Weakly cancellative universal algebras

Let $\underline{S}=(S;\cdot)$ be a semigroup, and Ω be an operator domain. Assume also that $\omega\mapsto\overline{\omega}=(\overline{\omega}_0,\ldots,\overline{\omega}_n)$ is a mapping from $\Omega(n)$ into S^{n+1} for any $n\geqslant 1$. An Ω-algebra $(S;\Omega)$ can be defined as follows:

$$\omega(a_1,\ldots,a_n)=\overline{\omega}_0a_1\overline{\omega}_1a_2\ldots\overline{\omega}_{n-1}a_n\overline{\omega}_n \tag{4.1}$$

for any $n\geqslant 1,\omega\in\Omega(n)$, $a_1,\ldots,a_n\in S$. Every subalgebra $\mathfrak{A}=(A;\Omega)$ of the algebra $(S;\Omega)$ is said to be a *polylinear subalgebra* of the given semigroup.

An Ω-algebra $\mathfrak{A}=(A;\Omega)$ is called *weakly cancellative* iff it is a polylinear subalgebra of a cancellative semigroup. Cancellative universal algebras and cancellative polysemigroup are special kinds of weakly cancellative universal algebras.

A universal polylinear covering semigroup can be associated to any universal algebra in the following way.

Let $\mathfrak{A}=(A;\Omega)$ be an Ω-algebra, and define a set $\overline{\Omega}$ by

$$\overline{\Omega}=\{(\omega,i)\,|\,\omega\in\Omega(n),\ n\geqslant 1,\ 0\leqslant i\leqslant n\}. \tag{4.2}$$

Further on, we shall write ω_i instead of (ω,i). Consider the free semigroup $(A\cup\overline{\Omega})^+$ on $A\cup\overline{\Omega}$, and we define the notion of an Ω-word and its value in $\mathfrak{A}$ as follows. First, if $u\in A$, then u is an Ω-word with a value $[u]=u$. If $u\in(A\cup\overline{\Omega})^+$ and $u\notin A$, then u is an Ω-word iff $u=\omega_0u_1\ldots\omega_{n-1}u_n\omega_n$, where $\omega\in\Omega(n)$, $u_1,u_2,\ldots,u_n$ are Ω-words with values $[u_1]=a_1,\ldots,[u_n]=a_n$, and the value of u is

$$[u]=\omega(a_1,\ldots,a_n).$$

The notion of maximal Ω-subwords of words (i.e. of elements of $(A\cup\overline{\Omega})^+$) is clear.

It can be easily shown ([6]) that if $u \in (A \cup \bar{\Omega})^+$, then u can be represented uniquely in the following form

$$u=\alpha_0 u_1 \alpha_1 u_2 \ldots \alpha_{p-1} u_p \alpha_p \tag{4.3}$$

where $\alpha_\nu \in \bar{\Omega}^*$, $p \geq 0$, and $u_1, \ldots, u_p$ are maximal Ω-subwords of u. We say that (4.3) is the canonical factorization of u. Let $v \in (A \cup \bar{\Omega})^+$ have the following canonical factorization

$$v=\beta_0 v_1 \beta_1 \ldots v_q \beta_q . \tag{4.3'}$$

then

$$u \approx v \Leftrightarrow p=q,\ \alpha_\nu=\beta_\nu, [u_\nu] = [v_\nu].$$

It can be shown ([6]) that $\approx$ is a congruence on $(A \cup \bar{\Omega})^+$, and it is clear that

$$a=\omega(a_1,\ldots,a_n) \text{ in } \mathfrak{A} \Rightarrow a \approx \bar{\omega}_0 a_1 \bar{\omega}_1 \ldots a_n \bar{\omega}_n,$$
$$a,b \in A \Rightarrow (a \approx b \Rightarrow a=b), \tag{4.4}$$

which implies that $\mathfrak{A}$ can be embedded as a polylinear subalgebra in the semigroup $\hat{\mathfrak{A}}=(A \cup \bar{\Omega})^+/\approx$. We say that $\hat{\mathfrak{A}}$ is the *universal polylinear covering semigroup* of the given Ω-algebra $\mathfrak{A}$.

The following assertion is clear:

Theorem 4.1. Let $\hat{\mathfrak{A}}$ be the universal polylinear covering semigroup of a universal Ω-algebra $\mathfrak{A}=(A;\Omega)$, and let $\equiv$ be the least congruence on $\hat{\mathfrak{A}}$ such that $\hat{\mathfrak{A}}/\equiv$ be cancellative. Then $\mathfrak{A}$ is a weakly cancellative Ω-algebra iff the following implication is satisfied:

$$a,b \in A \Rightarrow (a \equiv b \Rightarrow a=b). \tag{4.5}$$

We point that Theorem 4.1 is not the most satisfactory description of the quasivariety of weakly cancellative universal algebras, although a description of relation $\equiv$ is known ([1]);namely, we do not know an axiom system for that quasivariety. Rebane's description is not explicit enough either.

Proposition 4.2. If an Ω-algebra $\mathfrak{A}=(A;\Omega)$ is weakly cancellative and if $u,v \in (A \cup \bar{\Omega})^+$, $u',u'',v',v'' \in (A \cup \bar{\Omega})^*$ are such that $u'uu''$, $u'vu''$, $v'uv''$, $v'vv''$ are Ω-words, then the following implication is tru in $\mathfrak{A}$:

$$[u'uu'']=[u'vu''] \Rightarrow [v'uv'']=[v'vv''] . \tag{4.6}$$

Remark that in each of the special cases considered in the first three parts of the paper, the corresponding quasivarieties were defined with quasiidentities of the form (4.6)(together with a set of identities in 2 and 3).

Proposition 4.3. Let $\Omega=\Omega(1)$ consist of unary operators only. An Ω-algebra $\mathfrak{A}=(A;\Omega)$ is weakly cancellative iff it satisfies the following quasiidentity

$$\omega(x)=\omega(y)\Rightarrow x=y \qquad (4.7)$$

for any $\omega\in\Omega$, i.e. iff the transformations on A induced by Ω are injective.

Proof. Assume that any quasiidentity (4.7) holds, and let $\hat{\mathfrak{A}}$ be the universal polylinear covering semigroup of $\mathfrak{A}$. We are going to show that $\mathfrak{A}$ is cancellative.

Define a reduced form $\bar{u}$ of a word $u\in(A\cup\bar{\Omega})^+$ as follows. Let $u=\alpha_0u_1\alpha_1\ldots u_p\alpha_p$ be the canonical factorization of u, and $[u_i]=a_i$ for $i\in\{1,2,\ldots,p\}$. Then

$$\bar{u}=\alpha_0a_1\alpha_1\ldots a_p\alpha_p.$$

It is easily seen that if $u,v\in(A\cup\bar{\Omega})^+$, then

$$u\approx v\Leftrightarrow\bar{u}=\bar{v},\quad \overline{uv}=\overline{\bar{u}\bar{v}}. \qquad (4.8)$$

Now, it follows from (4.8) that it is enough to show if $u,v\in(a\cup\bar{\Omega})^+$, $w\in A\cup\bar{\Omega}$, then

$$\overline{uw}=\overline{vw}\Rightarrow\bar{u}=\bar{v},\quad \overline{wu}=\overline{wv}\Rightarrow\bar{u}=\bar{v}.$$

The only nontrivial cases are $w=\omega_1$ for the first implication, and $w=\omega_0$ for the second one, where $\omega\in\Omega$.

If $\bar{u}=u'\omega_0a$, $\bar{v}=v\omega_0b$, where $a,b\in A$, it follows from $\overline{u\omega_1}=\overline{v\omega_1}$ that $u'=v'$ and $\omega(a)=\omega(b)$. But, we have from (4.7) that $a=b$, i.e. $\bar{u}=\bar{v}$.

The second case is treated in the same manner, i.e. we obtain that $\hat{\mathfrak{A}}$ is cancellative.

5. Subalgebras of commutative cancellative semigroups

We can identify the cancellative and the weakly cancellative universal algebras when commutative cancellative semigroups are considered. A universal Ω-algebra $\mathfrak{A}=(A;\Omega)$ is said to be *commutative* iff any equality $\xi=\eta$, where ξ,η are Ω-terms such that any symbol $z\in A\cup\Omega$ occurs equal number of times in ξ and η, is an identity in $\mathfrak{A}$.

Theorem 5.1. An Ω-algebra $\mathfrak{A}=(A;\Omega)$ is commutative and cancellative iff it admits a commutative cancellative CR-semigroup.

Proof. Let $\Omega\neq\Omega(1)$, i.e. Ω does not consist of unary operators only. Assume that C is the free commutative semigroup generated by the sat

$A \cup \Omega$, and define a relation $\equiv$ on C by

$$u \equiv v \Leftrightarrow (\exists w \in C)\,[uw]=[vw],$$

where [ ] denotes the partial operation of evaluation of some elements of C in $\mathfrak{A}$. (If $u,v \in (A \cup \Omega)^{+}$, v is an Ω-word defined as in the proof of Theorem 1.1 and u=v in $\underline{C}$, then $[u]=[v]$). Now, it can be easily seen that $\equiv$ is a congruence on $\underline{C}$ such that $\underline{S}=\underline{C}/\equiv$ is a commutative cancellative CR-semigroup of $\mathfrak{A}$.

Consider now the case when Ω consists of unary operators only. If $\mathfrak{A}=(A;\Omega)$ admits a commutative cancellative CR-semigroup, then the formulas

$$\begin{aligned} &\omega_1\omega_2(x)=\omega_2\omega_1(x), \\ &\omega(x)=\omega(y) \Rightarrow x=y, \\ &\omega_1\dots\omega_p(x)=\tau_1\dots\tau_q(x) \Rightarrow \omega_1\dots\omega_p(y)=\tau_1\dots\tau_q(y) \end{aligned} \tag{5.1}$$

are true in $\mathfrak{A}$.

Suppose now that the formulas (5.1) are true in $\mathfrak{A}$, and let $\omega \mapsto \bar{\omega}$ be a bijection of Ω onto $\bar{\Omega}=\{\bar{\omega} \mid \omega \in \Omega\}$. We are going to show that there is a commutative CR-group for $\mathfrak{A}$. For proving the preceding statement, we will use the model theoretical methods, rather than the algebraic methods used till now. Therefore, let $\mathcal{T}$ be a first order theory with the language $\Omega'=\{\cdot, {}^{-1}, e\} \cup \Omega \cup \bar{\Omega}$, where $\cdot$ is a binary, ${}^{-1}$ and Ω are unary, e and $\bar{\Omega}$ are nullary operators, and with nonlogical axioms $x(yz)=(xy)z$, $xy=yx$, $xx^{-1}=e$, $\omega(x)=\bar{\omega}x$, for every $\omega \in \Omega$.

It is known ([9],[7]) that a universal algebra $\mathfrak{A}=(A;\Omega)$ admits a commutative CR-group iff every quasiidentity in $\mathcal{T}$, which is in the language Ω, holds in $\mathfrak{A}$. Let $\omega_\nu \in \Omega$, $\alpha_j, \beta_s, \alpha_{ij}, \beta_{rs}$ be nonnegative integers, and consider a quasiidentity in $\mathcal{T}$:

$$\begin{aligned} &\omega_1^{\alpha_{11}}\dots\omega_p^{\alpha_{1p}}(x_{i_1})=\omega_1^{\beta_{11}}\dots\omega_p^{\beta_{1p}}(x_{j_1})\,\&\dots \\ &\dots\&\ \omega_1^{\alpha_{k1}}\dots\omega_p^{\alpha_{kp}}(x_{i_k})=\omega_1^{\beta_{k1}}\dots\omega_p^{\beta_{kp}}(x_{j_k}) \Longrightarrow \\ &\Longrightarrow \omega_1^{\alpha_1}\dots\omega_p^{\alpha_p}(x_i)=\omega_1^{\beta_1}\dots\omega_p^{\omega_p}(x_j). \end{aligned} \tag{5.2}$$

Then

$$\bar{\omega}_1^{\alpha_{11}}\ldots\bar{\omega}_p^{\alpha_{1p}}\,x_{i_1}=\bar{\omega}_1^{\beta_{11}}\ldots\bar{\omega}_p^{\beta_{1p}}\,x_{j_1}\ \&\ldots$$
$$\ldots\&\ \bar{\omega}_1^{\alpha_{k1}}\ldots\bar{\omega}_p^{\alpha_{kp}}\,x_{i_k}=\bar{\omega}_1^{\beta_{k1}}\ldots\bar{\omega}_p^{\beta_{kp}}\,x_{j_k}\Longrightarrow$$
$$\Longrightarrow\bar{\omega}_1^{\alpha_1}\ldots\bar{\omega}_p^{\alpha_p}\,x_i=\bar{\omega}_1^{\beta_1}\ldots\bar{\omega}_p^{\beta_p}\,x_j$$

is a quasiidentity in the variety of commutative groups, which can be written as

$$\bar{\omega}_1^{\gamma_{11}}\ldots\bar{\omega}_p^{\gamma_{1p}}\,x_{i_1}x_{j_1}^{-1}=e\ \&\ldots\&\ \bar{\omega}_1^{\gamma_{k1}}\ldots\bar{\omega}_p^{\gamma_{kp}}\,x_{i_k}x_{j_k}^{-1}=e\Longrightarrow$$
$$\Longrightarrow\bar{\omega}_1^{\gamma_1}\ldots\bar{\omega}_p^{\gamma_p}\,x_ix_j^{-1}=e, \tag{5.3}$$

where $\gamma_{\nu\lambda}$ are integers.

The meaning of (5.3) is that the consequence in the last implication is an equality in the commutative group given by the presentation

$$<\bar{\omega}_1,\ldots,\bar{\omega}_p,x_i,x_j,x_{i_1},\ldots,x_{i_k},x_{j_1},\ldots,x_{j_k}\,|\,\bar{\omega}_1^{\gamma_{11}}\ldots\bar{\omega}_p^{\gamma_{1p}}\,x_{i_1}x_{j_1}^{-1},\ldots$$
$$\ldots,\bar{\omega}_1^{\gamma_{k1}}\ldots\bar{\omega}_p^{\gamma_{kp}}\,x_{i_k}x_{j_k}^{-1}>$$

in the variety of commutative groups. Therefore,

$$\bar{\omega}_1^{\gamma_1}\ldots\bar{\omega}_p^{\gamma_p}\,x_ix_j^{-1}$$

is obtained from the defining words by multiplications and inversions. It is clear that inverse corresponds to the changing of the left-hand side with the right-hand side in some of the equalites in (5.2). Also, there are some ν_μ,λ_μ such that

$$x_ix_j^{-1}=(x_{\nu_1}x_{\lambda_1}^{-1})\ldots(x_{\nu_m}x_{\lambda_m}^{-1}).$$

This means that

$$x_ix_j^{-1}=(x_ix_{\mu_1}^{-1})(x_{\mu_1}x_{\mu_2}^{-1})\ldots(x_{\mu_{r-1}}x_{\mu_r}^{-1})(x_{\mu_r}x_j^{-1}) \tag{5.4}$$

for some $\mu_1,\ldots,\mu_r$. Now, we can construct (5.4) such that some of the equalites in (5.2) would be multiplied by coresponding strings of operations, and we can use the transitive law after that.

Thus, we conclude that (5.2) is a quasiidentity which is true in $\mathfrak{A}$, and that completes the proof.

If $\mathfrak{A}$ is a commutative and cancellative Ω-associative, then it can be easily shown that its universal cancellative covering semigroup $\hat{\mathfrak{A}}$ is commutative as well, which implies the following

Theorem 5.2. An Ω-associative is an Ω-subassociative of a commutative group iff it is a cancellative and commutative Ω-associative.

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1) $\{\Omega(n)\,|\,n\geqslant 1\}$ is a disjoint family of operators $\Omega=\bigcup\{\Omega(n)\,|\,n\geqslant 1\}$, and any n-ary operator $\omega\in\Omega$ induces an n-operation (denoted also by ω) on the carrier of a given algebra. Let $X=\{x_1,\ldots,x_n,\ldots\}$ be an infinite denumerable set of elements which are called variables; the set TermΩ of Ω-terms is defined recursively by: (i) $X\subseteq$ TermΩ, (ii) If ξ is a nonempty finite sequence on $X\cup\Omega$ and $\xi\notin X$, then $\xi\in$ TermΩ iff $\xi=\omega\xi_1\ldots\xi_n$, where $\omega\in\Omega(n)$ and $\xi_1,\ldots,\xi_n\in$ TermΩ. If $\xi\in$ TermΩ and if any variable that occurs in ξ is an element of $\{x_1,\ldots,x_n\}$, then we often write $\xi(x_1,\ldots,x_n)$ instead of ξ; then the value $\xi(a_1,\ldots,a_n)=a$ of ξ on a "vector" $(a_1,\ldots,a_n)\in A^n$, where $(A;\Omega)$ is an Ω-algebra, is defined in the usual way; the Ω-algebra $\mathfrak{A}=(A;\Omega)$ satisfies an identity $\xi=\eta$ iff $\xi(a_1,\ldots,a_n)=\eta(a_1,\ldots,a_n)$, for any $a_1,\ldots,a_n\in A$; and $\mathfrak{A}$ satisfies a quasiidentity

$$\xi_1=\eta_1\,\&\ldots\&\,\xi_k=\eta_k \Longrightarrow \xi=\eta$$

iff for any $a_1,\ldots,a_n\in A$ such that $\xi_1(a_1,\ldots,a_n)=\eta_1(a_1,\ldots,a_n),\ldots$ $\ldots,\xi_k(a_1,\ldots,a_n)=\eta_k(a_1,\ldots,a_n)$, it follows $\xi(a_1,\ldots,a_n)=\eta(a_1,\ldots,a_n)$.

2) If B is a nonempty set, then B^+ is the set of all nonempty sequences on B, and it is in fact the free semigroup on B, where the operation is the usual catenation of sequences. $B^*=B^+\cup\{\emptyset\}$ is the free monoid on B. For any $u\in B^*$ we denote by $|u|$ the length of u.

3) The notions of CR-monoid and CR-groups are defined as in 1, when monoid and group instead of semigroup are considered.

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