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EMBEDDING OF ALGEBRAS IN DISTRIBUTIVE SEMIGROUPS

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Abstract. Subalgebras of different kinds of distributive semigroups are considered in [11], [8] and [9]. Here we make corresponding investigations concerning left (right) semigroups. We also esteblish some conections between ω-subalgebras and n-subsemigroups of each of the classes of distributive semigroups, whereas ω is an n-ary operator.

0. PRELIMINARIES

Necessary preliminary definitions and results will be stated first.

An Ω-algebra $\underline{A} = (A;\Omega)$ is an Ω-subalgebra of a semigroup $\underline{S} = (S;.)$ if $A \subseteq S$ and there is a mapping $\omega \mapsto \bar{\omega}$ from Ω into S, such that

$$(1) \qquad \omega(a_1,a_2,\dots,a_n) = \bar{\omega}a_1a_2\dots a_n$$

for every $\omega \in \Omega(n)$, $a_1,a_2,\dots,a_n \in A$ ($\Omega(i)$ denotes the set of all i-ary operators in Ω).

If $\{\omega\} = \Omega(n) = \Omega$, then instead of "$\Omega$-(sub)algebra" we say "$\omega$-(sub)algebra". An ω-algebra $\underline{A} = (A;\omega)$ is called an n-subsemigroup of a semigroup $\underline{S} = (S;.)$ if $\omega \in \Omega(n)$, $n \geqslant 3$, $A \subseteq S$ and

$$(2) \qquad \omega(a_1,a_2,\dots,a_n) = a_1a_2\dots a_n$$

for all $a_1,a_2,\dots,a_n \in A$.

Let V be a variety of semigroups. Then $V(\Omega)$)($V(n)$) denotes the class of Ω-subalgebras (n-subsemigroups, resp.) of semigroups in V and $\tilde{V}V(\Omega)$ ($\tilde{V}V(n)$) denotes the variety of Ω-algebras (ω-algebras, resp.) defined by the set of all identities valid in $V(\Omega)$ ($V(n)$, resp.). If $V(\Omega)$ ($V(n)$) is a variety then clearly $\tilde{V}V(\Omega) = V(\Omega)$ ($\tilde{V}V(n) = V(n)$, resp.). But in general $V(\Omega)$ ($V(n)$) is a quasivariety [10, pg.254]. In several papers ([2], [3],[4],[6],[7],[8],[9],[11],[12],[13]) special varieties V are considered and the corresponding answers whether $V(\Omega)$ ($V(n)$) is a proper quasivariety or a variety are given. One of the first results is that SEM(Ω) is the variety of all Ω-algebras [1], and the other is that SEM(n) is the variety of all n-semigroups [4] , whereas SEM denotes the variety of all semigroups.

Here we are dealing with the following four varieties of semigroups: The variety $\mathcal{D}^{\ell}$ ($\mathcal{D}^{r}$) of left (right, resp.) distributive semigroups, i.e. the variety defined by the left (right, resp.) distributive law

$$(3)\quad xyz = xyxz \qquad ((3')\quad xyz = xzyz) \ ,$$

the variety $\mathcal{D} = \mathcal{D}^{\ell} \cap \mathcal{D}^{r}$ of distributive semigroups and the variety $\mathcal{D}^{c}$ of commutative distributive semigroups.

It is shown in [11] that $\mathcal{D}(n)$ is a variety and that $\mathcal{D}^{\ell}(n)$, $\mathcal{D}^{r}(n)$ are proper quasivarieties of n-semigroups. We also know ([3]) that $\mathcal{D}^{c}$ is a member of an infinite set of varieties $\mathfrak{M}$ of commutative semigroups such that $\mathfrak{M}(n)$ is a variety. Concerning Ω-subalgebras, we have ([8]) that $\mathcal{D}^{c}(\Omega)$ is a variety of Ω-algebras for any operator domain Ω and ([9]) that $\mathcal{D}(\Omega)$ is a variety iff $|\Omega \setminus \Omega(0)| \leq 1$.

In this paper we are going to prove the following theorems:

THEOREM 1. $\mathcal{D}^{\ell}(\Omega)$ <u>is a variety iff</u> $\Omega = \Omega(0) \cup \Omega(1)$.

THEOREM 2. $\mathcal{D}^{r}(\Omega)$ <u>is a variety iff</u> $\Omega = \Omega(0) \cup \Omega(1)$ <u>and</u> $|\Omega(1)| \leq 1$.

THEOREM 3. <u>Let</u> ω <u>be an</u> n-<u>ary operator</u> ($n \geq 3$). <u>The following relations are satisfied</u>:

i) $\mathcal{D}^{c}(n) = \mathcal{D}^{c}(\omega)$

ii) $\mathcal{D}(\omega) \subset \mathcal{D}(n)$, <u>the inclusion is strict</u>

iii) <u>if</u> $p \in \{\ell, r\}$, <u>then neither of the classes</u> $\mathcal{D}^{p}(n), \mathcal{D}^{p}(\omega)$ <u>is a subclass of the other</u>.

Before giving the proofs of the theorems we shall state some lemmas which are obvious or easy to prove.

LEMMA 0.1. <u>Let</u> V <u>be an arbitrary variety of semigroups</u>. <u>If</u> $\Omega = \Omega(0)$, <u>then</u> $V(\Omega)$ <u>is a variety</u>. <u>If</u> $\Omega \neq \Omega(0)$, <u>then</u> $V(\Omega)$ <u>is a variety iff</u> $V(\Omega \setminus \Omega(0))$ <u>is a variety</u>.

Further on we assume that $\Omega(0) = \emptyset$ and that $\Omega \neq \emptyset$.

LEMMA 0.2. <u>If</u> $\Omega \subseteq \Omega'$ <u>and</u> $\mathcal{D}^{\ell}(\Omega)$ ($\mathcal{D}^{r}(\Omega)$) <u>is a proper quasivariety</u>, <u>then</u> $\mathcal{D}^{\ell}(\Omega')$ ($\mathcal{D}^{r}(\Omega')$) <u>is a proper quasivariety</u>.

Let ξ be a word in an arbitrary alphabet. Denote the number of occurrences of symbols in ξ by $d(\xi)$, the set of symbols occurring in ξ by $c(\xi)$ and the i-th symbol in ξ from left to the right (the right to the left) by $\xi(i)$ ($(i)\xi$, resp.).

Two words ξ and η in an arbitrary alphabet are said to be $\mathcal{D}^{\ell}$-*correlated* if:

a) $c(\xi) = c(\eta)$, $\xi(i) = \eta(i)$, $i = 1,2$, $(1)\xi = (1)\eta$

b) the sequences of the first occurrences of the symbols in ξ and η are equal (whereas $\xi(i)$ is the first occurrence of the symbol $\xi(i)$ in ξ if $\xi(j) \neq \xi(i)$ for every j, $j < i$)

c) if $\xi(k) \neq (1)\xi$ for every k, $0 < k \leqslant d(\xi)$, then $\eta(k) \neq$ $\neq (1)\eta$ for every k, $0 < k \leqslant d(\eta)$.

A word ξ is said to be the *inverse* of a word η if $d(\xi) =$ $= d(\eta)$ and $\xi(i) = (i)\eta$ for every i, $0 < i \leqslant d(\xi) = d(\eta)$.

Two words ξ and η are said to be $\mathcal{D}^{r}$-*correlated* if their inverses are $\mathcal{D}^{\ell}$-correlated.

LEMMA 0.3. ([11]) *A semigroup identity* $\xi = \eta$ *is valid in* $\mathcal{D}^{\ell}$ ($\mathcal{D}^{r}$) *iff* ξ *and* η *are* $\mathcal{D}^{\ell}$-*correlated* ($\mathcal{D}^{r}$-*correlated*).

LEMMA 0.4. *An* Ω-*identity* $\xi = \eta$ *is valid in* $\mathcal{D}^{\ell}(\Omega)$ ($\mathcal{D}^{r}(\Omega)$) *iff* ξ *and* η *are* $\mathcal{D}^{\ell}$-*correlated* ($\mathcal{D}^{r}$-*correlated*).

1. PROOF OF THEOREM 1

First, let $\Omega = \Omega(1)$.

Let $\underline{A} = (A;\Omega)$ belong to the variety $\tilde{V}\mathcal{D}^{\ell}(\Omega)$. We shall show that $\underline{A} \in \mathcal{D}^{\ell}(\Omega)$, so that $\tilde{V}\mathcal{D}^{\ell}(\Omega) = \mathcal{D}^{\ell}(\Omega)$.

Let $\bar{\Omega} = \{\bar{\omega}; \omega \in \Omega\}$ be a set of symbols such that $A \cap \bar{\Omega} = \emptyset$ and $\omega \neq \tau \Rightarrow \bar{\omega} \neq \bar{\tau}$ for every $\omega, \tau \in \Omega$. Let $F(.)$ be the free semigroup in the variety $\mathcal{D}^{\ell}$ generated by the set $\bar{\Omega} \cup A$. Say that $u, v \in F(.)$ are ω-*neighbours* or simply *neighbours* if $u = u_1 \cdot \bar{\omega} \cdot b \cdot u_2$, $v = u_1 \cdot a \cdot u_2$, for $\omega(b) = a$ in $\underline{A}$. Let $\approx$ be the transitive and reflexive extension of the relation of neighbourhood in $F(.)$.

LEMMA 1.1. *Relation* $\approx$ *is a congruence on* $F(.)$.

Proof Let $u_1 \approx v_1$ and $u_2 \approx v_2$. Then $u_1 u_2 \approx u_1 v_2 \approx v_1 v_2$. ::

Let $D(.) = F(.)/\approx$. We shall show that $\underline{A}$ is a subalgebra of $D(.)$.

Define a *value*, denoted by $[\ ]$, as a partial mapping from $F(.)$ into A by: $[\bar{\omega}_1 \bar{\omega}_2 \ldots \bar{\omega}_s a] = \omega_1 \omega_2 \ldots \omega_s(a)$.

It is easy to see that:

1°. [ ] is a well defined mapping, and that

2°. if u,v are neighbours and u is in the domain of [ ] , then v is also in the domain of [ ] and [u] = [v] .

The set A can be considered as a subset of D. For, if $a \approx b$ for some $a,b \in A$, then there is a sequence $a=u_0, u_1, \dots, u_{t-1}, u_t=b$ such that u_i, u_{i+1} are neighbours $(0 \leqslant i \leqslant t-1)$ and $a=[a]=[u_1]= \dots$ $\dots =[b]=b$.

The fact that $\omega(a)=\bar{\omega}a$ for every $\omega \in \Omega$, $a \in A$ is obvious.

Let $\Omega \neq \Omega(1)$.

If ω is an n-ary operator in Ω $(n \geqslant 2)$, then the quasiidentity

$$(4)\quad \omega x^n = \omega y^{n-1}x \rightarrow \omega xz^{n-1} = \omega y^{n-1}\omega xz^{n-1}$$

is valid in $\mathcal{D}^{\ell}(\Omega)$. Namely, for an arbitrary subalgebra $\underline{A}=(A;\Omega)$ of a semigroup S(.) belonging to $\mathcal{D}^{\ell}$ whose elements a,b satisfy the relation $\omega(a^n)=\omega(b^{n-1}a)$, we have: $\omega(ac^{n-1}) = \bar{\omega}.a.c^{n-1} = \bar{\omega}.a^n.c^{n-1} = (\omega(a^n)).c^{n-1} = (\omega(b^{n-1}a)).c^{n-1} = \bar{\omega}.b^{n-1}.a.c^{n-1} = \bar{\omega}.b^{n-1}.a.c^{n-1} = \bar{\omega}.b^{n-1}.\bar{\omega}.a.c^{n-1} = \bar{\omega}.b^{n-1}.(\omega(ac^{n-1})) = \omega(b^{n-1}(\omega(ac^{n-1}))$ for every $c \in A$. On the other hand, the quasiidentity (4) is not a consequence of the identities in $\mathcal{D}^{\ell}(\Omega)$. To prove that, consider the algebra $\underline{A}=(A;\{\omega\})$, belonging to the variety $\tilde{V}\mathcal{D}^{\ell}(\omega)$ and generated by the set $\{a,b,c\}$, with one defining relation between the generators: $\omega(a^n)=\omega(b^{n-1}a)$. The relation $\omega(b^{n-1}\omega(ac^{n-1})) = \omega(ac^{n-1})$ is not valid in $\underline{A}$. Roughly speaking, starting with $\omega(ac^{n-1})$, the element a remains in the second and the element c in the last place after using the identities in $\mathcal{D}^{\ell}(\omega)$. So, the defining relation can not be used to change the element a in the second place, because of the element c in the last.

Thus, by Lemma 0.2 we have shown Theorem 1.

2. PROOF OF THEOREM 2

Let $\Omega = \{\omega\} = \Omega(1)$. Utilizing Lemma 0.4 , we see that $\xi = \eta$ is an identity in $\mathcal{D}^r(\Omega)$ iff $(1)\xi = (1)\eta$. On the other hand, the class of Ω-algebras defined by the identity of that type is precisely $\mathcal{D}(\Omega)$ (see [9]). Thus, if $\underline{A} \in \tilde{V}\mathcal{D}^r(\Omega)$, then $\underline{A} \in \mathcal{D}(\Omega)$ and because $\mathcal{D}(\Omega) \subseteq \mathcal{D}^r(\Omega)$, $\underline{A} \in \mathcal{D}^r(\Omega)$. We can now conclude that $\mathcal{D}^r(\Omega) = \mathcal{D}(\Omega)$ and that $\mathcal{D}^r(\Omega)$ is a variety.

Let $\Omega = \{\omega,\tau\} = \Omega(1)$. The quasiidentity

$$(5) \quad \omega x = \tau x \rightarrow \omega^2 x = \tau^2 x$$

is valid in $\mathcal{D}^r(\Omega)$ (proceed as for the quasiidentity (4)). An example of an algebra $\underline{A}=(A;\{\omega,\tau\})$ belonging to $\tilde{V}\mathcal{D}^r(\Omega)$ and not satisfying (5) is the following: $A = \{a,b,c\}$, $\omega(x)=b$ for every $x\in A$, $\tau(a)=b$, $\tau(b)=c=\tau(c)$. We have $\omega(a)=\tau(a)$, but $\omega^2(a)=b\neq$ $\neq c=\tau^2(a)$. The algebra $\underline{A}$ belongs to $\tilde{V}\mathcal{D}^r(\Omega)$ because every Ω-term ξ, with $d(\xi)\geqslant 3$, has an interpretation in $\underline{A}$ equal to c for $\xi(1)=\tau$ and to b for $\xi(1)=\omega$.

Finally, let $\Omega = \{\omega\} = \Omega(n)$, $n\geqslant 2$. The quasiidentity

$$(6) \quad \omega xy^{n-1} = \omega x^n \rightarrow \omega xy^{n-1} = \omega y^{n-1}x$$

is valid in $\mathcal{D}^r(\Omega)$. To check that consider an Ω-algebra $\underline{A}$ belonging to $\mathcal{D}^r(\Omega)$ and its elements a and b satisfying the relation $\omega(ab^{n-1}) = \omega(a^n)$. We have: $\omega(ab^{n-1}) = \omega(a^n) = \bar{\omega}.a^n =$ $= \bar{\omega}.a^n.a = \omega(a^n).a = \omega(ab^{n-1}).a = \bar{\omega}.a.b^{n-1}.a = \bar{\omega}.b^{n-1}.a =$ $= \omega(b^{n-1}a)$.

In order to prove that the quasiidentity (6) is not a consequence of the identities valid in $\mathcal{D}^r(\Omega)$ define an ω-algebra $\underline{A}=(A;\omega)$ as follows: $A = \{a,b,c\}$,

$$\omega(d_1,d_2,\ldots,d_n) = \begin{cases} c & \text{if } d_n=c \text{ or } d_n=d_{n-1}=b \\ a & \text{otherwise} \end{cases} .$$

We have $\omega(bc^{n-1}) = c = \omega(b^n)$ and $\omega(bc^{n-1}) = c \neq a = \omega(c^{n-1}b)$. Thus (6) is not valid in $\underline{A}$ and it is obvious that $\underline{A}\in\tilde{V}\mathcal{D}^r(\Omega)$.

Now we can use Lemma 0.2 and the proof of Theorem 2 is completed.

3. PROOF OF THEOREM 3

1°. It is easy to see that $u = v$ is an identity in $\mathcal{D}^c$ iff $c(u) = c(v)$ and $d(u),d(v)\geqslant 3$ or it is a trivial one. Thus $\xi =$ $= \eta$ is an identity in $\mathcal{D}^c(\omega)$ or $\mathcal{D}^c(n)$ iff $c(\xi) = c(\eta)$ or it is a trivial one. So, bearing in mind that both $\mathcal{D}^c(\omega)$ and $\mathcal{D}^c(n)$ are varieties ([8],[11]) we have proved the first part of Theorem 3.

2°. An identity $u = v$ is valid in $\mathcal{D}$ iff it is trivial or $c(u) = c(v)$, $u(1) = v(1)$, $(1)u = (1)v$ and $d(u),d(v)\geqslant 3$. Thus:

a) $\xi = \eta$ is valid in $\mathcal{D}(\omega)$ iff $c(\xi) = c(\eta)$, $(1)\xi = (1)\eta$

and $d(\xi), d(\eta) \geqslant 3$, or it is trivial.

b) $\xi = \eta$ is valid in $\mathcal{D}(n)$ iff $c(\xi) = c(\eta)$, $(1)\xi = (1)\eta$, $d(\xi), d(\eta) \geqslant 3$ and $\xi(i) = \eta(j)$ where $\xi(i)$ and $\eta(j)$ are the first variable symbols occurring in ξ and η respectively, or it is trivial.

We can see that every identity valid in $\mathcal{D}(n)$ is valid in $\mathcal{D}(\omega)$. Thus, because both classes are varieties ([9],[11]), every algebra belonging to $\mathcal{D}(\omega)$ belongs to $\mathcal{D}(n)$. The converse assertion is evidently not true. For example, the identity $\omega x y^{n-1} = \omega y x y^{n-2}$ is valid in $\mathcal{D}(\omega)$ but not in $\mathcal{D}(n)$.

3°. For an ω-term ξ, denote by $\bar{\xi}$ the semigroup term obtained from ξ by deleting every occurrence of an operator symbol in ξ. An analogue of Lemma 0.4 is the following assertion: an identity $\xi = \eta$ is valid in $\mathcal{D}^{\ell}(n)$ ($\mathcal{D}^{r}(n)$) iff $\bar{\xi}$ and $\bar{\eta}$ are $\mathcal{D}^{\ell}$-correlated ($\mathcal{D}^{r}$-correlated, resp.). Thus, it is easy to check that:

a) The identity $\omega x y^{n-1} = \omega x^{n-1} \omega x y^{n-1}$ is valid in $\mathcal{D}^{\ell}(\omega)$ and not in $\mathcal{D}^{\ell}(n)$. Conversely, $\omega x^{n} = \omega^{2} x^{2n-1}$ is valid in $\mathcal{D}^{\ell}(n)$ but not in $\mathcal{D}^{\ell}(\omega)$.

b) The identity $\omega x \omega y x^{2n-3} = \omega^{2} y x^{2n-2}$ is valid in $\mathcal{D}^{r}(\omega)$ but not in $\mathcal{D}^{r}(n)$. Conversely, $\omega x \omega y^{2n-2} = \omega^{2} x y^{2n-2}$ is valid in $\mathcal{D}^{r}(n)$ but not in $\mathcal{D}^{r}(\omega)$.

Theorem 3 is proved.

REFERENCES

1. P.M.Cohn, *Universal Algebra*, D.Reidel, Publ.Comp. Dordrecht, Holand 1981

2. G.Čupona, n-*subsemigroups of semigroups satisfying the identity* $x^{r}=x^{r+m}$, God.Zbor.Mat.Fak. Skopje, 30, 1979, 5-14

3. G.Čupona, n-*subsemigroups of some varieties of abelian semigroups*, God.Zbor.Mat.Fak. Skopje, 31, 1981

4. G.Čupona, S.Markovski, *Polynomial subalgebras*, Algebraic conference, Novi Sad, 1981, 1-7

5. G.Čupona, G.Vojvodić, S.Crvenković, *Subalgebras of abelian torsion groups*, Algebraic conference, Novi Sad, 1981, 141-148

6. G.Čupona, G.Vojvodić, S.Crvenković, *Subalgebras of semilattices*, Univ. u Novom Sadu Zb.Rad.Prirod.-Mat.Fak. 10, 1980, 191-195

7. G.Čupona, S.Crvenković, G.Vojvodić, Subalgebras of commutative semigroups satisfying the law $x^r=x^{r+m}$, Univ. u Novom Sadu Zb.Rad. Prirod.-Mat.Fak. 11, 1981

8. S.Kalajdžievski, Subalgebras of commutative distributive semigroups, God.Zbor.Mat.Fak. Skopje,32, 1981

9. S.Kalajdžievski, Subalgebras of distributive semigroups "n-ary structures",Skopje, 1982, 223-228

10. А.И.Мальцев, Алгебраические системы, Мир, Москва, 1980

11. С.Марковски, За дистрибутивните полугрупи, Год.Збор. Мат.Фак. Скопје,30, 1979

12. Ю.К.Ребане, О претставлений универсальних алгебр в коммутативных полугрупах, Сибирск.Мат.Ж. 7, 1966, 878-885

13. Ю.К.Ребане, О претставлении универсальних алгебр в нильпотентних полугрупах, Сибирск Мат.Ж. 10, 1969, 947-949

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