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SEMIGROUPS WITH n-PROPERTIES

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Semigroups with n-property were introduced in [8] in the following way: a semigroup S posseses the n-property iff every n-subsemigroup of S is a subsemigroup, i.e. $Q \subseteq S,\ Q^{n+1} \subseteq Q \Rightarrow Q^2 \subseteq Q$. The problem of describing the structure of a semigroup with n-property is a special case of a problem formulated in [1]. Nevertheless, this special case is not easy to deal with and a structure description is given in [8] only for unipotent semigroups of that type. Using the idea of involving an (n+1)-ary operation, $n > 1$, in a semigroup, in this paper we introduce several classes of semigroups and give structure descriptions which follow the same pattern of structure description for unipotent semigroups with n-property.

First we collect some of the results from [8] in the following

Theorem 1. (i) Every semigroup with n-property is periodic;

(ii) If S is a group, then S posseses the n-property iff the order of every element of S is relatively prime with n;

(iii) Let H be a group with n-property, P a set such that $H \cap P = \emptyset$ and $\phi : P \to H$ a mapping. Extend ϕ to a mapping from $S^* = H \cup P$ onto H by $\phi(x) = x$ for all $x \in H$ and define an operation in S^* by

$$x o y = \phi(x)\phi(y).$$

Then $S^* = S[H,P,\phi]$ will be a unipotent semigroup with n-property. Conversely, every unipotent semigroup with n-property can be obtained in that way. ▯

The general pattern sugested by (iii) of the above theorem is the structure set $[H,P,\phi]$. To be more precise, and for convenience, we bring out the following

Lemma 1. Let P be a partial semigroup, E a semigroup such that $P \cap E = \emptyset$ and $\phi : P \to E$ a homomorphism. Extend ϕ to a mapping $\phi^* : S = P \cup E \to E$ by $\phi^*(e) = e$ for all $e \in E$ and define an operation in S by

$$x \circ y = \begin{cases} xy \text{ if } x,y \in P \text{ and } xy \text{ is defined in } P, \\ \phi^*(x)\phi^*(y) \text{ otherwise.} \end{cases}$$

Then $S(\circ)$ will be a semigroup with E as an ideal and ϕ^*-epimorphism.

Proof. This is in fact Lemma III.4.1 of [3]. Note that a mapping ϕ from a partial semigroup P into a semigroup E is a homomorphism if $\phi(xy) = \phi(x)\phi(y)$, $x,y \in P$, whenever xy is defined in P.

We will denote the semigroup constructed in Lemma 1 by $S[P,E,\phi]$.

A subclass of the class of semigroups with n-property can be defined in the following way: a semigroup S is said to be a λ_o^n-semigroup iff $Q \subseteq S$, $Q^{n+1} \subseteq Q \Rightarrow SQ \subseteq Q$. The structure of a λ_o^n-semigroup is very simple which is seen from

Lemma 2. ([7], Theorem 1). A semigroup S is a λ_o^n-semigroup iff S is periodic and $xy = e_y$ for all $x,y \in S$, where e_y is the corresponding idempotent in $\langle y \rangle$. (Here $n > 1$).

In order to obtain more interesting classes of semigroups we can substitute the left-ideality by corresponding n-ideality and, alternatively, taking subsemigroups, beside n-subsemigroups, to posseses the ideal property. In that way we can introduce the following two classes of semigroups: a semigroup S is said to be a λ_1^n-semigroup (λ_2^n-semigroup) iff $Q \subseteq S$, $Q^{n+1} \subseteq Q \Rightarrow S^nQ \subseteq Q$ ($Q \subseteq S$, $Q^2 \subseteq Q \Rightarrow S^nQ \subseteq Q$). Each of this two classes, for $n = 1$, represents the class of λ-semigroups (see, for example, [2], [4]). So, dealing with λ_1^n-or λ_2^n-semigroups we asume that $n > 1$. For any semigroup which belongs to either of this two classes we say that posseses "left-ideal n-property". Observe that

Lemma 3. (i) Every λ_o^n-semigroup is a λ-semigroup and a semigroup with n-property;

(ii) Every λ-semigroup is a λ_2^n-semigroup;

(iii) Every λ_1^n-semigroup is a λ_2^n-semigroup.

The following is almost abvious:

Lemma 4. Every subsemigroup and every homomorphic image of a λ_o^n-, λ_1^n-, λ_2^n-semigroup is a λ_o^n-, λ_1^n-, λ_2^n-semigroup, respectively.

Lemma 5. Let S be a semigroup. Then:

(i) S is a λ-semigroup iff $Sa \subseteq \langle a \rangle$ for every $a \in S$;

(ii) S is a λ_1^n-semigroup iff $S^n a \subseteq \langle a \rangle_n$ for every $a \in S$, where $\langle a \rangle_n = \{a^{kn+1} \mid k \in \mathbb{N}^o\}$ is the cyclic n-subsemigroup of S generated by a; if S is a λ_1^n-semigroup then $S^n a \subseteq \langle a \rangle$ for every $a \in S$;

(iii) S is a λ_2^n-semigroup iff $S^n a \subseteq \langle a \rangle$ for all $a \in S$.

Proof. For (i) and (ii) see [2] Lemma 2 and [7] Lemma 3. From (ii) and Lemma 3 it follows that if S is a λ_2^n-semigroup then $S^n a \subseteq \langle a \rangle$ for all $a \in S$. Conversely, if Q is a subsemigroup of a λ_2^n-semigroup S and $q \in Q$, from $S^n q \subseteq \langle q \rangle \subseteq Q$ if follows that $S^n Q \subseteq Q$.

Lemma 6. Let S be a λ-, λ_1^n-, λ_2^n-semigroup. Then:

(i) S is periodic;

(ii) The set E of all idempotents of S is a right-zero subsemigroup of S and is an ideal in S;

(iii) For all $a \in S$, $m_a = 1$ where m_a is the period of a and: if S is a λ-semigroup, then $|\langle a \rangle| \le 3$, if S is a λ_1^n-semigroup then $|\langle a \rangle| \le n + 2$, if S is a λ_2^n-semigroup then $|\langle a \rangle| \le 2n + 1$.

Proof. For λ- and λ_1^n-semigroups see [2] and [7]. Let S be a λ_2^n-semigroup, a S and $\langle a \rangle = \{a, a^2, \ldots\}$. If $\langle a \rangle$ were infinite, then $Q = \langle a^{n+1} \rangle$ would be a subsemigroup of S which does not contain a^{2n+1} but, on the other hand, $a^{2n+1} = a^n a^{n+1} \in S^n Q \subseteq Q$ and this is a contradiction. So, S is periodic since $\langle a \rangle$ is finite. Let e_a be the idempotent in $\langle a \rangle$ and $x \in S$. Then

$$xe_a = xe_a \ldots e_a \in S^n e_a \subseteq \langle e_a \rangle = \{e_a\},$$

i.e.

$$xe_a = e_a. \tag{1}$$

From (1) it follows that E is a right-zero subsemigroup of S. If $e \in E$ and $x \in S$, again from (1) it follows that $exex = e.e.x = ex$, so $ex \in E$, i.e. $ES \subseteq E$ which proves (ii). Let K_a be the periodic part of $\langle a \rangle$ and $y \in K_a$. From (1) we have that $y = ye_a = e_a$ and $K_a = \{e_a\}$. Let $\langle a \rangle = \{a, a^2, \ldots, a^s = e_a\}$ and $Q = \langle a^{n+1} \rangle$; if $s \geq 2n+1$ then $a^{2n+1} \in Q$ which is a contadiction and so we have $|\langle a \rangle| \leq 2n+1$.

In what follows S will be a semigroup of any of the classes λ_o^n, λ, λ_1^n, λ_2^n. Let us put

$$P = S \setminus E$$

where E is as before, the set of all idempotents of S. Then P will be a partial semigroup such that for every $a \in P$ there exists some $k \in \mathbb{N}$ with a^k not defined in P, which is a consequence of the periodicity of S; we may call such a partial semigroup a power breaking partial semigroup. We have therefore seen that

a) P is a power breaking partial semigroup.

Let us define a mapping $\phi : S \to E$ by $\phi(x) = e_x$, e_x the idempotent in $\langle x \rangle$, and let $xy = z$, $x, y, z \in S$. For some $m \in \mathbb{N}$, $m > \frac{n}{2}$, we have that $z^m = e_z$ and then, by Lemma 3 and 5,

$$e_z = xy \ldots xy \in S^n y \subseteq \langle y \rangle$$

which implies that $e_z = e_y$ and

$$\phi(xy) = e_y = e_x e_y = \phi(x)\phi(y),$$

and ϕ is an epimorphism from P onto E. The restriction $\psi = \phi_{|P}$ then is a homomorphism from P into E, which establishes

b) There is a homomorphism $\psi : P \to E$.

The operation in S can be, now, expressed of follows:

$$\text{c) } xy = \begin{cases} xy & \text{if } x, y \in P \text{ and } xy \text{ is defined in } P \\ \phi(x)\phi(y) & \text{otherwise.} \end{cases}$$

Finaly, from Lemma 2 and 6 it follows that P posseses the left-ideal property which can be introduced in the following way:

d) (i) If S is a λ_o^n-semigroup then P is just a set; (ii) if S is a λ-semigroup then $xy = y^2$ whenever xy is defined in P; (iii) if S is a λ_1^n-semigroup then $x_o x_1 \ldots x_n = x_n^s$, $s < n+2$, whenever $x_o x_1 \ldots x_n$

is defined in P; (iv) if S is a λ_2^n-semigroup then as in (iii) with $s < 2n+1$.

Conversely, let E be a left-zero semigroup, P a power breaking partial semigroup, $P \cap E = \emptyset$, and $\phi: P \to E$ a homomorphism. Extend ϕ to a mapping $\phi^*: S = P \cup E \to E$ by $\phi(e) = e$ for all $e \in E$ and define an operation in S as in Lemma 1. According to Lemma 1 S(o) will be a semigroup and ϕ^* an epimorphism. It is easily seen that S is periodic. Finaly: (i) if P is a set without operation defined on it, then S(o) will be a λ_o^n-semigroup; (ii) if $xy = y^2$ whenever xy is defined in P, then $Sy \subseteq \{\phi(y), y^2\} \subseteq \langle y \rangle$ since $\phi(y)$ is the corresponding idempotent to y (if y^s is not defined in P then $y^s = [\phi(y)]^s = \phi(y)$) and so, S(o) will be a λ-semigroup; for (iii) and (iv), similarly as in (ii) we can see that S(o) will be a λ_1^n-, λ_2^n-semigroup, respectively.

From the above discusion follows

<u>Theorem 2</u>. A semigroup S posseses the left-ideal n-property iff $S = S[P,E,\phi]$ where E is a left-zero semigroup, P a power breaking partial semigroup and: (i) S is a λ_o^n-semigroup, (ii) λ-semigroup, (iii) λ_1^n-semigroup, (iv) λ_2^n-semigroup iff (i) P is a set without operation defined on it, (ii) $xy = y^2$, $x,y \in P$, whenever xy is defined in P, (iii) $x_o x_1 \ldots x_n = x_n^s$, $s < n+2$, whenever $x_o x_1 \ldots x_n$ is defined in P and (iv) $x_o x_1 \ldots x_n = x_n^s$, $s < 2n+1$ whenever $x_o x_1 \ldots x_n$ is defined in P.

Let us observe that it is very easy to formulate right dual of left-ideal n-property and, by symetry, to translate all results. Also, we can, now, obtain structure description for semigroups with ideal n-property which can be introduced in an obvious way. For example, for the corresponding class of λ_o^n-semigroups we will come to zero semigroups while in all other classes with ideal n-property E reduces to one idempotent and some additional identities will be needed: instead of $xy = y^2$ or $x_o x_1 \ldots x_n = x_n^s$ we will have $x_y = x^2 = y^2$ or $x_o x_1 \ldots x_n = x_o^s = x_n^s$ whenever xy, respectively $x_o x_1 \ldots x_n$ is defined in P. Let us observe that the class of λ_1^n-semigroups can be interpreted as a class of n-semigroups (for a structure description see [9]).

R E F E R E N C E S

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