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COMMUTATIVE SEMIGROUPS WITH n-PROPERTY

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A semigroup S posseses the n-property iff $Q \subseteq S$, $Q^{n+1} \subseteq Q \Rightarrow Q^2 \subseteq Q$ ([2]). A structure description is given in [2] only for unipotent semigroups with n-property. In this paper we give a structure description for commutative semigroups with n-property when they have more than one idempotent.

First we recall some of the properties of semigroups with n-property stated in [2].

Theorem 1. (i) Every semigroup S with n-property is periodic; the index of $\langle a\rangle$, for every $a \in S$, is not greater than 2 and the period of $\langle a\rangle$ is relatively prime with n.

(ii) If S is a group, then S posseses the n-property iff the order of every element of S is relatively prime with n.

(iii) Let H be a group with n-property, P a set, $H \cap P = \emptyset$, and $\phi: P \to H$ a mapping. Extend ϕ to a mapping from $S = H \cup P$ onto H by $\phi(x) = x$ for all $x \in H$ and define an operation on S by

$$x o y = \phi(x)\phi(y).$$

Then $S = [S\ H, P, \phi]$ will be a unipotent semigroup with n-property. Conversely, every unipotent semigroup with n-property can be obtained in that way. ∎

In what follows S will denote a commutative semigroup with n-property, $n \geq 2$, E - the subsemigroup of S consisting of all idempotents of S and

$$S_e = \{x \in S \mid (\exists m \in N) x^m = e\}, \quad e \in E,$$

H_e - the maximal subgroup of S containing the idempotent e,

$$I_e = S_e \backslash H_e,$$

$$H = \cup\{H_e \mid e \in E\}, \quad I = \cup\{I_e \mid e \in E\}.$$

It is easily seen that

Lemma 1. H is a subsemigroup of S. ∎

Let us define $A_e \subseteq I_e$ by

$$A_e = \{x \in I_e \mid (\exists f \subset E)\, xf \in I\},$$

$$A = \cup\{A_e \mid e \in E\}, \quad S^* = S\backslash A.$$

Lemma 2. S^* is a subsemigroup of S such that, for every $x,y \in S^*$, $xy \in H$.

Proof. Let $x,y \in S^*$ and $x \in S_e$, $y \in S_f$, $ef = g$. Since $x \in S^*$ it follows that $xf \in H_g$ and similarly $ey \in H_g$. Let $Q = \{x,y\} \cup H_g$ and $q = q_o q_1 \ldots q_n$, $q_j \in Q$.

(i) If some q_j belongs to H_g, then $q_j = q_j g = q_j ef$ and every x in the representation of q can be substituted by xe, every y in q can be substituted by yf. So, since $xe \in H_e$, $yf \in H_g$ we will have that $q \in H_e H_f H_g \subseteq H_g$. Here we use the properties $H_e H_f \subseteq H_g$, $H_e H_g \subseteq H_g$, $H_f H_g \subseteq H_g$ ([1], §1.6, Example 4).

(ii) If none q_j belongs to H_g, then at least one of x^2 and y^2 will appear in q; if x^2 appears in q, since $x^2=x^2e$ and $ey \in H_g$, we will have that $q \in H_e H_g \subseteq H_g$; similarly if y^2 appears in q.

We have proved that Q is an n-subsemigroup of S and, as S posseses the n-property, Q is a subsemigroup of S, too. So, $xy \in Q$; since $xy \in S_g$ it follows that $xy \in H_g$. ∎

We call S^* a reduced subsemigroup of S with n-property. Now we are ready to give a structure description for S^*.

If S^* is a reduced commutative semigroup with n-property then:

(a) H is an inverse commutative periodic semigroup which is a union of groups and the order of each element of H is relatively prime with n, $n \geq 2$.

(b) Let $B = I\setminus A$; then B is a set such that $H \cap B = \emptyset$.

(c) If we put $\psi(x) = xe$, $x \in S_e^* = S_e \setminus A_e$, then ψ will be a mapping from S* onto H such that $\psi_{|H} = 1_H$, and the operation in S* can be expresed as follows:

$$xy = \psi(x)\psi(y), \tag{1}$$

since according to Lemma 2, $xy \in H$ and, if $x \in S_e^*$, $y \in S_f^*$, $ef=g$, then $xy \in H_g$ and so

$$xy = xyg = xeyf = \psi(x)\psi(y).$$

Conversely, let H be an inverse commutative periodic semigroup which is a union of groups and the order of each element of H is relatively prime with n $(n \geq 2)$; let B be a set, $H \cap B = \emptyset$, and ψ be a mapping from $S^* = H \cup B$ onto H such that $\psi_{|H} = 1_H$. If in S* we define an operation by (1), then S* will turn out to be a comm ta-tive semigroup (which is easily seen). Let Q be an n-subsemigroup of S* and $x \in Q$; then $\psi(x) \in H_e$ (for some $e \in E$) has an order, say m, which is relatively prime with n, and so, $sm = rn+1$, s,r - integers, from which it follows that

$$e = [\psi(x)]^{sm} = [\psi(x)]^{rn+1} = x^{rn+1} \in Q,$$

according to the definition of the operation in S*. Now, if $x,y \in Q$, since $e \in Q$, we have that $xy = \psi(x)\psi(y) = e^{n-1}\psi(x)\cdot\psi(y) \in Q$ which shows that Q is a subsemigroup of S*, too, i.e. S* is a commutative semigroup with n-property. It follows by the definition of the operation in S* that S* is reduced.

Denoting the above constructed semigroup S* by $S^* = [H,B,\psi]$, we have:

Theorem 2. A semigroup S* is a reduced commutative semigroup with n-property iff $S^* = [H,B,\psi]$, where H is as in (a), B a set such that $H \cap B = \emptyset$ and $\psi : H \cup B \to H$ a mapping such that $\psi_{|H} = 1_H$.

If we combine, now, Theorem 2 and Theorem 4.11 of [1], we can give another description of a reduced commutative semigroup with n-property which we will make use of in describing the structure of a non-reduced semigroup of this type. Namely, let E be a semilattice and, for each $e \in E$, let be given a commutative group G_e with n-property and a set B_e such that $G_e \cap B_f = \emptyset$ for all $e,f \in E$ and $G_e \cap G_f = B_e \cap B_f = \emptyset$ for $e \neq f$. Let us denote $\mathcal{G} = \{G_e \mid e \in E\}$, $\mathcal{B} = \{B_e \mid e \quad E\}$, $G = \cup\{G_e \mid e \in E\}$, $B = \cup\{B_e \mid e \in E\}$ and let $\psi : S = G \cup B \to G$ be a mapping such that $\psi_{|G} = 1_G$. Let $\Phi = \{\phi_{ef} \mid e,f \in E\}$ be a family of homomorphisms between the groups from $\mathcal{G}$ defined as follows: if $e,f \in E$ and $ef=fe=e$, then ϕ_{ef} is a homomorphism from G_f into G_e and, if $ef=e$, $fg=g$ then $\phi_{fg}\phi_{ef} = \phi_{eg}$; $\phi_{ee} = 1_{|G_e}$. By putting $\psi_{ef} = \phi_{ef}\psi$ we define a family of mappings $\Psi = \{\psi_{ef} \mid e,f \in E\}$ from S into G. We define an operation in S by

$$xoy = \psi_{eg}(x)\psi_{fg}(y)$$

if $x \in S_e = G_e \cup B_e$, $y \in S_f$ and $ef=g$. Let us denote the semigroup we have now constructed by $S = [\mathcal{G}, \mathcal{B}, \Phi, \Psi]$. From Theorem 2 and Theorem 4.11 of [1] it follows that:

Theorem 3. A semigroup S is a reduced commutative semigroup with n-property iff $S = [\mathcal{G}, \mathcal{B}, \Phi, \Psi]$. ∎

We now turn to the non-reduced case.

Lemma 3. Let S be a commutative semigroup with n-property, let $e,f,g \in E$, $ef=g$ and $y \in S_f$. Then one and only one of the following statements holds:

$$\text{(a) } S_e y \subseteq H_g; \text{ (b) } S_e y \subseteq A_g.$$

Proof. Let $uy \quad H_g$ for some $u \quad S_e$. Then:

$$uy = uyg = uy\cdot ge = uyg.e = uye = ue\cdot y.$$

Since $ue \in H_e$, there exists $v \in H_e$ such that $v.ue=e$ and then

$$ey = v\cdot ub\cdot y = v\cdot uey = v\cdot uy \in H_e H_g \subseteq H_g,$$

so, $ey \in H_g$.

Let $x \in S_e$ is arbitrary chosen and let $R = \{x,y,x^n y,x^{2n}y,\ldots,,\ldots,\} \cup H_g \cup H_e$. Let $r = r_o r_1 \ldots r_n$, $r_j \in R$.

(i) If some r_j belongs to H_g, then $r \in H_g$ as in Lemma 2.

(ii) If in r y apears two or more times, then $y^k \in H_g$, $k \geq 2$ and we will have the case (i).

(iii) It remains, now, the case $r=x^k y^r$, where r=0,1; if r=0 then $r \in H_e$ and then $r \in R$, and if r=1, again $r \in R$.

We have shown that R is an n-subsemigroup of S; now R is a subsemigroup, too, since S posseses the n-property. So, $xy \in R$. For xy we have two possibilities: $xy \subset H_g$ or $xy = x^{kn}y$ since $xy \in S_g$. If $xy=x^{kn}y$ then $(k \geq 1)$ $xy=ex^{kn}\cdot y=e\cdot x^{kn}y=e\cdot xy=xey=xe\cdot ey \in H_eH_g \subset H_g$ since $ey \in H_g$. In any case we have that $xy \in H_g$. ∎

<u>Lemma 4</u>. Let S be a commutative semigroup with n-property, $x \in S_e$, $y \in S_f$ and $xy \in A$. Then one and only one of the following is true:

$$\text{(a) } xy = xe\cdot y; \quad \text{(b) } xy = xyf.$$

<u>Proof</u>. As in the proof of Lemma 3 we can show that $Q = \{x,y,x^n y,x^{2n}y,\ldots,xy^n,xy^{2n},\ldots,\} \cup H_e \cup H_f \cup H_g$ is an n-subsemigroup of S, ef=g. If $xy \in A$ then $xy=x^{kn}y=ex^{kn}y=exy$ or $xy=xy^{rn}=$ $=xy^{rn}f=xyf$. It is not possible to hold (a) and (b) together since in that case we would have that

$$xy = e.xy = \sout{e}xyf = xyef \in S_gH_g \subseteq H_g$$

which is a contradiction. ∎

Let us, now, define a function $\rho: E \times A \to \{0,1\}$ as follows: $\rho(e,x) = 0$ iff $ex \in G$ and $\rho(e,x) = 1$ iff $ex \in A$.

(1) Let $\rho(e,x) = 0$, $y \in S_e$, $x \in S_f$, ef=g. According to Lemma 3 we have that $yx \in G_g$. If we put $\chi(u)=uh$ where $u \in S_h$, we can define a mapping $\chi: S \to G$ and express the product xy in the following way:

$$xy = xy\cdot g = xy\cdot ef = xf\cdot ye = \chi(x)\chi(y).$$

Observe that $\chi_{|S^*}=\psi$, where ψ is defined as is Theorem 2.

(2) Let $\rho(e,x) = 1$; then to each $y \in S_e$ we can correspond a partial mapping $\lambda_y^e : A \to A$ which is defined on x by putting $\lambda_y^e(x) = xy$. In this way we obtain a family $\Lambda = \{\lambda_y^e \mid e \in E, y \in S_e\}$ of partial mappings from A to A which satisfy the following properties:

(2a) If $x \in S_e$, $y \in S_f$, $ef=g$ then

$$\lambda_{xy}^g = \lambda_x^e \lambda_y^f = \lambda_{yx}^g,$$

which follows from $(xy)z = x(yz)$ and, λ_{xy}^g is defined on z iff λ_x^e is defined on z, and λ_y^f is defined on $\lambda_x^e(z)$ iff λ_y^f is defined on z and λ_x^e is defined on $\lambda_y^f(z)$.

(2b) If $x \in S_e$, $y \in S_f$, $xy \in A$ then, according to Lemma 4, one and only one of the following statements is satisfied: $\lambda_x^e = \lambda_e^e \lambda_x^e$ or $\lambda_y^f = \lambda_f^f \lambda_y^f$.

(3) A is a partial semigroup such that, if x,y belong to the same A_e, then xy is not defined in A.

Conversely, let $S^* = [\mathcal{G}, \mathcal{B}, \Phi, \Psi]$ be a reduced commutative semigroup with n-property and let $\mathcal{A} = \{A_e \mid e \in E\}$ be a family of disjoint sets such that $A_e \cap G_f = A_e \cap B_f = \emptyset$ for all $e,f \in E$, where E is as in Theorem 2. Let $A = \cup\{A_e \mid e \in E\}$ and let a partial operation be defined in A so that A will become a partial commutative semigroup with the property: if $x,y \in A_e$ for some $e \in E$, then xy is not defined in A.

Let $\rho : E \times A \to \{0,1\}$ be a function and $\Lambda = \{\lambda_y^e \mid e \in E, y \in S_e\}$, where $S_e = S_e^* \cup A_e$, be a family of partial mappings from A into A which satisfies the following conditions:

(i) If $\rho(e,x) = 1$ then for all $y \in S_e$, λ_y^e is defined for x.

(ii) $\lambda_{xy}^g = \lambda_x^e \lambda_y^f = \lambda_{yx}^g$ where $x \in S_e$, $y \in S_f$, $ef=g$ and λ_{xy}^g is defined for $z \in A$ iff λ_y^f is defined for z and λ_x^e is defined for $\lambda_y^f(z)$ iff λ_x^e is defined for z and λ_y^f is defined for $\lambda_x^e(z)$. Here, xy is the product in A if $x,y \in A$ and xy is defined in A, or xy is the product in S^* if $x,y \in S^*$, or $xy = \lambda_x^e(y)$ if $x \in S_e^*$, $y \in A_f$, or, similarly, $xy = \lambda_y^f(x)$ if $x \in A_e$, $y \in S_f^*$.

(iii) Let $x \in S_e$, $y \in S_f$ and xy is defined in A or one of the following conditions is satisfied: $\rho(e,y) = 1$ or $\rho(f,x) = 1$. Then one and only one of the following assertions is true:

$$\lambda_x^e(y) = \lambda_e^e \lambda_x^e(y),$$
$$\lambda_y^x(x) = \lambda_f^f \lambda_y^f(y).$$

Let $S = S^* \cup A$ and $\chi : S \to G$ be a mapping such that $\chi_{|S^*} = \psi$, ψ defind before Theorem 2. We define in S an operation "o" by:

1) $xoy = xy$ as in S^* if $x,y \in S^*$,

2) $xoy = xy$ as in A if $x,y \in A$ and xy is defined in A,

3) $xoy = \chi(x)\chi(y)$ if $\rho(e,y) = 0$, $x \in S_e$,

4) $xoy = yox = \lambda_x^e(y)$ if $\rho(e,y) = 1$, $x \in S_e$.

Then S(o) will be a commutative semigroup. To prove this it is enough to show that $(xoy)oz = xo(yoz)$ only in the case when not all x,y and z are in S^* or in A and when one of the elements $(xoy)oz$, $xo(yoz)$ is in A, since for the other cases the associativity is easily seen. Let, for example, $(xoy)oz$ is in A where $x \in S_e, y \in S_f$, $ef=g$. Then $\rho(f,z) = 1$ and according to (ii) we will have that $\rho(f,z) = 1$ and $\rho(e,\lambda_y^f(z)) = 1$. So, by the definition of "o" we will have that $xo\lambda_y^f(z) = xo(yoz)$ is in A and,

$$(xoy)oz = \lambda_{xy}^g(z) = \lambda_x^e \lambda_y^f(z) = \lambda_x^e(yoz) = xo(yoz,.$$

Let us prove, now, that S(o) posseses the n-property; let Q be an n-subsemigroup of S and $x,y \in Q$. If $x,y \in S^*$, then as in the proof of Theorem 2 we will have that $xoy \in Q$. Let $\rho(e,y) = 0$ or $\rho(f,x) = 0$ and $xoy = \chi(x)\chi(y)$ and again as in Theorem 2 we will have that $xoy \in Q$. Finally, if $xoy \in A$, according to (iii) we have that

$$xoy = \lambda_x^e(y) = \lambda_e^e \lambda_x^e(y) = eoxoy = \ldots = \underbrace{eo\ldots oe}_{n-1}oxoy \in Q$$

or

$$xoy = \lambda_y^f(x) = \lambda_f^f \lambda_y^f(x) = foyox = \ldots = \underbrace{fo\ldots of}_{n-1}oyox \in Q$$

since in the first case $e \in Q$ and in the second one $f \in Q$ which can be proved as in Theorem 2.

Let us denote the semigroup S(o) which we constructed above by $[\mathcal{G}, \mathcal{B}, ;\Phi, \Psi, \Lambda; \rho]$. We have proved the following:

Theorem 4. A semigroup S is a commutative semigroup with n-property iff S is isomorphic with a semigroup $[\mathcal{G}, \mathcal{B}, \mathcal{A}; \Phi, \Psi, \Lambda; \rho]$. ∎

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