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n-SUBSEMIGROUPS OF CANCELLATIVE SEMIGROUPS

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We show in this paper that any cancellative n-semigroup is an n-subsemigroup of a cancellative semigroup. Furthermore, for any cancellative n-semigroup $\underline{S}$ there exists a universal covering cancellative semigroup $\underline{S}^{\vee}$. It is shown that $\underline{S}^{\vee}$ is very useful in the investigation of the class of n-subsemigroups of groups; we prove that a cancellative n-semigroup is an n-subsemigroup of a group iff its universal covering cancellative semigroup is a subsemigroup of a group.

1. Cancellative n-semigroups. An algebra $\underline{S} = (S,f)$ with an n+1-ary operation f is called an *n-semigroup* if it satisfies the following identities

$$(1.1) \qquad f(f(x_o,\ldots,x_n),x_{n+1},\ldots,x_{2n}) = \\ = f(x_o,\ldots,x_{i-1},f(x_i,\ldots,x_{i+n}),\ldots,x_{2n})$$

for all $i=1,2,\ldots,n$.

Suppose that $\underline{S} = (S,f)$ is an n-semigroup and consider any two derived products $\Pi_1(a_o,\ldots,a_p)$ and $\Pi_2(a_o,\ldots,a_p)$ on the same sequence $a_o,\ldots,a_p$ of elements of S. Then, by (1.1), we have that $\Pi_1 = \Pi_2$ is a valid equality in $\underline{S}$. Thus, any n-semigroup $\underline{S}$ can be considered as a kn-semigroup, where $k = 1,2,\ldots$. From now on an n-semigroup $\underline{S} = (S,f)$ will be denoted by S[ ], where $[a_o\ldots a_{kn}]$ denotes the value of the corresponding products in $\underline{S}$ on the sequence $a_o,\ldots,a_{kn}$ of elements of S. In this notation we will put $[a] = a$ and $[a^{n+1}] = [\underbrace{a\ldots a}_{n+1}]$. We will use also a shorter notation $[\underline{a}]$, where $\underline{a} = a_o\cdots a_{kn}$.

An n-semigroup is cancellative if it satisfies all the quasiidentities of the form

(1.2) $[x_o \dots x_{i-1} y x_{i+1} \dots x_n] = [x_o \dots x_{i-1} z x_{i+1} \dots x_n] \Rightarrow y = z$,

where $i = 0,1,2,\dots,n$.

1.1. The following conditions are equivalent for an n-semigroup $S[\ ]$ $(n \geq 2)$:

(i) $S[\ ]$ is cancellative;

(ii) $S[\ ]$ satisfies (1.2) for some $i: 1 \leq i \leq n-1$;

(iii) $S[\ ]$ satisfies (1.2) for $i=0$ and $i=n$;

(iv) the quasiidentity

$$[x^i y x^{n-i}] = [x^i z x^{n-i}] \Longrightarrow y = z$$

is valid in $S[\ ]$ for some $i: 1 \leq i \leq n-1$;

(v) the implication

$$[x^n y] = [x^n z] \vee [y x^n] = [z x^n] \Longrightarrow y = z$$

is valid in $S[\ ]$.

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv), (i) $\Rightarrow$ (iii) $\Rightarrow$ (v).

(iv) $\Longrightarrow$ (i): Let $[x_o \dots x_{j-1} y x_{j+1} \dots x_n] = [x_o \dots x_{j-1} z x_{j+1} \dots x_n]$. If i=j, then by multiplying to the left-hand side and to the right-hand side, we obtain:

$$[x_{i+1} \dots x_n x x_o \dots x_{i-1} y x_{i+1} \dots x_n x x_o \dots x_{i-1} t^{n-2}] =$$
$$= [x_{i+1} \dots x_n x x_o \dots x_{i-1} z x_{i+1} \dots x_n x x_o \dots x_{i-1} t^{n-2}],$$

i.e.

$$[[x_{i+1} \dots x_n x x_o \dots x_{i-1}] y [x_{i+1} \dots x_n x x_o \dots x_{i-1}] t^{n-2}] =$$
$$= [[x_{i+1} \dots x_n x x_o \dots x_{i-1}] z [x_{i+1} \dots x_n x x_o \dots x_{i-1}] t^{n-2}].$$

Now letting $t = [x_{i+1} \dots x_n x x_o \dots x_{i-1}]$ from the last equation and (iv) we have $y = z$.

If $j < i$ (and similarly if $i < j$), then we have

$$[x^{i-j}x_o\ldots x_{j-1}yx_{j+1}\ldots x_{n-i+j-1}[x_{n-i+j}\ldots x_n x^{n-i+j}]] =$$
$$[x^{i-j}x_o\ldots x_{j-1}zx_{j+1}\ldots x_{n-i+j-1}[x_{n-i+j}\ldots x_n x^{n-i+j}]].$$

Hence, we can apply the above considerations again, i.e. we conclude that $y = z$ in any case.

(v) $\Longrightarrow$ (iv): Let $[x^i y x^{n-i}] = [x^i z x^{n-i}]$. Then we have

$$[x^{n-i}[x^i y x^{n-i}]x^i] = [x^{n-i}[x^i z x^{n-i}]x^i],$$

i.e.

$$[[x^n y]x^n] = [[x^n z]x^n],$$

which implies $y = z$.∎

The following property will be very useful in the further considerations.

1.2. Let $\underline{x}' = x_1\ldots x_i$, $\underline{x}'' = x_{i+1}\ldots x_{n-k}$, $\underline{t}' = t_1\ldots t_j$, $\underline{t}'' = t_{j+1}\ldots t_{n-k}$, $\underline{y} = y_o\ldots y_k$, $\underline{z} = z_o\ldots z_k$, where $0 \le i,\ j \le n-k$, $0 \le k \le n$. Then the quasiidentity

$$[\underline{x}'\underline{y}\underline{x}''] = [\underline{x}'\underline{z}\underline{x}''] \Rightarrow [\underline{t}'\underline{y}\underline{t}''] = [\underline{t}'\underline{z}\underline{t}'']$$

is valid in any cancellative n-semigroup.

Proof. $[\underline{x}'\underline{y}\underline{x}''] = [\underline{x}'\underline{z}\underline{x}''] \Rightarrow$

$[x^{n-i}\underline{x}'\underline{y}\underline{x}''x^i] = [x^{n-i}\underline{x}'\underline{z}\underline{x}''x^i] \Rightarrow$

$[x^{n-i}\underline{x}'[\underline{y}\underline{x}''x^i]] = [x^{n-i}\underline{x}'[\underline{z}\underline{x}''x^i]] \Rightarrow [\underline{y}\underline{x}''x^i] = [\underline{z}\underline{x}''x^i] \Rightarrow$

$[t^k[t^{n-j-k}\underline{t}'\underline{y}]\underline{x}''x^i] = [t^k[t^{n-j-k}\underline{t}'\underline{z}]\underline{x}''x^i] \Rightarrow$

$[t^n[t^{n-j-k}\underline{t}'\underline{y}]\underline{x}''x^{i+k}] = [t^n[t^{n-j-k}\underline{t}'\underline{z}]\underline{x}''x^{i+k}] \Rightarrow$

$[t^{n-j-k}\underline{t}'\underline{y}] = [t^{n-j-k}\underline{t}'\underline{z}] \Rightarrow [t^n\underline{t}'\underline{y}\underline{t}''] = [t^n\underline{t}'\underline{z}\underline{t}''] \Rightarrow$

$[\underline{t}'\underline{y}\underline{t}''] = [\underline{t}'\underline{z}\underline{t}'']$.∎

2. Universal covering cancellative semigroup of a cancellative n-semigroup. An n-semigroup S[ ] is said to be an n-subsemigroup of a semigroup P = (P,*) if $S \subseteq P$ and for

all $a_o,\dots,a_n$ S

$$[a_o\dots a_n] = a_o*\dots*a_n.$$

Let S[ ] be a cancellative n-semigroup and denote by $\underline{F} = (F,\cdot)$ the free semigroup generated by the set S. We can assume that the elements of F are all nonempty finite sequences of elements of the set S and that the operation in $\underline{F}$ is the usual concatenation of sequences. If $u\in F$, then we denote by $|u|$ the length of u. If $u\in F$ and $|u| = kn+1$ for some $k\geq 0$, then we denote by [u] the product in the n-semigroup S[ ], corresponding to the sequence u.

Define a relation $\sim$ in the set F by

(2.1) $\quad u,v\in F \Rightarrow (u\sim v \Leftrightarrow (\exists\, w\in F)[u\,w] = [v\,w])$.

2.1. **The relation** $\sim$ is a congruence on the semigroup $\underline{F}$.

Proof. First we note that the existential quantifier in (2.1) can be replaced by the universal one, according to 1.2. (In that case $\forall w\in F$ means that the length of w should be good for the sign [ ] .) We also have that $u\sim v$ implies $|u| \equiv |v|$ (mod n). Now it is easy to prove that $\sim$ is a congruence on $\underline{F}$. (For instance, if $u,v,w\in F$ and $u\sim v$, then there is some $t\in F$ such that [uwt] = [vwt] and [wut] = [wvt]. This implies that $uw\sim vw$ and $wu\sim wv$.) ∎

Denote by $\underline{S}^\sim = (S^\sim,\bullet)$ the quotient semigroup $\underline{F}/\sim$ and denote by $u^\sim$ $(u\in F)$ the corresponding equivalence class. Then $S^\sim = F/\sim = \{u^\sim \mid u\in F\}$.

2.2. The semigroup $\underline{S}^\sim$ is cancellative.

Proof. Let $wu\sim wv$ for some $u,v,w\in F$. Then, there is some $t\in F$ such that [wut] = [wvt], and 1.2 implies that $u\sim v$. In the same manner one can prove that $uw\sim vw \Rightarrow u\sim v$. ∎

2.3. The mapping $\phi: S\to S^\sim$ defined by $\phi(a) = a^\sim$ is injective and for all $a_o,\dots,a_n\in S$

(2.2) $$\phi([a_o\dots a_n]) = \phi(a_o)\dots\phi(a_n).$$

(A mapping of an n-semigroup to a semigroup is said to be an n-homomorfism if it satisfies (2.2). An injective n-homomorphism is called an n-monomorfism.)

Proof. If $a,b \in S$ and $a \sim b$, then there is some $u \in F$ such that $[ua] = [ub]$, but this implies $a=b$. ∎

As a consequence of 2.2 and 2.3 we have that every cancellative n-semigroup is an n-subsemigroup of a cancellative semigroup, i.e.

2.4. The class of n-subsemigroups of cancellative semigroups is equal to the quasivariety of cancellative n-semigroups. ∎

The cancellative semigroup $\underline{S}^{\sim}$ is called the universal covering cancellative semigroup of the n-semigroup $S[\ ]$. The reason for this is the following theorem:

2.5. Let $S[\ ]$ be a cancellative n-semigroup, $P = (P,*)$ be a cancellative semigroup and $\psi : S \to P$ be an n-homomorphism of $S[\ ]$ to $\underline{P}$. If ϕ is the n-monomorphism defined as in 2.3, then there is a homomorphism θ of $\underline{S}^{\sim}$ to $\underline{P}$ such that $\psi = \theta\phi$.

Proof. Suppose that $u=a_1 \ldots a_i \in F$, where $a_1, \ldots, a_i \in S$, and define θ by

$$\theta(u^{\sim}) = \psi(a_1) * \ldots * \psi(a_i).$$

It is enough to show that v is a well defined mapping. If $b_1, \ldots, b_j \in S$, $v=b_1 \ldots b_j \in F$ and $u \sim v$, then there is some $c \in S$, such that $[a_1 \ldots a_i c^k] = [b_1 \ldots b_j c^k]$ and $i+k \equiv j+k \equiv 1 \pmod n$, $k \geq 0$. This implies that $\psi(a_1) * \ldots * \psi(a_i) * \psi(c)^k =$ $= \psi(b_1) * \ldots * \psi(b_j) * \psi(c)^k$ is a valid equality in $\underline{P}$, i.e. $\psi(a_1) * \ldots * \psi(a_i) = \psi(b_1) * \ldots * \psi(b_j)$. ∎

We can give a better description of the semigroup $\underline{S}^{\sim}$. We identity $a^{\sim}$ with a for $a \in S$, and so we have $S \subseteq S^{\sim}$. If $u = a_0 \ldots a_k \in F$, $k \geq n$ and $[a_0 \ldots a_n] = b$, then $u \sim v$, where $v = b a_{n+1} \ldots a_k$. So we can write

$$S^{\sim} = S \cup S^2 \cup \ldots \cup S^n,$$

where S^i contains all the products on $\underline{S}^{\sim}$ of the form $a_1 \bullet \ldots \bullet a_i = a_1 \ldots a_i$ $(a_1, \ldots, a_i \in S)$. Also $i \neq j$ implies $S^i \cap S^j = \emptyset$ and if $a_1 \ldots a_i$, $b_1 \ldots b_i \in S^i$, then $a_1 \ldots a_i = b_1 \ldots b_i$ in $\underline{S}^{\sim}$ iff there is some $c \in S$ such that $[a_1 \ldots a_i c^{n-i+1}] = [b_1 \ldots b_i c^{n-i+1}]$ in $S[\ ]$.

<u>3</u>. <u>n-subsemigroups of groups</u>. In what follows we will use the universal covering cancellative semigroup $\underline{S}^{\sim}$ of a cancellative n-semigroup $S[\ ]$ in investigating the class of n-subsemigroups of groups. We note at first that any n-subsemigroup of a group should be cancellative.

An n-semigroup is said to be <u>commutative</u> if it satisfies the identity

$$[x_0 \ldots x_n] = [x_{p(0)} \ldots x_{p(n)}]$$

for any permutation p of the set $\{0, 1, \ldots, n\}$.

<u>3.1</u>. If $S[\ ]$ is a cancellative commutative n-semigroup, then its universal cancellative covering semigroup $\underline{S}$ is also commutative.

<u>Proof</u>. Assume that $S[\ ]$ is a cancellative commutative n-semigroup and $a_1 \ldots a_i$, $b_1 \ldots b_j \in S^{\sim}$. Then there is a $c \in S$ such that

$$[a_1 \ldots a_i b_1 \ldots b_j c^{2n-i-j+1}] = [b_1 \ldots b_j a_1 \ldots a_i c^{2n-i-j+1}],$$

which implies that $\underline{S}^{\sim}$ is commutative. ∎

We will generalize the following result ([2], p. 58):

<u>3.2</u>. If a cancellative semigroup $\underline{P} = (P, \cdot)$ satisfies the condition $Pa \cap Pb \neq \emptyset$ for any $a, b \in P$, then $\underline{P}$ is a subsemigroup of a group. ∎

Namely, we have:

<u>3.3</u>. Let $S[\ ]$ be a cancellative n-semigroup which satisfies the condition $[S^{n-i}a_o\ldots a_i] \cap [S^{n-j}b_o\ldots b_j] \neq \emptyset$ for any $a_o,\ldots,a_i,b_o,\ldots,b_j \in S$. Then $S[\ ]$ is an n-subsemigroup of a group.

<u>Proof</u>. It is clear that if $\underline{S}^{\sim}$ is a subsemigroup of a group $\underline{G}$, then $S[\ ]$ is an n-subsemigroup of $\underline{G}$ as well. So, by <u>3.2</u> and the definition of $\underline{S}^{\sim}$, it is enough to be shown that for any $a_o\ldots a_i,\ b_o\ldots b_j \in S^{\sim}$ there are integers $p,q: 1\leq p,q\leq n$, $p+1 \equiv q+j \pmod n$, such that $S^p \cdot a_o\ldots a_i \cap S^q \cdot b_o\ldots b_j \neq \emptyset$. But, this is satisfied by the hypothesis of the proposition. ∎

As a consequence of <u>3.1</u> and <u>3.3</u>, we have

<u>3.4</u>. Any cancellative commutative n-semigroup is an n-subsemigroup of a group. ∎

We already noted that if the universal covering cancellative semigroup of a cancellative n-semigroup $S[\ ]$ is a subsemigroup of a group, then $S[\ ]$ is also an n-subsemigroup of a group. Next we proceed to the opposite implication.

Let $S[\ ]$ be an n-semigroup and let $F_{S\cup S^{-1}} = F_1$ be the free monoid generated by the set $S\cup S^{-1}$. Define a congruence $\approx$ in F_1 by $u \approx v$ iff v is obtained from u by using the following transformations:

replace a by $a_o\ldots a_n$ ($a_o\ldots a_n$ by a), where $a = [a_o\ldots a_n]$ in $S[\ ]$,

replace aa^{-1} and $a^{-1}a$ by 1,

replace 1 by aa^{-1} or $a^{-1}a$.

Then $\underline{G}_S = F_1/\approx$ is the group generated by the set S with the set of defining relation $\{a=a_o\ldots a_n \mid a = [a_o\ldots a_n]$ in $S[\ ]\}$.

If $u = a_1^{k_1}\ldots a_p^{k_p} \in G_S$, where $a_1,\ldots,a_p \in S$, then we write

$$d_u = \sum_{i=1}^{p} k_i.$$

It is clear that if u=v in $\underline{G}_S$, then

(3.1) $$d_u \equiv d_v \pmod n.$$

3.5. A cancellative n-semigroup S[ ] is an n-subsemigroup of a group iff $\underline{S}^\sim$ is a subsemigroup of a group.

Proof. Let the cancellative n-semigroup S[ ] be an n-subsemigroup of a group $\underline{G}$. Concider the group $\underline{G}_S = \langle S; \{a=a_o\ldots a_n \mid a = [a_o\ldots a_n] \text{ in } S\}\rangle$. Then $S \subseteq G_S$ and $a = [a_o\ldots a_n]$ in S[ ] implies $a = a_o\ldots a_n$ in $\underline{G}_S$. In such a way the mapping $a \mapsto a$ $(a \in S)$ induces a homomorphism of $\underline{G}_S$ into $\underline{G}$. Hence, we can conclude that S[ ] is an n-subsemigroup of the group $\underline{G}_S$ as well.

We can regard the group $\underline{G}_S$ as a cancellative semigroup and by 2.5 it follows that there is a homomorphism θ of $\underline{S}^\sim$ to $\underline{G}_S$ such that for any $a_1\ldots a_i \in S^\sim$

$$\theta(a_1\ldots a_i) = a_1\ldots a_i.$$

If $\theta(a_1\ldots a_i) = \theta(b_1\ldots b_j)$, then we have $a_1\ldots a_i = b_1\ldots b_j$ in $\underline{G}_S$. Now (3.1) implies $i = j$, and as a consequence we have that for any $c \in S$ the equality

$$a_1\ldots a_i c^{n-i+1} = b_1\ldots b_i c^{n-i+1}$$

is valid in $\underline{G}_S$, i.e.

$$[a_1\ldots a_i c^{n-i+1}] = [b_1\ldots b_i c^{n-i+1}]$$

holds in S[ ]. Thus $a_1\ldots a_i = b_1\ldots b_i$ in $\underline{S}^\sim$ and θ is injective. ∎

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