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SUBALGEBRAS OF DISTRIBUTIVE SEMIGROUPS

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An Ω-algebra $\underline{A} = (A;\Omega)$ is called an Ω-subalgebra of a semigroup S if $A \subseteq S$, and there is a mapping $\omega \mapsto \bar{\omega}$ from Ω in S such that:

$$\omega(a_1,\dots,a_n) = \bar{\omega}a_1\dots a_n \qquad (1)$$

for all $a_1,a_2,\dots,a_n \in A$, $\omega \in \Omega(n)$. ($\Omega(n)$ is the set of n-ary operators belonging to Ω, i.e. Ω is a union of disjoint sets $\Omega(0),\Omega(1),\dots,\Omega(n),\dots$.) Denote by $\mathcal{D}$ the variety of distributive semigroups, i.e. semigroups that satisfy the following distributive laws:

$$xyz = xyxz, \quad xyz = xzyz, \qquad (2)$$

and by $\mathcal{D}(\Omega)$ the class of Ω-subalgebras of distributive semigroups. By a more general result (see for example [1, p. 272], or [3]), for every operator domain Ω, $\mathcal{D}(\Omega)$ is a quasivariety. We show in this note that $\mathcal{D}(\Omega)$ is a variety iff $\Omega\setminus\Omega(0)$ contains at most two operators.

1. The complete system of identities in the variety $\mathcal{D}$ is described in [2]. Namely, if $x_1,x_2,\dots$ are variables and p,q,i_ν,j_ν are positive integers, then:

$$x_{i_1}x_{i_2}\dots x_{i_p} = x_{j_1}x_{j_2}\dots x_{j_q} \qquad (3)$$

is a nontrivial identity equation in $\mathcal{D}$ (i.e. in every semigroup $S \in \mathcal{D}$) iff:

$$p,q \geq 3,\ i_1=j_1,\ i_p=j_q,\ \{i_1,\dots,i_p\} = \{j_1,\dots,j_q\}. \qquad (4)$$

((3) is a trivial identity if p=q and $i_\nu=j_\nu$.)

Now, we can give a description of the complete system of identity equations that hold in $\mathcal{D}(\Omega)$ (i.e. in every algebra $\underline{A} \in \mathcal{D}(\Omega)$). Let ξ be an Ω-term, i.e. a term in the first order language in which the operator symbols belonging to Ω are the only non-logical symbols. Then we denote by $|\xi|$ the length of ξ, by $c(\xi)$ the set of symbols that occur in ξ, by $f(\xi)$ the first symbol of ξ and by $\ell(\xi)$ the last symbol of ξ. Then, if ξ and η are two Ω-terms the formula $\xi=\eta$ is a non-trivial identity in $\mathcal{D}(\Omega)$ iff:

$$|\xi|,\ |\eta| \geq 3,\ c(\xi) = c(\eta),\ f(\xi) = f(\eta),\ \ell(\xi) = \ell(\eta). \tag{5}$$

The following propositions are obvious.

<u>1.1</u>. $\mathcal{D}(\Omega)$ is a variety iff: $\underline{A}$ satisfies all identities of $\mathcal{D}(\Omega)$ implies that $\underline{A} \in \mathcal{D}(\Omega)$.

<u>1.2</u>. $\mathcal{D}(\Omega)$ is a variety iff $\mathcal{D}(\Omega\backslash\Omega(0))$ is a variety.

<u>1.3</u>. If $\Omega' \subseteq \Omega$, then: $\mathcal{D}(\Omega')$ is a proper quasivariety implies that $\mathcal{D}(\Omega)$ is a proper quasivaricty.

Thus, if we have an Ω-algebra $\underline{A}$ which satisfies all identities of $\mathcal{D}(\Omega)$ but does not satisfy a quasiidentity of $\mathcal{D}(\Omega)$, then we can conclude, by <u>1.1</u>, that $\mathcal{D}(\Omega)$ is a proper quasivariety. By <u>1.2</u> we can assume that $\Omega(0) = \emptyset$, and finally, by <u>1.3</u>, if $\mathcal{D}(\rho,\tau)$ is a proper quasivariety, where $\rho,\tau \in \Omega$, then $\mathcal{D}(\Omega)$ is a proper quasivariety as well.

<u>2</u>. Assume that ρ and τ are two different operators, such that ρ is m-ary and τ is n-ary.

Let us show that

$$\rho(x^m) = \tau(x^n) \implies \rho^2(x^{2m-1}) = \tau^2(x^{2n-1}) \tag{6}$$

is a quasiidentity in $\mathcal{D}(\rho,\tau)$ for any $m,n \geq 1$, and that if $m,n \geq 2$, then the following quasiidentity also holds in $\mathcal{D}(\rho,\tau)$:

$$\rho(x^m) = \tau(x^n) \implies \rho^2(x^m y^{m-1}) = \tau\rho(x^m y^{n-1}). \tag{7}$$

Namely, let $\underline{A} = (A;\tau,\rho)$ be a $\{\tau,\rho\}$-subalgebra of a distributive semigroup S and let $a \in A$ be such that $\rho(a^m) = \tau(a^n)$. Then we have:

$$\rho^2(a^{2m-1}) = \bar{\rho}^2\ a^{2m-1} = \bar{\rho}\ a^m\ \bar{\rho}\ a^m = \rho(a^m)\rho(a^m) = \tau(a^n)\tau(a^n) =$$
$$= \bar{\tau}\ a^n\ \bar{\tau}\ a^n = \bar{\tau}^2\ a^{2n-1} = \tau^2(a^{2n-1}),$$

i.e. we obtain that $\underline{A}$ satisfies the quasiidentity (6).

Assume now that $m,n \geq 2$. Then, $\rho(a^m)=\tau(a^n)$ implies that:

$$\rho^2(a^m b^{m-1}) = \bar{\rho}^2\ a^m b^{m-1} = \bar{\rho}\ a^m b^{n-1} = \rho(a^m)b^{n-1} =$$
$$= \tau(a^n)b^{n-1} = \bar{\tau}\ a^n b^{n-1} = \bar{\tau}\ \tau(a^n)b^{n-1} =$$
$$= \bar{\tau}\ \rho(a^m)b^{n-1} = \bar{\tau}\ \bar{\rho}\ a^m b^{n-1} = \tau\ \rho(a^m b^{n-1}),$$

for any $b \in A$. Therefore, if $m,n \geq 2$, then (7) holds in $\mathcal{D}(\rho,\tau)$.

Let $n=1$, and let $\underline{A} = (\{a,b,c\};\rho,\tau)$ be a $\{\rho,\tau\}$-algebra such that $a \neq b \neq c \neq a$, and:

$$\rho(x_1,\ldots,x_m) = b, \text{ for any } x_1,\ldots,x_m \in \{a,b,c\},$$

$\tau(a) = b,\ \tau(b) = \tau(c) = c.$

The algebra $\underline{A}$ does not satisfy (6), for $\rho(x^m) = \tau(x)$ implies $x=a$, but then:

$$\rho^2(a^{2m-1}) = b \neq c = \tau^2(a).$$

Let ξ and η be two $\{\rho,\tau\}$-terms such that (5) holds. If the first symbol of ξ and η is ρ, then any value of ξ and η is b. If $f(\xi) = f(\eta) = \tau$ and if the symbol ρ occurs in ξ and η, then any value of ξ and η is c. And, finally if $\xi = \tau^i x$, $\eta = \tau^j x$, $i,j \geq 2$, then also any value of ξ and η is c. Therefore the algebra $\underline{A}$ satisfies all the identities of $\mathcal{D}(\rho,\tau)$, but it does not belong to $\mathcal{D}(\rho,\tau)$, for it does not satisfy the quasiidentity (6).

It remains the case when $m \geq 2$, $n \geq 2$. Now we consider the following algebra $\underline{A} = (\{a,b,c\};\rho,\tau)$, where $a \neq b \neq c \neq a$, and:

$$\rho(c^m) = \tau(c^n) = c,$$

$$\rho(x_1,\ldots,x_m) = a, \quad \tau(x_1,\ldots,x_n) = b,$$

if $x_i \neq c$ for some i.

The obtained algebra $\underline{A}$ does not satisfy the quasiidentity (7), for we have $\rho(c^m)=c=\tau(c^n)$, but $\rho^2(c^m b^{m-1})=a\neq b=\tau\rho(c^m b^{n-1})$. Therefore $\underline{A}\notin\mathcal{D}(\rho,\tau)$ but it can be easily seen that $\underline{A}$ satisfy all the identities of $\mathcal{D}(\rho,\tau)$. Namely, let $\xi(x_1,x_2,\dots,x_p)$, $\eta(x_1,x_2,\dots,x_p)$ be two $\{\rho,\tau\}$-terms such that (5) holds. Then we have: $\xi(c^p)=c=\eta(c^p)$. And, if $(c^p)\neq(x_1,\dots,x_p)$, then $f(\xi)=f(\eta)=\rho$ $\Longrightarrow$ $\xi(x_1,\dots,x_p)=a=\eta(x_1,\dots,x_p)$, $f(\xi)=f(\eta)=\tau\Longrightarrow\xi(x_1,\dots,x_p)=b=$ $=\eta(x_1,\dots,x_p)$. From this example it follows that $\mathcal{D}(\rho,\tau)$ is a proper quasivariety.

Thus, we have shown that if $\Omega\backslash\Omega(0)$ contains at least two operators, then $\mathcal{D}(\Omega)$ is a proper quasivariety.

3. It remains the case when Ω consists of only one n-ary operator ω, where $n\geq 1$. In this case $\mathcal{D}(\Omega)=\mathcal{D}(\omega)$ is a variety.

Let $\underline{A}=(A;\omega)$ satisfy all the identities of $\mathcal{D}(\omega)$, and let e be a symbol which does not belong to A. Consider the semigroup D which is freely generated by $B=A\cup\{e\}$ in the variety $\mathcal{D}$. Thus, D consists of all "formal products" $b_1b_2\dots b_p$ of elements of B, and

$$b_1b_2\dots b_p=b_1'b_2'\dots b_q'$$

in D iff

$q=p$, $b_\nu=b_\nu'$ or $p,q\geq 3$, $b_1=b_1'$, $b_p=b_q'$ and

$$\{b_1,\dots,b_p\}=\{b_1',\dots,b_q'\}.$$

If $u=\dots a\dots\in D$, a A, $a=\omega(a_1,\dots,a_n)$ in $\underline{A}$, and $v=\dots ea_1\dots a_n\dots$, then we say that (u,v) and (v,u) are two pairs of neighbours. Two elements $u,v\in D$ are said to be <u>equivalent</u>, and this is denoted by $u\approx v$, iff there is a sequence $u_o,\dots,u_t\in D$ such that $t\geq 0$, $u_o=u$, $u_t=v$, and (u_{i-1},u_i) is a pair of neighbours if $i\geq 1$. It is clear that $\approx$ is a congruence on D.

We will show that the following statement is satisfied:

$$a,b\in A\Longrightarrow(a\approx b\Longrightarrow a=b)\qquad(*)$$

and this will implies that $\underline{A}$ can be embedded as an Ω-subalgebra in the semigroup $S = D/\approx$.

Consider first the case $n \geq 2$. An element $u \in D$ will be called an $\underline{\omega\text{-word}}$ if $u \in A$, or $u = e\, b_1 b_2 \ldots b_p a$, where $b_\nu \in B$, $p \geq 1$, $a \in A$. The "$\underline{\text{value}}$" $[u]$ of an ω-word is defined in the following way. First, if $u \in A$, then $[u] = u$. And, if $u = eb_1 \ldots b_p a$, then $u = e^i b_1 b_2 \ldots b_p a^j$ for any $i \geq 1$, $j \geq 1$, and we can choose i,j in such a way that the term ξ, which is obtained from u by replacing any occuring of e by ω is a "continued product" in (A,ω); then, $[u]$ is the "value" of this "product". The fact that $(A;\omega)$ satisfies all the identities of $D(\omega)$ implies that the value $[u]$ of an ω-word is uniquelly determined.

Let $a,b \in A$ and $a \approx b$. Then, there exist $u_o, u_1, \ldots, u_t \in D$ such that $a=u_o$, $b=u_t$, $t \geqslant 0$ and (u_{i-1}, u_i) is a pair of neighbours for any $i \in \{1,\ldots,t\}$. If u_i is an ω-word for any i, then $[u_{i-1}] = [u_i]$, and $a=[a]=[u_1]=\ldots=[u_{t-1}]=[b]=b$.

Assume now that u_i is an ω-word if $i \in \{0,\ldots,p\}$, u_j is not an ω-word if $i \in \{p+1,\ldots,p+q-1\}$, and u_{p+q} is an ω-word. Put $u'_{p+\nu} = eu_{p+\nu}$, if $\nu \in \{1,\ldots,q-1\}$. Then we have that $(u_p, u_{p+1}), \ldots, (u_{p+q-1}, u_{p+q})$ are pairs of neighbours and thus we have $[u_p]=[u'_{p+1}]=\ldots=[u'_{p+q-1}]=[u_{p+q}]$, and therefore we consider the sequence $a=u_o,\ldots,u_p,u'_{p+1},\ldots,u'_{p+q-1},u_{p+q}$, instead of $u_o,\ldots,u_p,u_{p+1},\ldots,u_{p+q-1},u_{p+q}$. This shows that it can be assumed that all the members of the sequence $u_o,u_1,\ldots,u_t$ are ω-words which implies that a=b.

It remains the case n=1. Let $a=u_o,u_1,\ldots,u_t=b$, be as above. If t=0, then a=b, and if $t \geq 1$, then $t \geq 2$. Let t=2. Then $u_1=ec$, where $c \in A$, and $a=\omega(c)=b$. Let $n \geq 3$ and $u_2 \neq a$; by a standard induction it can be shown that if $c \in A$ and $u_j = \ldots c \ldots$, then $a=\omega^2(c)=b$.

This completes the proof of (*).

4. From 1, 2, and 3 we obtain the following main result of this note:

Theorem. $\mathcal{D}(\Omega)$ is a variety of Ω-algebras iff $\Omega\backslash\Omega(0)$ contains at most one operator.

R E F E R E N C E S

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