Proceedings of the Symposium
n-ARY STRUCTURES, Skopje 1982

REPRESENTATIONS OF SEMIGROUPS OF OPERATIONS IN SEMIGROUPS OF LEFT TRANSLATIONS

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The well known Cohn-Rebane Theorem ([1.IV.4], [8.§12]) states that any universal algebra can be embedded in a semigroup in such a way that the operations of the algebra are induced by inner left translations of semigroups. Several results concerning this Theorem are obtained in the following papers: [2] - [7], [9] - [12].

In this paper we make some investigations about representations of semigroups of finitary operations in semigroups of left translations. Namely, three classes $\underline{C}^{(1)}, \underline{C}^{(2)}, \underline{C}^{(3)}$ of semigroups of operations are associated to a class $\underline{C}$ of semigroups. In general we have that $\underline{C}^{(3)} \subseteq \underline{C}^{(2)} \subseteq \underline{C}^{(1)}$, and we consider some sets of classes of semigroups such that $\underline{C}^{(1)} = \underline{C}^{(2)} = \underline{C}^{(3)}$, or $\underline{C}^{(3)} \subset \underline{C}^{(2)} = \underline{C}^{(1)}$.

0. First we state necessary preliminary definitions.

Let A be a nonempty set, $n \geq 1$, and $O_n(A)$ be the set of n-ary operations on A. Denote by $O(A)$ the set of all non-nullary operations on A, and define a binary operation "o" on $O(A)$ in the usual way, i.e. by:

$$f o g(x_1,\dots,x_{m+n-1}) = f(g(x_1,\dots,x_m),\dots,x_{m+n-1}), \quad (1)$$

where $g \in O_m(A)$, $f \in O_n(A)$. Then, $(O(A),o)$ is a semigroup, and any subsemigroup of this semigroup is called a <u>semigroup of operations</u> on A.

Let $\underline{C}$ be a class of semigroups. A semigroup Γ of operations on a set A is said to be a member of $\underline{C}^{(1)}$ if there is a semigroup $S \in C$ and a mapping $f \mapsto \overline{f}$ of Γ into S such that $A \subseteq S$, and

$$f(a_1,\dots,a_n) = \overline{f}a_1\dots a_n, \tag{2}$$

for each $f \in \Gamma_n = \Gamma \cap O_n(A)$, $n \geq 1$, and $a_1,\dots,a_n \in A$. (The right hand side of (2) is a "product" in the semigroup S, and the left one is the element $a \in A$ such that $f:(a_1,\dots,a_n) \mapsto a$). Replacing the word "mapping" by "homomorphism", "injective homomorphism" we get the class $\underline{C}^{(2)}$, $\underline{C}^{(3)}$ respectively.

Throughout the paper the class of all: semigroups, cancellative semigroups, nilpotent semigroups, abelian semigroups, abelian cancellative semigroups, abelian nilpotent semigroups, abelian groups, abelian torsion groups, abelian groups in which orders of elements are divisors of a given integer $m \geq 2$, will be denoted respectively by: SEM, CANSEM, NILSEM, ABSEM, ABCANSEM, ABNILSEM, ABGP, ABTG, $\underline{A}_m$.

Now we can state the main results of the paper.

Theorem 1. If $\underline{C} \in \{$SEM, CANSEM, ABSEM, ABCANSEM, ABGP$\}$ then

$$\underline{C}^{(1)} = \underline{C}^{(2)} = \underline{C}^{(3)}.$$

Theorem 2. If $\underline{C} \in \{$ABTG, NILSEM, ABNILSEM$\} \cup \{\underline{A}_m \mid m \geq 2\}$, then $\underline{C}^{(3)} \subset \underline{C}^{(2)} = \underline{C}^{(1)}$, and the inclusion is strict.

We note that for any class of semigroups $\underline{C}$ the following inclusions hold: $\underline{C}^{(3)} \subseteq \underline{C}^{(2)} \subseteq \underline{C}^{(1)}$.

1. Here we will show that $SEM^{(3)}$ consists of all the semigroups of finitary operations and that will imply the equalities $SEM^{(1)} = SEM^{(2)} = SEM^{(3)}$.

Let Γ be a semigroup of operations on a set A, and let $F_\Gamma = (A \cup \Gamma)^*$ be the free monoid on $A \cup \Gamma$, i.e. the set of all words on $A \cup \Gamma$ (including the empty word), and the operation is the usual concatenation. A word $u \in F_\Gamma$ is said to be a

Γ-word if $u \in A$, or $u = fu_1 \dots u_n$, where $f \in \Gamma_n$, and $u_1, \dots, u_n$ are Γ-words. There is a mapping $u \mapsto [u]$ of the set of Γ-words in A defined in the following way: (a) if $u \in A$, then $[u] = u$; (b) if $u = fu_1 \dots u_n$, then $[u] = f([u_1], \dots, [u_n])$.

We define a relation $\vdash$ in F_Γ in the following way:
(c) if $a = f(a_1, \dots, a_n)$, then $\dots a \dots \vdash \dots fa_1 \dots a_n \dots$,
(d) if $f = goh$ in Γ then $\dots f \dots \vdash \dots gh \dots$.

Let $\vdash\dashv$ be the symmetric extension of $\vdash$ (i.e. $u \vdash\dashv v$ iff $u \vdash v$ or $v \vdash u$), and $\approx$ the transitive and reflexsive extension of $\vdash\dashv$, i.e.:

$$u \approx v \Leftrightarrow (\exists u_0, \dots, u_p \in F_\Gamma) p \geq 0,\ u = u_0,\ v = u_p,\ u_{i-1} \vdash\dashv u_i,\ i \geq 1.$$

Clearly, $\approx$ is a congruence on F_Γ, and the following conditions are satisfied:

(i) $f \in \Gamma \Rightarrow (f \approx u \Leftrightarrow u = f_1 \dots f_k$ and $f = f_1 o \dots o f_k$ in $\Gamma)$

(ii) $a \in A \Rightarrow (a \approx u \Leftrightarrow u$ is a Γ-word such that $[u] = a)$.

This implies that we may assume that Γ is a subsemigroup of $\Gamma^\wedge = F_{\Gamma/\approx}$, and that A is a subset of Γ. Moreover, then any equation of the form (2) is satisfied, with $f = \overline{f}$.

Thus, we have proved that any semigroup of operations belongs to $SEM^{(3)}$.

2. The problem of embeddings of universal algebras in cancellative semigroups is considered in [11]. A detailed proof of the statements: $CANSEM^{(3)} = CANSEM^{(2)} = CANSEM^{(1)}$, $ABCANSEM^{(3)} = ABCANSEM^{(2)} = ABCANSEM^{(1)}$, will be given in [5], and here we will only state some results.

Assume first that $\Gamma \in CANSEM^{(1)}$. Let F_Γ be defined as in 1, and let $u', u'', u_1, u_2, v, w \in F_\Gamma$ be such that $u'vu''$, $u'wu''$, u_1vu_2, u_1wu_2 be Γ-words. Then:

(α) $\qquad [u'vu''] = [u'wu''] \Rightarrow [u_1vu_2] = [u_1wu_2]$.

Conversely, let Γ be a semigroup of operations on a set A such that all the implications (α) are satisfied. Consider first the case $\Gamma \neq \Gamma_1$, and define a relation $\equiv$ in F_Γ in the following way: $v \equiv w$ iff there exist $u', u'' \in F_\Gamma$ such that $u'vu'', u'wu''$ are Γ-words and $[u'vu''] = [u'wu'']$. Then, $\equiv$ is a congruence on F_Γ, the semigroup $\Gamma = F_{\Gamma/\equiv}$ is a cancellative semigroup. Moreover we have:

$$a,b \in A \Rightarrow (a \equiv b \Rightarrow a = b)$$

$$f,g \in \Gamma \Rightarrow (f \equiv g \Rightarrow f = g)$$

$$f,g,h \in \Gamma \Rightarrow (f = goh \Rightarrow f \equiv gh)$$

$$f \in \Gamma_n,\ a_1,\ldots,a_n \in A \Rightarrow (f(a_1,\ldots,a_n) \equiv fa_1\ldots a_n),$$

and this implies that $\Gamma \in \text{CANSEM}^{(3)}$.

It remains the case when $\Gamma = \Gamma_1$. We may assume that Γ is a monoid. Define an operation $\bullet$ on the set $S = A^* \times \Gamma$ (A^* is the free monoid on A) in the following way:

$$(\underline{a},f)\bullet(b\underline{c},g) = (\underline{a}f(b)\underline{c},g)$$

$$(\underline{a},f)\bullet(1,g) = (\underline{a},fog),$$

where $\underline{a},\underline{c} \in A^*$, $b \in A$. Then we get a cancellative semigroup and moreover we may assume that $A \subseteq S$, and that Γ is a subsemigroup of S such that

$$f \in \Gamma,\ a \in A \Rightarrow f(a) = (f(a),1) = (1,f)(a,1) = f\bullet a.$$

Thus, $\Gamma \in \text{CANSEM}^{(3)}$, i.e.

$$\text{CANSEM}^{(1)} = \text{CANSEM}^{(3)}.$$

<u>3</u>. Let $\Gamma \in \text{ABSEM}^{(1)}$. Then the following condition is satisfied:

(β) If $u,v \in F_\Gamma$ are two Γ-words such that v is a permutation of u then $[u] = [v]$.

Conversely, let Γ satisfy (β). Clearly, we may assume that Γ is a monoid, for if it is not one, we can consider the monoid $\Gamma^1 = \Gamma \cup \{1_A\}$, which also satisfies the condition (β). Denote by B_A the free abelian monoid on A, and consider the direct sum $L = \Gamma \oplus B_A$. If $\approx$ is the minimal congruence on L such that:

$$f(a_1,\dots,a_n) = a \Longrightarrow fa_1\dots a_n \approx a,$$

(i.e. $\approx$ is the reflexive and transitive extension of $\vdash\!\dashv$, where $\vdash\!\dashv$ is defined as in 1), then it distinguishes the elements of $\Gamma \cup A$. This implies that Γ can be embedded in $K = L/\approx$ in the desired way.

4. It is shown in [5] that $ABCANSEM^{(3)} = ABSEM^{(3)} \cap$ $\cap\, CANSEM^{(3)}$, and this implies that:

$$ABCANSEM^{(3)} = ABCANSEM^{(2)} = ABCANSEM^{(1)}.$$

Clearly, $ABGP^{(i)} = ABCANSEM^{(i)}$.

Thus, the proof of Theorem 1 is completed.

5. Here we will show that $NILSEM^{(3)} \subset NILSEM^{(2)} =$ $= NILSEM^{(1)}$.

Assume that $\Gamma \in NILSEM^{(1)}$. Then, it is clear that the following condition is satisfied.

(i) There exists a positive number $m \geq 2$, such that if u and v are Γ-words with lengths not less than m, then $[u] = [v]$.

Let Γ be a semigroup of operations on a set A, such that (γ) is satisfied. Then, there exists an element $e \in A$ such that:

$$f(a_1,\dots,a_{i-1},e,a_{i+1},\dots,a_n) = e, \; [u] = e,$$

for any Γ-word u with a length $\geq m$, any $f \in \Gamma_n$, $a_j \in A$, and $i \in N_n$.

Consider the semigroup $\Gamma^\dagger = \Gamma^\wedge \setminus \{1\}$, where $\Gamma^\wedge$ is defined in 1 and recall that if $a \in A$, $u \in F_\Gamma$, then $a \approx u$ iff u is a

Γ-word such that $[u] = a$. Let $I = \{\overline{u}_1\overline{u}_2\ldots\overline{u}_m \mid u_1,\ldots,u_m \in F_\Gamma^\dagger\}$, where $\overline{u}$ is the element of $\Gamma^\dagger$ defined by $u \in F_\Gamma^\dagger$. Then $I \cap \overline{A} = \{\overline{e}\}$, and this implies that we may assume that $A \subset \Gamma^\dagger/_I$. The semigroup $S = \Gamma^\dagger/_I$ is obviously nilpotent. The mapping $f \mapsto \overline{f}$ is injective iff $\Gamma = \Gamma_1$. This shows that

$$\mathrm{NILSEM}^{(3)} \subset \mathrm{NILSEM}^{(2)} = \mathrm{NILSEM}^{(1)},$$

and the inclusion is strict.

In the same way it can be shown that

$$\mathrm{ABNILSEM}^{(3)} \subset \mathrm{ABNILSEM}^{(2)} = \mathrm{ABNILSEM}^{(1)},$$

with a strict inclusion. We note that $\mathrm{NILSEM}^{(3)}$ ($\mathrm{ABNILSEM}^{(3)}$) consists of (abelian) nilpotent semigroups of unary operations.

6. The class $\mathrm{ABTG}^{(3)}$ will be described here. First, if $f \in \Gamma\setminus\Gamma_1$ then f has an infinite order, and therefore, if $\Gamma \in \mathrm{ABTG}^{(3)}$, then $\Gamma = \Gamma_1$ is a semigroup of transformations on a set A. It is also clear that then $\Gamma \in \mathrm{ABTG}$, and that the following condition is satisfied:

$$(\gamma) \qquad (\forall f,g \in \Gamma)\{[(\exists x \in A) f(x) = g(x)] \Rightarrow f = g\}.$$

We will show now that if Γ is a group of permutations on A such that $\Gamma \in \mathrm{ABTG}$ and the condition (γ) is satisfied, then $\Gamma \in \mathrm{ABTG}^{(3)}$.

Define a relation on A in the following way:

$$a \approx b \iff (\exists f \in \Gamma)\ b = f(a).$$

Then $\approx$ is an equivalence on A. Let B be a subset of A such that for each $a \in A$ there exists exactly one element $b \in B$ such that $a \approx b$.

Then the following statement is satisfied:

(i) For any $a \in A$ there exists one and only one pair $(f,b) \in \Gamma \times B$ such that $a = f(b)$.

Namely, if $a \in A$ then, there exist $b \in B$, $f \in \Gamma$ such that $f(b) = a$. If $g \in \Gamma$, $c \in B$ are such that $g(c) = a$, first we get $b = c$, and then by (γ) $f = g$.

Let P_B be an abelian torsion group generated by B, and let $G = \Gamma \times P_B$. A typical element of G will be denoted by $fb_1^{\beta_1}\ldots b_s^{\beta_s}$, $b_\nu \in B$, $b_\nu \neq b_\lambda$ if $\nu \neq \lambda$. If $b \in B$, $f \in \Gamma$ and $f(b) = a \in A$, then we will assume that $a = fb$. Let $b \in B$, $g, f \in \Gamma$ and $f(b) = a$. Then we have

$$g(a) = g(f(b)) = gfb = g(fb) = ga.$$

This completes the proof that $\Gamma \in ABTG^{(3)}$.

It is not difficult to give an example of a group of permutations $\Gamma \in ABTG$ on A such that (γ) is not satisfied. For example, if $A = \{a,b,c\}$, and $f = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$. then $\{1_A, f\} \in ABTG$, but (γ) is not satisfied.

7. Let $\Gamma \in ABTG^{(1)}$. Then, there is a mapping $m: z \mapsto m(z)$ of $A \cup \Gamma$ in the set of positive integers such that the following condition is satisfied.

(δ) Let $u, v \in F_\Gamma$ be two Γ-words such that $|u|_z \equiv |v|_z (\mathrm{mod}(m(z)))$ for every $z \in A \cup \Gamma$. Then, $[u] = [v]$. (Here, $|u|_z$ is the number of occurences of z in u).

We shall show now that if $\Gamma \neq \Gamma_1$ and if Γ satisfies the condition (δ) then $\Gamma \in ABTG^{(2)}$, and this will imply that $ABTG^{(1)} = ABTG^{(2)}$.

First we note that if Γ is not a monoid and if we put $m(1_A) = 1$, then the monoid $\Gamma^1 = \Gamma \cup \{1_A\}$ satisfies the condition (δ). Further on we will assume that Γ is a monoid.

Define a relation $\sim$ in Γ in the following way: $f \sim g$ iff there exists a sequence $f_1, \ldots, f_p$ of elements of Γ such that $p \geq 3$ and:

$$
\begin{aligned}
&f = f_1 o f_2^{m(f_2)} \\
&f_1 o f_3^{m(f_3)} = f_4 o f_5^{m(f_5)} \\
&\dots\dots\dots\dots\dots\dots\dots \\
&f_{p-2} o f_p^{m(f_p)} = g.
\end{aligned}
\qquad (3)
$$

It can be easily seen that:

(i) The relation $\sim$ is a congruence on Γ and

$$\overline{\Gamma} = \Gamma/\sim \ \in \text{ABTG}.$$

Now we will prove the following proposition:

(ii) If $f \in \Gamma_{n'}$, $g \in \Gamma_{n''}$, $a_1, \dots, a_q \in A$ and α_ν, β_ν are nonnegative integers such that $f \sim g$, $\alpha_\nu \equiv \beta_\nu \pmod{m(a_\nu)}$, and $\alpha_1 + \dots + \alpha_q = n'$, $\beta_1 + \dots + \beta_q = n''$, then

$$f(a_1^{\alpha_1}, \dots, a_q^{\alpha_q}) = g(a_1^{\beta_1}, \dots, a_q^{\beta_q}).$$

Assume that equations (3) are satisfied. For technical reasons we will consider the case when $p = 6$, i.e.

$$f = f_1 o f_2^{m_2},\ f_1 o f_3^{m_3} = f_4 o f_5^{m_5},\ f_4 o f_6^{m_6} = g,$$

where $m_\nu = m(f_\nu)$.

Let $f_\nu \in \Gamma_{n_\nu}$, and $n_3, n_6 \geq 2$. Then, we have

$$n' = n_1 + m_2(n_2 - 1),\ n_1 + m_3(n_3 - 1) = n_4 + m_5(n_5 - 1),$$

$$n_4 + m_6(n_6 - 1) = n''.$$

Let $s_3, t_3, s_6, t_6 > 0$ be such that

$$s_3 m_3 (n_3 - 1) = t_3 m_1',\ s_3 m(f_6)(n_6 - 1) = t_6 m_1',$$

where $m_1' = m(a_1)$.

Then we have:

$$f(a_1^{\alpha_1}, \dots, a_q^{\alpha_q}) = f_1 f_2^{m_2} f_3^{s_3 m_3} (a_1^{t_3 m_1' + \alpha_1} a_2^{\alpha_2}, \dots, a_q^{\alpha_q}) =$$

$$= f_2^{m_2} f_3^{(s_3 - 1) m_3} f_1 f_3^{m_3} (a_1^{t_3 m_1' + \alpha_1}, a_2^{\alpha_2}, \dots, a_q^{\alpha_q}) =$$

$= f_2^{m_2} f_3^{(s_3-1)m_3} f_4 f_5^{m_5}(a_1^{t_3m_1'+\alpha_1}, a_2^{\alpha_2}, \dots, a_q^{\alpha_q}) =$

$= f_2^{m_2} f_3^{(s_3-1)m_3} f_4 f_5^{m_5} f_6^{s_6m_6}(a_1^{(t_3+t_6)m_1'+\alpha_1}, a_2^{\alpha_2}, \dots, a_q^{\alpha_q}) =$

$= f_2^{m_2(s_3-1)m_3} f_5^{m_5} f_6^{(s_6-1)m_6} g(a_1^{(t_3+t_6)m_1'+\alpha_1}, a_2^{\alpha_2}, \dots, a_q^{\alpha_q}) =$

$= g(a_1^{\beta_1}, a_2^{\beta_2}, \dots, a_q^{\beta_q})$.

If $n_3 = 1$ then $s_3 = 1$, $t_3 = 0$, and also if $n_6 = 1$ then $s_6 = 1$, $t_6 = 0$.

As a corollary of (ii) we get that:

(iii) If $f \sim g$ and $f, g \in \Gamma_n$ then $f = g$.

Let $a \in A$ and let C_a be the cyclic group with a generator a and order $m(a)$. Consider the direct sums

$$H = \oplus \sum_{a \in A} C_a, \quad K = \overline{\Gamma} \oplus H.$$

A typical element of K will be denoted by $\overline{f} a_1^{\alpha_1} a_2^{\alpha_2} \dots a_q^{\alpha_q}$, where $f \in \Gamma$, $a_\nu \in A$, and $a_\nu \neq a_\lambda$ if $\nu \neq \lambda$. Then:

$$\overline{f} a_1^{\alpha_1} \dots a_q^{\alpha_q} = \overline{g} a_1^{\beta_1} \dots a_q^{\beta_q} \Leftrightarrow f \sim g \text{ and } \alpha_\nu \equiv \beta_\nu (\mathrm{mod}\, m(a_\nu)).$$

An element $u = \overline{f}\underline{b} \in K$ is called a Γ-word with a value $a = [u]$ iff the corresponding element $f\underline{b} \in F_\Gamma$ is a Γ-word with a value a. By (ii) the value $[u]$ of a Γ-word $u \in K$ is uniquely determined.

Define a relation $\vdash$ in K in the following way: if $u = \overline{g} a_1^{\alpha_1} \dots a_q^{\alpha_q}$, $\alpha_\nu \geq 0$, $\alpha_1 > 0$ and $a_1 = h(a_1^{\gamma_1}, \dots, a_q^{\gamma_q})$, then:

$$u \vdash \overline{gh}\, a_1^{\alpha_1+\gamma_1-1} a_2^{\alpha_2+\alpha_2} \dots a_q^{\alpha_q+\alpha_q}.$$

Let $\vdash\!\dashv$ be the symmetric extension of $\vdash$, and $\approx$ the transitive extension of $\vdash\!\dashv$. Then $\approx$ is a congruence on K which satisfies the following propositions.

(iv) $\qquad f(a_1, \dots, a_n) = a \Rightarrow a \approx \overline{f} a_1 \dots a_n$,

and

(v) $a,b \in A \Rightarrow (a \approx b \Rightarrow a = b)$,

from which it follows that $\Gamma \in ABTG^{(2)}$.

The proposition (iv) is namely obvious, and (v) is an immediate consequence from the following statement:

(vi) Let $u,v \in K$ be such that $u \vdash v$. Then u is a Γ-word iff v is a Γ-word, and then $[u] = [v]$.

Assume first that $f_0 \in \Gamma_{n_0}$, when $n_0 \geq 2$, and that u is a Γ-word. Then we have:

$$u = \overline{f}a_1^{\alpha_1}\ldots a_q^{\alpha_q} = \overline{g}a_1^{\beta_1}\ldots a_q^{\beta_q},$$

where $\alpha_\nu \equiv \beta_\nu (\mathrm{mod}\, m(a_\nu))$, $\alpha_\nu, \beta_\nu \geq 0$, $\beta_1 > 0$, $f \sim g$, $\alpha_1+\ldots+\alpha_q = n$, $f \in \Gamma_n$ and:

$$v = \overline{gh}a_1^{\beta_1+\gamma_1-1} a_2^{\beta_2+\gamma_2}\ldots a_q^{\alpha_q+\gamma_q},$$

where $h \in \Gamma_{n'}$, $n' = \gamma_1+\ldots+\gamma_q$, $a_1 = h(a_1^{\gamma_1},\ldots,a_q^{\gamma_q})$.

Denote $m(f_0)$ by m_0, and $m(a_\nu)$ by m_ν'.

For any $s,t \geq 0$ we have:

$$v = \overline{fh}a_1^{\alpha_1+\gamma_1-1+r_1m_1'} a_2^{\alpha_2+\gamma_2}\ldots a_q^{\alpha_q+\gamma_q} =$$
$$= \overline{fhf_0^{sm_0}}a_1^{\alpha_1+\gamma_1-1+(r_1+t)m_1'} a_2^{\alpha_2+\gamma_2}\ldots a_q^{\alpha_q+\gamma_q},$$

where $\beta_1 = \alpha_1+r_1m(a_1)$. If we chose $s,t > 0$ such that $r_1+t \geq 0$ and $sm_0(n_0-1) = (r_1+t)m_1'$, then we get that:

$$fhf^{sm_0} \in \Gamma_{n+n'-1+sm_0(n_0-1)},$$

and:

$$n+n'-1+sm_0(n_0-1) = \alpha_1+\ldots+\alpha_q+\gamma_1+\ldots+\gamma_q+(r_1+t)m_1'-1.$$

Therefore, v is Γ-word and:

$$[\overline{v}] = fhf^{sm_0}(a_1^{\alpha_1+\gamma_1-1+(r_1+t)m_1'}, a_2^{\alpha_2+\gamma_2},\ldots,a_q^{\alpha_q+\gamma_q}) =$$
$$= ff_0^{sm_0}(a_1^{\alpha_1-1+(r_1+t)m_1'}, a_2^{\alpha_2},\ldots,a_q^{\alpha_q}, h(a_1^{\gamma_1},\ldots,a_q^{\gamma_q})) =$$

$= ff_0^{sm_0}(a_1^{\alpha_1+(r_1+t)m_1'}, a_2^{\alpha_2}, \ldots, a_q^{\alpha_q}) = f(a_1^{\alpha_1}, \ldots, a_q^{\alpha_q}) =$

$= [u]$.

It remains the case when v is a Γ-word. Namely, let $v = \overline{fa_1^{\alpha_1} \ldots a_q^{\alpha_q}} = \overline{gha_1^{\beta_1} \ldots a_q^{\beta_q}}$, $u = \overline{ga_1^{\beta_1-\gamma_1+1} a_2^{\beta_2-\gamma_2} \ldots a_q^{\beta_q-\gamma_q}}$, where $f \in \Gamma_n$, $g \in \Gamma_{n'}$, $h \in \Gamma_{n''}$, $n = \alpha_1+\ldots+\alpha_q$, $n'' = \gamma_1+\ldots+\gamma_q$, $\alpha_\nu \equiv \beta_\nu (\mathrm{mod}\, m_\nu')$, α_ν, $\beta_\nu-\gamma_\nu \geq 0$, and $f \sim gh$.

By definition of the relation $\sim$, there exist $f_\nu \in \Gamma_{n_\nu}$, $1 \leq \nu \leq p$, $(p \geq 3)$ such that

$$f = f_1 o f_2^{m_2},\ f_1 o f_3^{m_3} = f_4 o f_5^{m_5}, \ldots, f_{p-2} o f_p^{m_p} = gh,$$

and this implies the following equation:

$$\begin{aligned} &n+m_3(n_3-1)+m_6(n_6-1)+\ldots+m_r(n_p-1) = \\ &= n'+n''-1+m_2(n_2-1)+m_5(n_5-1)+\ldots+m_{p-1}(n_{p-1}-1). \end{aligned} \quad (4)$$

As in the proof of (ii), we will consider the special case $p = 3$. Then (4) reduces to

$$n+m_3(n_3-1)+m_6(n_6-1) = n'+n''+m_2(n_2-1)+m_5(n_5-1)-1. \quad (4')$$

Clearly, there exist positive integers s,t such that:

$$s[m_0(n_0-1)+m_3(n_3-1)+m_6(n_6-1)] = t(m_1'+m_2'+\ldots+m_q')$$

and $\alpha_\nu+t\, m_\nu' > \gamma_\nu$. Then:

$$v = \overline{ff_0^{sm_0}f_3^{sm_3}f_6^{sm_6}} a_1^{\alpha_1+tm_1'} \ldots a_q^{\alpha_q+tm_q'}$$

and

$$n+sm_0(n_0-1)+sm_3(n_3-1)+sm_6(n_6-1) = \alpha_1+\ldots+\alpha_q+tm_1'+\ldots+tm_q'.$$

By (4'), from the last equation we obtain:

$$\begin{aligned} &n''+sm_0(n_0-1)+m_2(n_2-1)+m_5(n_5-1)+(s-1)m_3(n_3-1)+(s-1)m_6(n_6-1) = \\ &= (\alpha_1-\gamma_1+1)+(\alpha_2-\gamma_2)+\ldots+(\alpha_q-\gamma_q)+tm_1'+\ldots+tm_q', \end{aligned}$$

and this implies that u is a Γ-word, for

$$u = \overline{gf_0^{sm_0}f_2^{m_2}f_5^{m_5}f_3^{(s-1)m_3}f_6^{(s-1)m_6}} a_1^{\alpha_1-\gamma_1+1+tm_1'} a_2^{\alpha_2-\gamma_2+tm_2'} \ldots a_q^{\alpha_q-\gamma_q+tm_q'}.$$

Then we also have

$$u = gf_0^{sm_0} f_2^{m_2} f_5^{m_5} f_3^{(s-1)m_3} f_6^{(s-1)m_6} (a_1, a_1^{\alpha_1-\gamma_1+tm_1'}, \dots, a_q^{\alpha_q-\gamma_q+tm_q'}) =$$
$$= ghf_0^{sm_0} f_2^{m_2} f_5^{m_5} f_3^{(s-1)m_3} f_6^{(s-1)m_6} (a_1^{\alpha_1+tm_1'}, \dots, a_q^{\alpha_q+tm_q'}) = [v].$$

This completes the proof of (vi), and therefore the equation $ABTG^{(1)} = ABTG^{(2)}$ is proved.

Let us give an example of a semigroup of operations Γ on A such that $\Gamma \in ABTG^{(2)} \setminus ABTG^{(3)}$.

Let $A = \{0,1\} = Z_2$ and $f(x,y) = x+y \pmod 2$. Then the semigroup $\Gamma = \{f, f^2, \dots, f^k, \dots\}$ belongs to $ABTG^{(2)}$, for

$$f^k(x_0, x_1, \dots, x_k) = x_0 + x_1 + \dots + x_k,$$

and $\overline{f^k} = 0$.

8. From the results obtained in 7 it follows that a semigroup of operations Γ on A such that $\Gamma \neq \Gamma_1$ belongs to $\underline{A}_m^{(1)} = \underline{A}_m^{(2)}$ if the following condition is satisfied:

(δ_m) If $u, v \in F_\Gamma$ are Γ-words such that $|u|_z \equiv |v|_z \pmod{m(z)}$ for every $z \in A \cup \Gamma$, then $[u] = [v]$.

And, from 6 it follows that if $\Gamma = \Gamma_1$ then $\Gamma \in \underline{A}_m^{(3)} \Leftrightarrow$ $\Leftrightarrow \Gamma \in \underline{A}_m^{(2)}$. Moreover $\Gamma \in \underline{A}_m^{(3)}$ if $\Gamma = \Gamma_1 \in \underline{A}_m$ and if the condition (δ) stated in 7 is satisfied.

Therefore: $\underline{A}_m^{(3)} \subset \underline{A}_m^{(2)} = \underline{A}_m^{(1)}$.

This completes the proof of Theorem 2.

9. (i) Convenient descriptions of $SEM^{(1)}$, $ABSEM^{(1)}$, $CANSEM^{(1)}$, $NILSEM^{(1)}$, $ABTG^{(1)}$, $\underline{A}_m^{(1)}$ are given in [1], [10], [11], [12], [4], [4] respectively. But, these papers do not deal with semigroup of operations. Namely, let Ω be a set of finitary operators and $\underline{C}$ a class of semigroups. Then $\underline{C}(\Omega)$ is the class of Ω-algebras (A, Ω) with the following property. There exists a semigroup $S \in \underline{C}$ and a mapping $f \mapsto \overline{f}$ of Ω into $\underline{C}$

such that (2) is satisfied. Namely, the classes SEM(Ω), ABSEM(Ω), CANSEM(Ω), NILSEM(Ω), ABTG(Ω), A_m(Ω) are described in the mentioned papers. And, the class $\underline{A}_{r,m}$(Ω) is described in [3].

(ii) From the results of the papers [3], [4] and [6] it follows that there exists a variety $\underline{C}$ of semigroups and a set of operators Ω such that $\underline{C}$(Ω) is not a variety. We do not know any convenient description of the set of varieties $\underline{C}$ of semigroups such that $\underline{C}$(Ω) is also a variety, for any $\underline{S} \in C$.

(iii) We do not know any class $\underline{C}$ of semigroups such that $\underline{C}^{(2)}$ is a proper subclass of $\underline{C}^{(1)}$.

(iv) We find it interesting to look for corresponding description of $\underline{C}$(Ω), $\underline{C}^{(1)}$, $\underline{C}^{(2)}$, $\underline{C}^{(3)}$, when $\underline{C}$ is the class of: groups, periodic groups, idempotent semigroups, inverse semigroups, regular semigroups, finite semigroups.

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