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WORD PROBLEM FOR n-GROUPS

Biljana Janeva

In this paper original results will not be presented. As a direct consequence of a property for presentations of n-groups a connection between some known resuts for solvability of the word problem for presentations of groups and presentations of n-groups will be given. Also, some interesting open problems for presentations of n-groups and n-semigroups will be pointed out.

1. Presentations of n-groups

Let $B\neq\emptyset$, $B'=B\cup B^{-1}$ and $F'=F_{B'}^{(n)}$ be the free n-semigroup generated by B'. Let Λ be a set of words $w=b_1^{\alpha_1}b_2^{\alpha_2}\ldots b_k^{\alpha_k}$, $b_i\in B$, $\alpha_i\in Z$, such that $|w|=\alpha_1+\ldots+\alpha_k\equiv 0\,(\text{mod } n-1)$.

Define a relation $\sim$ of two words u and v in F', as follows:

(i) $u=u_1bb^{-1}u_2$, $b\in B'$, $v=u_1u_2 \Longrightarrow u\sim v$

(ii) $u=u_1wu_2$, $v=u_1u_2$, $w\in\Lambda \Longrightarrow u\sim v$

(iii) $u\sim v$ iff there exists a sequence $u_o,u_1,\ldots,u_s$ such that $u=u_o$, $v=u_s$, and $u_i\sim u_{i+1}$ $(i=0,1,\ldots,s-1)$ by (i) or (ii).

The relation $\sim$ is a congruence on F' and $F'/\!\sim$ is an n-group. We say that the n-group $F'/\!\sim$ has a presentation $\langle B;\Lambda\rangle_n$. The presentation $\langle B;\Lambda\rangle_2$ is a presentation of a group and we will denote it simply by $\langle B;\Lambda\rangle$.

For the presentations $\langle B;\Lambda\rangle_n$ and $\langle B;\Lambda\rangle$ the following properties are valid ([1]):

1.1° $\langle B;\Lambda\rangle_n^{\wedge}=\langle B;\Lambda\rangle$, where $\langle B;\Lambda\rangle_n^{\wedge}$ is the universal covering of $\langle B;\Lambda\rangle_n$.

1.2° The presentation $\langle B;\Lambda\rangle_n$ has a solvable word problem iff its universal covering $\langle B;\Lambda\rangle$ has.

Having in mind the properties 1.1° and 1.2° for presentations of n-groups and the fact that every presentation of group with one defining relator has a solvable word problem ([3]) it is obvious that the presentations of n-groups with one defining relator have a solvable word problem. The algorithm is the same as for presentations of groups with one defining relator.

In the "small cancellation" theory the symmetrized set of defining relator is used. Clearly, if Λ is a set of words $a_1^{\alpha_1}a_2^{\alpha_2}\dots a_k^{\alpha_k}$ such that $\sum_{i=1}^{k}\alpha_i\equiv 0(\mathrm{mod}\ n-1)$, then the symmetrized set Λ^S of Λ also satisfies this condition, and, moreover, $\langle B;\Lambda^S\rangle_n$ is a presentation of the same n-group. Thus, the Greendlinger's lemma and the Lyndon's theorem ([2]) give sufficient conditions for solvability of the word problem for presentations of n-groups, as well.

We can notice here again that the defining relators of the set Λ of the n-group $\langle B;\Lambda\rangle_n$ have a special form, i.e. the sum of their exponents is congruent modulo n-1. The known examples of finite presentations of groups with unsolvable word problem do not have this property, so we do not have examples of presentations of n-groups, $n\geq 3$, with an unsolvable word problem.

2. Presentations of n-semigroups

Let $B\neq\emptyset$ and $F=F_B^{(n)}$ be the free n-semigroup generated by B, (i.e. consists of all words $w=b_1\dots b_p$, $b_i\in B$, such that $p\equiv 1(\mathrm{mod}\ n-1)$), and let the n-ary operation [ ] on F be defined by:

$$[w_1 \ldots w_n] = w_1 \ldots w_n$$

Let $\Lambda \subseteq F \times F$ and Λ^* be the congruence over F generated by Λ. Then $Q=F/\Lambda^*$ is an n-semigroup, and we say that $\langle B;\Lambda\rangle_n$ is a presentation of the n-semigroup Q.

Like before, denote the presentation of the semigroup $\langle B;\Lambda\rangle_2$ by $\langle B;\Lambda\rangle$. For the presentations $\langle B;\Lambda\rangle_n$ and $\langle B;\Lambda\rangle$ the following property is valid ([1]):

2.1° $\langle B;\Lambda\rangle_n^{\wedge} = \langle B;\Lambda\rangle$

Concerning the property 1.2° for n-groups, for the presentations of n-semigroups it is only clear that if the semigroup $\langle B;\Lambda\rangle$ has a solvable word problem so does $\langle B;\Lambda\rangle_n$. Also, if $\langle B;\Lambda\rangle_n$ is an n-semigroup with a left (right) cancellation property, then if $\langle B;\Lambda\rangle_n$ has a solvable word problem, so does $\langle B;\Lambda\rangle$.

We point out the following open problems for presentations of n-semigroups:

I. Is there a presentation of an n-semigroup with an unsolvable word problem?

II. Let there exist an algorithm for solving the word problem for $\langle B;\Lambda\rangle_n$. Is there an algorithm for solving the word problem of its universal covering?

R E F E R E N C E S

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Faculty of Mathematics, Skopje,
P.O.Box 504, Yugoslavia