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# SUBALGEBRAS OF DISTRIBUTIVE AND COMMUTATIVE SEMIGROUPS

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Let $\mathbf{A} = (A; \Omega)$ be an arbitrary $\Omega$ — algebra. Say that $\mathbf{A}$ is a *subalgebra of a semigroup* $S(.)$ if $A \subseteq S$ and there exists a mapping $\omega \to \bar{\omega}$ from $\Omega$ in $S$, such that:

$$(1) \qquad \omega(a_1 . a_2, \ldots, a_n) = \bar{\omega} \cdot a_1 \cdot a_2 \cdot \ \ldots \ . a_n$$

for every $\omega \in \Omega(n)$, $a_1 a_2 \ldots, a_n \in A$ ($\Omega(i)$ stands for the all i-ary operators in $\Omega$).

In this note we are dealing with the class of subalgebras of distributive and commutative semigroups, i.e. the semigroups satisfying the identities:

$$(2) \qquad x \cdot y = y \cdot x$$

$$(3) \qquad x \cdot y \ z = x^2 \cdot y \cdot z$$

Denote the variety of the distributive and commutative semigroups by $\mathfrak{D}_c$ and by $\mathfrak{D}_c(\Omega)$ the class of subalgebras of the semigroups belonging to $\mathfrak{D}_c \cdot \mathfrak{D}_c$ $(\Omega)$ is a quasivariety ([1], p. 274). Let $\mathfrak{D}$ be the class of distributive semigroups, i.e. the semigroups satisfying the identities $x \cdot y \cdot z = x \cdot y \cdot x \cdot z$ and $x \cdot y \cdot z = x \cdot z \cdot y \cdot z$. The class $\mathfrak{D}(\Omega)$ of the $\Omega$-subalgebras of the semigrops in $\mathfrak{D}$ is a variety iff (if and only if) $|\Omega \setminus \Omega(o)| = 1$ ([2]). The fact that $\mathfrak{D}_c(\Omega)$ is a subcalss of $\mathfrak{D}(\Omega)$ can not be used directly in order to conclude whether $\mathfrak{D}_c(\Omega)$ is, or is not a variety. Namely, we shall show in this notice that $\mathfrak{D}_c(\Omega)$ is always a variety and it is well known that every algebra is a subalgebra of some semigroup ([3]).

Let $\zeta$ be an $\Omega$-term, or a term interpretation in a certain algebra. Then $\hat{\zeta}$ will stand for the set of symbols occuring in $\zeta$ and $|\zeta|$ for the length of $\zeta$.

Using the last notation we can determine the complete system of identites in $\mathfrak{D}_c$ in a rather simple manner: $\zeta = \eta$ is an identity (valid) in $\mathfrak{D}_c$ iff $\hat{\zeta} = \hat{\eta}$ and $|\zeta|, |\eta| \geqslant 3$, or it is a trivial one.

Let us state the following easy-to-check properties:

**Proposition 1.** *$\mathfrak{D}_c(\Omega)$ is a variety iff $\mathfrak{D}_c(\Omega \setminus \Omega(o))$ is a variety.*

**Proposition 2.** $\zeta = \eta$ is an identity in $\mathfrak{D}_c(\Omega)$ iff $\hat{\zeta} = \hat{\eta}$ and $|\zeta|, |\eta| \geqslant 3$, or it is a trivial one.

The first property enables us to suppose that $\Omega(o) = \emptyset$.

Let the algebra $\mathbf{A} = (A; \Omega)$ satisfy the identities in $\mathfrak{D}_c(\Omega)$ and let $\overline{\Omega} = \{\overline{\omega} \mid \omega \in \Omega\}$ be a set disjointed from the set $A$, its elements satisfying the implication $\overline{\omega} = \overline{\tau} \Rightarrow \omega = \tau$. Let $S(\cdot)$ be the free semigroup in $\mathfrak{D}_c$ generated by the set $A \cup \overline{\Omega}$. If $u_1, u_2 \in S$, we say that $u_1$ and $u_2$ are neighbours (that is $u_1$ is a neighbour to $u_2$ and vice versa) if $u_1 = \ldots a \ldots$ and $u_2 = = \ldots \overline{\omega} . a_1 . a_2 . \ldots . a_n \ldots$ where $\omega \in \Omega(n)$ and $\omega(a_1, a_2, \ldots, a_n) = a$. We denote this by $(u_1, \omega, u_2)$ or by $(u_2, \omega, u_1)$. The fact that $u_i$ and $u_{i+1}$ are neighbours for every $i \in \{1, 2, \ldots, m-1\}$ is designated by $(u_1, \omega_2, u_2, \omega_3, u_3, \ldots, \omega_m, u_m)$ being clear the meaning of the operators $\omega_2, \omega_3, \ldots, \omega_m$. Now, let $\approx$ stand for the reflexive and transitive extension of the relation of neighbourhoodness in $S(\cdot)$. $\approx$ is a congruence on $S(\cdot)$. Let $D(\cdot) = S(\cdot)/\approx$. We shall show that the origin algebra $\mathbf{A}$ is a subalgebra of the semigroup $D(\cdot) \in \mathfrak{D}_c$, this fact being independent on the signature $\Omega$. There by we can conclude that $\mathbf{A} \in \mathfrak{D}_c(\Omega)$ and moreover, that $\mathfrak{D}_c(\Omega)$ is a variety defined by all identities $\xi = \eta$ for $\hat{\xi} = \hat{\eta}$ and $|\xi|, |\eta| \geqslant 3$.

It will be convenient an element $s \in S(.)$ to stand for the class $s^{\approx} \in D(\cdot)$ as well, being clear from the context which of the cases is applied. Now, it is easy to see that $\omega(a_1, a_2, \ldots, a_n) = a$ in $\mathbf{A}$ implies that $\overline{\omega} \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_n = a$ in $D(\cdot)$ for any $\omega \in \Omega(n)$, $n \in N$ and $a_1, a_2, \ldots, a_n \in A$. It remains to check out whether the implication

$$a \approx b \Rightarrow a = b \tag{4}$$

s satisfied for any $a, b \in A$.

So, let $a \approx b$, i.e. there is a sequence $u_0, u_1, \ldots, u_t$ $(t \geqslant 0)$ of elements in $S$, such that

$$(a = u_0, \omega_1, u_1, \omega_2, \ldots, \omega_t, u_t = b) \tag{*}$$

The number of appearances of the operators in (*) is a *distance* between $a$ and $b$, denoted by $d(a, b)$.

Let $U = \bigcup_{i=0}^{t} \hat{u}_i$ and $W$ be the set of the all operators occuring in (*).

**1°.** Let $W \subseteq \Omega(1)$.

**Proposition 3.** *Let $e \in A$, $W'$ be an arbitrary subset of $W$ and $\omega^{-1}(e) \neq \neq \varnothing$, for every $\omega \in W'$. Then $\{\omega(e); \omega \in W'\}$ is an one-element set.*

**Proof.** Let $\omega, \tau, \in W'$ and $\omega(e) = v$. We have: $\tau(e) = \tau\omega(c_1) = = \tau\omega\omega(c_1) = \tau\omega(e) = \tau(v) = \tau\omega(e) = \omega\tau(e) = \omega\tau\tau(c_2) = \omega\tau(c_2) = \omega(e) = v$.

**Proposition 4.** *Let $i \in \{0, 1, \ldots, t\}$ and $d \in A \cap \hat{u}_i$. Then $a = \omega_{i_1}\omega_{i_2} \ldots \ldots \omega_{i_s}(d)$ for some $i_1, i_2, \ldots, i_s \in \{1, 2, \ldots, i\}$, or $a = d$ (Simmetricaly, $b = \omega_{j_1}\omega_{j_2} \ldots \omega_{j_r}(d)$ for some $j_1, j_2, \ldots, j_r \in \{i+1, i+2, \ldots, t\}$, or $b = d$).*

**Proof.** First, exclude this case:

$$(a = u_0, \omega_1, \overline{\omega}_1 a_1, \omega_1, a, \omega_3, \overline{\omega}_3 a_3, \omega_3, a, \ldots).$$

Namely, then we have immediately: $d = a$, of $a = \omega_i(d)$ for $d \in A \cap \hat{u}_i$, $1 \leqslant i \leqslant t$.

Otherwise, $u_n = \overline{\omega}_{i_2}\overline{\omega}_{i_1} a'$ and $a = \omega_{i_2}\omega_{i_1}(a')$ for some $i_1, i_2$, $1 \leqslant i_1 < < i_2 \leqslant t - 2$. Moreover, $\omega_{i_1}(a) = \omega_{i_2}(a) = a$, and by Proposition 3, $\omega_j(a) = a$ for every $\omega_j \in W$, such that $\omega_j^{-1}(a) \neq \varnothing$.

Now, if $d \in A \cap \hat{u}_0$, then $d = a$. If $d \in A \cap \hat{u}_1$, then $\omega_1(d) = a$. Let the statement in the proposition be true for the elements $u_0, u_1, \ldots, u_{k-1}$. The corelation between the elements $u_{k-1}$ and $u_k$ is described by one of the following cases:

(a) $u_{k-1} = \ldots \overline{\omega}_k c \ldots, \quad u_k = \ldots d \ldots, \ \omega_k(c) = d$

(b) $u_{k-1} = \ldots c \ldots, \quad u_k = \ldots \overline{\omega}_k d \ldots, \ c = \omega_k(d)$.

By the hypothesis, $\omega_{k_1}\omega_{k_2} \ldots \omega_{k_s}(c) = a$ for some $k_1, k_2, \ldots, k_s \in \in \{1, 2, \ldots, k-1\}$ or $c = a$. If (b) then $a = \omega_{k_1}\omega_{k_2} \ldots \omega_{k_s}\omega_k(d)$ or $a = = \omega_k(d)$. Let (a) be applied. Then $\omega_k = \omega_j$ for some $j < k$. Let $i$ be the minimal number such that $\omega_k = \omega_i$. Then, $u_{i-1} = \ldots c' \ldots$, $u_i = \ldots \overline{\omega}_i d' \ldots$ $\omega_i(d') = c'$. By the hypothesis, $\omega_{i_1}\omega_{i_2} \ldots \omega_{i_r}(c') = a$ for some $i_1, i_2, \ldots, i_r < i$ or $c' = a$. So $\omega_{i_1}\omega_{i_2} \ldots \omega_{i_r}\omega_i(d') = a$ or $\omega_i(d') = a$ and certainly $\omega_i^{-1}(a) \neq \varnothing$. There by $a = \omega_i(a) = \omega_k(a) = \omega_k\omega_{k_1}\omega_{k_2} \ldots \omega_{k_s}(c) = = \omega_{k_1}\omega_{k_2} \ldots \omega_{k_s}(d)$ or $a = \omega_i(a) = \omega_k(a) = \omega_k(c) = d$.

**Proposition 5.** *If $\omega \in W$, then $\omega^{-1}(a) \neq \varnothing$.*

**Proof.** Let $W = \{\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_s}\}$. It is clear that $s \leqslant k$. Let $1 \leqslant j \leqslant s$, $1 \leqslant r \leqslant k$ and $r$ be the minimal number such that $\omega_{i_j} = \omega_r$.

Then $u_{r-1}=\ldots c\ldots, u_r=\ldots\bar{\omega}_r d\ldots$ and $\omega_r\ (d)=c$. By Proposition 4, $a=\omega_{i_1}\omega_{i_2}\ldots\omega_{i_m}(c)=\omega_r\omega_{i_1}\omega_{i_2}\ldots\omega_{i_m}(d)=\omega_{i_j}\omega_{i_1}\omega_{i_2}\ldots\omega_{i_m}(d)$, or $a=c=\omega_{i_j}(d)\cdot$ ⁘

Now, in order to prove that $a=b$ we can use an induction on the distance between $a$ and $b$. If $d(a,b)=0$, trivialy $a=b$. If $d(a,b)\geqslant 1$ then $d(a,b)\geqslant 2$ If $d(a,b)=2$ we have $(a,\omega_1,\bar{\omega}_1,\ \omega_1,\ b)$ and clearly $a=b$ Let $a=b$ for $d(a,b)<t$. We have two posibilities:

i) $(a,\omega_1,\bar{\omega}_1 a_1,\omega_1,a_1,\ldots)$

ii) $(a,\omega_1,\bar{\omega}_1 a_1,\ \omega_2,\bar{\omega}_1\bar{\omega}_2 a_2,\ \ldots)$

If i), then apply the inductive hypothesis.

If ii), then $\omega_1(a)=\omega_2(a)=a$. By the Propositions 5. and 3. $\omega(a)=a$ for every $\omega\in W$. By Proposition 4., $\omega_{i_1}\omega_{i_2}\ldots\omega_{i_k}(a)=b$ for some $i_1,i_2,\ldots,i_k\leqslant t$. So $a=\omega_{i_1}\omega_{i_2}\ldots\omega_{i_k}(a)=b$.

**2°.** $W\subseteq\Omega(1)$, i.e. $\Omega\cap W\neq\Omega(1)$.

Previously we ought to define „value" of some of the elements in $D(\cdot)$.

Let $u\in D(\cdot)$, $u=\bar{\omega}_1\bar{\omega}_2\ldots\bar{\omega}_k a_1a_2\ldots a_m$, $|u|>3$ and at least one of the operators $\omega_1,\omega_2,\ldots,\omega_k$ (supose $\omega_1$) does not belong to $\Omega(1)$ (here $u$ stands for the class $u^{\approx}$). *Value* of the element $u$, designed by $[u]$, is the element $\omega_1^r\omega_2\ldots\omega_k\ (a_1,a_2,\ldots,a_{m-1},a_m^s)$, where $r$ and $s$ are any positive integers such that $\omega_1^r\omega_2\ldots\omega_k x_1x_2\ldots x_m^s$ is an $\Omega$-term, Further, if $\omega_1\omega_2\ldots\omega_k x_1x_2\ldots x_m$ is an arbitrary $\Omega$-term, then the *value* of the element $u=\bar{\omega}_1\omega_2\ldots\omega_k a_1a_2\ldots a_m\in D(\cdot)$ is the element

$$[u]=\omega_1\omega_2\ldots\omega_k(a_1,a_2,\ldots,a_m).$$

It can be easily seen that:

**Proposition 6.** *The value is a well defined mapping from a subset of D in the set A*$\cdot$ ⁘

**Proposition 7.** If $(u_1,\omega_1,u_2)$, $u_1$ has a value and $u_2$ has a value then $[u_1]=[u_2]$. ⁘

Let $\omega_i\in\Omega(n)$, $n\geqslant 2$, $1\leqslant i\leqslant t$ and $\omega_j\in\Omega(1)$ for $j<i$. That means that $u_{i-1}=\ldots c\ldots$, $u_i=\ldots\bar{\omega}_i\cdot c_1\cdot c_2\ldots c_n\ldots$ and $c=\omega_i(c_1,c_2,\ldots,c_n)$. Using the similar reasoning as in Proposition 4, it can be shown that $a=\omega_{i_1}\omega_{i_2}\ldots\omega_{i_s}(c)$ for some $i_1,i_2,\ldots,i_s<i$. We have: $\omega_i(a,a,\ldots,a)=\omega_i(\omega_{i_1}\ldots\omega_{i_s}(c),\ldots,\omega_{i_1}\ldots\omega_{i_s}(c))=\omega_{i_1}\ldots\omega_{i_s}\omega_i(c,c,\ldots,c)=\omega_{i_1}\ldots\omega_{i_s}\omega_i(c_1,c_2,\ldots,c_n)=\omega_{i_1}\ldots\omega_{i_s}(c)=a$. So, $(u_i=\bar{\omega}_i^2 u_i,\omega_i,\bar{\omega}_i^2 u_{i-1},\ \omega_{i-1},\ldots,\bar{\omega}_i^2 u_1,\omega_1,\bar{\omega}_i^2 a=\bar{\omega}_i\ a_n,\ a)$, and by Proposition 7., $[u]=[a]=a$.

Similarly, if $\omega_k \in \Omega(m)$, $m \geqslant 2$, $i \leqslant k \leqslant t$ and $\omega_j \in \Omega(1)$ for $j > k$, then $[u_k] = b$.

Let $u_k = \ldots \bar{\omega}_k \cdot d_1 \cdot d_2 \cdot \ldots \cdot d_m \ldots, \omega_k \in \Omega(m)$. We have:

$$(u_i = \bar{\omega}_i c_1 \ldots c_n u_i, \omega_{i+1}, \bar{\omega}_i c_1 \ldots c_n u_{n+1}, \ldots,$$

$$\ldots, \bar{\omega}_i c_1 \ldots c_n u_k = \bar{\omega}_i c_1 \ldots c_n \bar{\omega}_k d_1 \ldots d_m u_k, \omega_k, \bar{\omega}_i c_1 \ldots$$

$$\ldots c_n \bar{\omega}_k d_1 \ldots d_m u_{k-1}, \ldots, \bar{\omega}_i c_1 \ldots c_n \bar{\omega}_k d_1 \ldots d_m u_i =$$

$$= \bar{\omega}_k d_1 \ldots d_m u_i, \omega_{i+1}, \bar{\omega}_k d_1 \ldots d_m u_{i+1}, \ldots, \bar{\omega}_k d_1 \ldots$$

$$\ldots d_m u_k = u_k).$$

Again, by Proposition 7., $[u_j] = [u_k]$. Thus, finaly $a = b$ and $\mathbf{A} \in \mathcal{D}_c(\Omega)$.

Thus we have proved the following theorem:

**Theorem.** *The class of subalgebras of the distributive and commutative semigroups is a variety defined by the all identities $\zeta = \eta$ for $\hat{\zeta} = \hat{\eta}$ and $|\zeta|, |\eta| \geqslant 3$.* ⁚⁚

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## ПОДАЛГЕБРИ ОД ДИСТРИБУТИВНИ И КОМУТАТИВНИ ПОЛУГРУПИ


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**Резиме**

Алгебрата $\mathbf{A} = (A; \Omega)$ е подалгебра од полугрупата $S(\cdot)$ ако $A \subseteq S$ и ако за секоја $n$-арна ($n \geqslant 0$) операција $\omega \in \Omega$ постои елемент $\bar{\omega} \in S$ така што $\omega$ $(a_1, a, \ldots, a_n) = \bar{\omega} \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_n$. Нека со $\mathcal{D}_c$ ја означиме многукратноста на дистрибутивни комутативни полугрупи, т.е. класата полугрупи дефинирана со идентитете $x \cdot y = y \cdot x$ и $x \cdot y \cdot z = x^2 \cdot y \cdot z$. Покажана е следнава

**Теорема.** Класата алгебри во произволен јазик, кои се подалгебри од полугрупи во $\mathcal{D}_c$, е многукратност дефинирана со сите идентитети $\zeta = \eta$ такви што $\zeta$ има иста содржина како и $\eta$ и $\zeta$ и $\eta$ имаат должини не помали од 3.