Algebraic conference
Skopje 1980

POLYADIC SUBSEMIGROUPS OF SEMIGROUPS

G. Čupona, N. Celakoski

This work is an attempt to systematize a part of the results on generalized subsemigroups of semigroups obtained up to now mainly by the authors of this paper. The majority of the results is already published and we usually quote the paper where the corresponding result is published. The results published here for the first time are 3.11° and the most part of §5 and §6. We give also proofs to some known results in §4 with a purpose to illustrate the so called indirect method, i.e. the usefulness of Post Coset Theorem for investigation of n-groups by using binary groups.

§1. Universal enveloping semigroups

Let $\underline{A} = (A; F)$ be an algebra with a carrier A and a nonempty set of finitary operators, $F = F_2 \cup F_3 \cup \dots \cup F_n \cup \dots$, where F_n consists of n-ary operators belonging to F. Denote by Λ the set of "semigroup defining relations" $a = a_1 a_2 \dots a_n$, where $a = fa_1 \dots a_n$ in $\underline{A}$. Then the semigroup $A^\wedge$ with a presentation $\langle A; \Lambda \rangle$ (in the variety of semigroups) is called a *universal enveloping semigroup* for the algebra A. If

$u = a_1 \ldots a_k$ $(a_\nu \in A)$ is a word on A, then by $u^\wedge$ is denoted the element of $A^\wedge$ determined by u. Define a mapping $\lambda : A \to A^\wedge$ by $\lambda : a \to a^\wedge$. Then we have the following universal property of $A^\wedge$.

1.1°. If $\underline{A} = (A;F)$ and $\underline{A}' = (A';F)$ are F-algebras and $\phi : \underline{A} \to \underline{A}'$ a homomorphism, then there exists a unique homomorphism $\phi^\wedge : A^\wedge \to A'^\wedge$ such that $\lambda'\phi = \phi^\wedge\lambda$.

(We note that if ϕ is an epimorphism or isomorphism, then $\phi^\wedge$ has the corresponding property but it may happen ϕ to be a monomorphism and $\phi^\wedge$ not to be; [7].)

We say that $\underline{A}$ is an F-semigroup if there exists a semigroup S such that $A \subseteq S$ and

$$fa_1a_2 \ldots a_n = a_1a_2 \ldots a_n$$

for every n-ary operator $f \in F$ and $a_1, a_2, \ldots, a_n \in A$. If, in addition, S is generated by the set A, then S is called a covering semigroup of $\underline{A}$.

1.2°. $\underline{A}$ is an F-semigroup iff the mapping $\lambda : a \mapsto a^\wedge$ is an injection from A into $A^\wedge$. (Then we can assume that $A^\wedge$ is a covering semigroup of $\underline{A}$ and it is now called the universal covering of $\underline{A}$.)

The cardinal number $|A_1|$ of the set A_1 of all elements $a^\wedge$ of $A^\wedge$, when $a \in A$, is called the semigroup order of the algebra $\underline{A}$, and if $|A_1| = 1$, then $\underline{A}$ is called a semigroup singular algebra.

1.3°. Every algebra is a subalgebra of a semigroup singular algebra ([4]).

It is easy to determine the universal enveloping semigroup of a semigroup singular algebra $\underline{A} = (A;F)$.

First, let J_F be the set of naturals defined by

$$J_F = \{n-1 \mid F_n \neq \emptyset\}, \tag{1.1}$$

and let d_F be the greatest common divisor of the elements of J_F. Then:

1.4°. If $\underline{A}$ is a semigroup singular algebra, then $A^\wedge$ is the cyclic group with order d_F. ([4]).

It is natural to ask the question: what are the implications of the statement that the universal enveloping semigroup $A^\wedge$ of an algebra $\underline{A}$ has corresponding given properties. We do not think that (in general) a property of $A^\wedge$ gives a good information about the structure of A, but if $\underline{A}$ belongs to some special class of algebras (like polyadic semigroups or groups) we usually have a better situation.

§2. Associatives

The class of F-semigroups is a quasivariety and a convenient system of axioms (in the form of quasiidentities) of this quasivariety is given in [2]. Here we shall state some results concerning F-semigroups.

2.1°. The class of F-semigroups is a variety iff $d_F \in J_F$. (J_F and d_F are defined in §1, (1.1).) ([19])

An algebra $\underline{A} = (A;F)$ is called a weak associative if for any $f \in F_n$, $g \in F_m$ and $i \in \{2,\dots,m\}$ the following identities are satisfied:

$$gfx_1 \dots x_{m+n-1} = fgx_1 \dots x_{m+n-1} \tag{2.1}$$

$$= gx_1 \dots x_{i-1}\, fx_i \dots x_{m+n-1}.$$

And, a weak associative is called an associative if for any $f_\nu \in F_{n_\nu+1}$, $g_\lambda \in F_{m_\lambda+1}$, $\nu \in \{1,\dots,r\}$, $\lambda \in \{1,\dots,s\}$

such that

$$n_1 + n_2 + \ldots + n_r = n = m_1 + m_2 + \ldots + m_s,$$

the following identity is satisfied in $\underline{A}$:

$$f_1 \ldots f_r x_o x_1 \ldots x_n = g_1 \ldots g_s\, x_o \ldots x_n. \quad (2.2)$$

(In other words, $\underline{A}$ is a weak associative if continued products do not change by any replacement of operator symbols, and an associative if continued products with same length are equal.)

Some connections between the classes of F-semigroups, weak associatives and F-associatives are given in the next statements.

2.2°. Every weak F-associative is an F-associative iff $F = F_n = \{f\}$ consists of only one n-ary operator. (In that case, the associative $(A;f)$ is called an n-semigroup.) ([4])

2.3°. The class of F-semigroups is a subclass of the class of F-associatives. ([19])

2.4°. Every F-associative is an F-semigroup iff $d_F \in J_F$. (In this case, it may be assumed that an F-associative is, in fact, a d_F-semigroup.) ([19])

Example. Assume that $d_F \notin J_F$ and let n be the least element of J_F and p the least element of J_F which is not divisible by n. Define an algebra $\underline{A} = (\{0,1,2\};F)$ in the following way:

- if $g \in F_m$, $m \neq p+1$, then $g x_1 x_2 \ldots x_m = 0$,
- if $f \in F_{p+1}$, then $f 22 \ldots 2 = 1$, and

$$f x_o x_1 \ldots x_p = 0 \text{ when } (x_o,\ldots,x_p) \neq (\underbrace{2,\ldots,2}_{p+1}).$$

Then A is an associative but it is not an F-semigroup.

Thus, if $d_F \notin J_F$, then the class of F-semigroups is a proper subclass of the variety of F-associatives. Below we shall state some sufficient conditions under which an F-associative is an F-semigroup.

2.5°. If an F-associative A satisfies some of the following conditions, then it is an F-semigroup ([2] [20]):

(i) A is surjective, i.e.

$(\forall a \in A)(\exists f \in F)(\exists a_1,\ldots,a_n \in A) \quad a = fa_1\ldots a_n.$

(ii) A is cancellative, i.e. for each n-ary operator $f \in F$, $i \in \{1,\ldots,n\}$ and $a_1,\ldots,a_{i-1}$, a_{i+1}, $\ldots$, $a_n \in A$ the mapping

$$x \mapsto fa_1 \ldots a_{i-1}xa_{i+1} \ldots a_n$$

is an injection.

An associative A is cyclic if it is generated by one of its elements. A description of the class of cyclic F-associatives is given in [3]. We note that the associative A in the above example is cyclic and it is generated by the element 2.

As concerns the singularity of associatives we have the following propositions.

2.6°. If an associative A is semigroup singular, then $|A| = 1$. ([4]).

2.7°. If $|F| \geq 2$, then there exists semigroup singular weak F-associatives; the class of subalgebras of semigroup singular weak F-associatives is a proper subclass of the class of weak F-associatives. ([4]).

§3. n-subsemigroups of semigroups

As we mentioned in 2.2°, an n-*semigroup* Q is an algebra (Q,f) with an associative n-ary operation f. Instead of $fx_1 \dots x_n$, we shall write $[x_1\ x_2\ \dots\ x_n]$. If $a_o, a_1, \dots, a_{s(n-1)}$ (s≥1) is a sequence of elements of Q, then all continued products $\Pi(a_o, a_1, \dots, a_{s(n-1)})$ are equal in Q, and thus we may denote by $[a_o a_1 \dots a_{s(n-1)}]$ any such a product; if s=0, then we also write $[a_o] = a_o$. The universal covering semigroup Q^ of an n-semigroup Q can be characterized in the following manner ([7], [17], [18]):

3.1°. $Q\hat{} = Q_1 \cup Q_2 \cup \dots \cup Q_{n-1}$, where $Q_1 = Q$, $Q_i = \{a_1 \dots a_i \mid a_1, \dots, a_i \in Q\}$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$.

3.2°. Two sequences $\underline{a} = (a_1, \dots, a_i)$ and $\underline{b} = (b_1, \dots, b_i)$ (a_ν, $b_\lambda \in Q$, $i \le n$) are said to be *strongly linked*, then we write $\underline{a}\ s\ell\ \underline{b}$, iff there is a sequence $\underline{c} = (c_1, c_2, \dots, c_t) \in Q^t$ and nonnegative integers $p_o, p_1, \dots, p_i$, $q_o, q_1, \dots, q_i$, such that

$$0=p_o<p_1<\dots<p_i,\quad 0=q_o<q_1<\dots<q_i,\quad p_i=q_i=t$$
$$p_{\nu+1}-p_\nu \equiv 1 \pmod{n-1},\quad q_{\nu+1}-q_\nu \equiv 1 \pmod{n-1}, \quad (3.1)$$
$$a_\nu=[c_{p_{\nu-1}+1}\dots c_{p_\nu}],\quad b_\nu=[c_{q_{\nu-1}+1}\dots c_{q_\nu}].$$

The transitive extension of $s\ell$ will be denoted by ℓ. Then, if $a_1, \dots, a_i$, $b_1, \dots, b_i \in Q$, the equation $a_1 \dots a_i = b_1 \dots b_i$ holds in Q iff $(a_1, \dots, a_i)\,\ell\,(b_1, \dots, b_i)$.

Below we shall state some connections between the properties of an n-semigroup Q and its universal covering semigroup Q^; it is assumed here that $n \ge 3$.

3.3°. The semigroup Q^ is commutative iff Q satisfies the following conditions:

(i) Q is commutative (i.e. for every permutation $\nu \to i_\nu$ of $1,2,\ldots,n$ the identity $[x_1 \ldots x_n] = [x_{i_1} \ldots \ldots x_{i_n}]$ holds in Q),

(ii) $P = Q \setminus \{[a_1 \ldots a_n] \mid a_1,\ldots,a_n \in Q\}$ contains at most one element. ([18])

3.4°. Some covering semigroup of an n-semigroup Q is commutative iff Q is commutative. ([18])

3.5°. Some covering semigroup of an n-semigroup Q is cancellative iff Q is cancellative. ([15])

3.6°. $Q^\wedge$ is a group iff Q is an n-group (i.e.

$$(\forall a_1,\ldots,a_{n-1} \in Q)(\exists x,y \in Q)\{[xa_2 \ldots a_n] = a_1,\ [a_1 \ldots a_{n-1}y] = a_n\}).$$ ([17])

3.7°. Some covering semigroup of Q is a group iff Q is an n-group. ([1])

3.8°. Q is freely generated (in the variety of n-semigroups) by B iff $Q^\wedge$ is freely generated (in the variety of semigroups) by B. ([2:§4])

Assume now that $\mathcal{C}$ is a class of semigroups and denote by $\mathcal{C}(n)$ the class of n-semigroups which are n-subsemigroups of semigroups which belong to $\mathcal{C}$. We think that the question of description of $\mathcal{C}(n)$ is interesting when $\mathcal{C}$ is given in a convenient way.

As a consequence of a general result from the model theory ([14; p.274]) it follows that if $\mathcal{C}$ is a variety, then $\mathcal{C}(n)$ is a quasivariety. We do not known any convenient description of the set of varieties of semigroups such that $\mathcal{C}(n)$ is also a variety. We shall state here some partial results concerning this problem.

3.9°. Let $\mathcal{P}_{r,m}$ be the variety of semigroups that satisfy the identity $x^r = x^{r+m}$ and $\mathcal{C}_{r,m}$ the variety of commutative $\mathcal{P}_{r,m}$-semigroups. Then:

(i) $\mathcal{P}_{r,m}(n)$ is a variety iff n-1 is a divisor of m or $r \in \{0,1\}$.

(ii) $\mathcal{C}_{r,m}(n)$ is a variety for any r,m,n. ([16])

3.10°. Let $\mathcal{D}^{\ell}$ be the variety of left distributive semigroups (i.e. semigroups which satisfy the identity xyz = xyxz), and $\mathcal{D}$ the variety of distributive (both left and right) semigroups. Then $\mathcal{D}^{\ell}(n)$ is not a variety for any $n \geq 3$, and $\mathcal{D}(n)$ is a variety for any n. ([16])

Let $\xi = x_{i_1} x_{i_2} \ldots x_{i_p}$ be a (semigroup) term, where $i_\nu \in \{1,2,3,\ldots\}$. Then $|\xi|_i$ is the number of i_ν such that $i_\nu = i$.

3.11°. If $\mathcal{C}$ is a variety of semigroups defined by a set of identities $\xi = \eta$ such that $|\xi|_i \equiv |\eta|_i$ (mod n-1) for each $i = 1,2,\ldots$, then $\mathcal{C}(n)$ is a variety.

We note that in [5] it is given a description of the class $\mathcal{P}(n)$, where $\mathcal{P}$ is the class of periodic semigroups, and in [15] it is given a description of the class $\mathcal{G}_p(n)$ where $\mathcal{G}_p$ is the class of groups.

If $\mathcal{C}$ is a class of (binary) semigroups, it is natural to ask for a convenient set of classes, $\{\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_n, \ldots\}$, such that $\mathcal{C} = \mathcal{C}_2$ and $\mathcal{C}_n$ is a class of n-semigroups. There are several possibilities of solution of that problem. Let us mention the following three of them:

(A) $Q \in \mathcal{C}_n'$ iff $Q^\wedge \in \mathcal{C}$,

(B) $Q \in \mathcal{C}_n''$ iff $S \in \mathcal{C}$ for some covering semigroup S of Q,

(C) $\mathcal{C}_n$ is defined directly.

It is clear that:

3.12°. $\mathcal{C}_n' \subseteq \mathcal{C}_n''$ for every class $\mathcal{C}$ of semigroups.

From 3.6°-3.8° it follows that (A), (B) and (C) give same solutions if $\mathcal{C}$ is the class of groups or the class of free semigroups. If $\mathcal{C}$ is the class of commutative semigroups (or the class of cancellative semigroups), then (B) and (C) give the same result, but the results of (B) and (A) are not identical (namely, $\mathcal{C}_n'$ is a proper subclass of $\mathcal{C}_n''$).

We note that if we try to define the notion of periodic n-semigroup by a definition of type (A) or (B) we would obtain unsatisfactory results. For example, if a ternary operation [xyz] is defined on $\mathbb{Z}$ by [xyz] = x-y+z, then we obtain a periodic ternary semigroup, but no covering semigroup of ($\mathbb{Z}$; [ ]) is periodic.

We also note that the classes of completely regular and inverse n-semigroups are defined in [11] and [13] respectively by definitions of type (C).

§4. Universal covering groups

The well-known Post Coset Theorem ([8; p. 218]) gives a connection between the polyadic groups and binary groups.

The proposition 3.6° is, in fact, a modification of that theorem and, by this proposition, if Q is an n-group, then Q^ is a group and it is called the <u>universal covering group</u> of Q. We shall state here some known results (4.1°-4.7°) about Q^, assuming as in §3 that $n \geq 3$.

<u>4.1°</u>. Let Q^ be the universal covering of an n-group Q. Then

(i) If $a_1,\dots,a_i,\ b_1,\dots,b_i \in Q,\ 1 \leq i \leq n-1$, then $a_1 \dots a_i = b_1 \dots b_i$ in Q^ iff there exist $c_1,\dots,c_{n-i} \in Q$ such that $[c_1\dots c_{n-1}a_1\dots a_i] = [c_1\dots c_{n-i}b_1\dots b_i]$.

(ii) If Q* is a covering semigroup of Q such that:

$$a_1,\dots,a_i,\ b_1,\dots,b_j \in Q \Rightarrow$$
$$\Rightarrow (a_1 \dots a_i = b_1 \dots b_j \text{ in } Q^* \Leftrightarrow i \equiv j \pmod{n-1}),$$

then Q* is the universal covering group of Q, Q^ = Q*.

(iii) Q_{n-1} (see 3.1°) is a normal subgroup of Q^ and the factor group $Q^\wedge/Q_{n-1}$ is cyclic with order n-1.

(iv) If $a_1,\dots,a_k \in Q,\ \alpha_1,\dots,\alpha_k \in \mathbb{Z}$, then

$$a_1^{\alpha_1}a_2^{\alpha_2} \dots a^{\alpha_k} \in Q_i \Leftrightarrow \alpha_1 + \dots + \alpha_k \equiv i \pmod{n-1}.$$

The proposition 3.6° is used in [1] for obtaining several axiom systems for n-groups. Some of them are:

<u>4.2°</u>. Let Q be an n-semigroup. The following statements are equivalent:

(i) Q is an n-group.

(ii) For some $k \in \{1,2,\dots,n-2\}$,

$$(\forall x_1,\dots,x_k \in Q)(\exists x_1',\dots,x_{n-k-1}' \in Q)(\forall y \in Q)$$
$$[x_1\dots x_k x_1'\dots x_{n-k-1}'y] = y = [yx_1'\dots x_{n-k-1}'x_1\dots x_k].$$

(iii) There exists an (n-2)-ary operation $(\ )^{-1}$ on Q such that for any $x_1,\dots,x_{n-2},y \in Q$ the following identity equalities hold:

$$[yx_1\dots x_{n-2}(x_1\dots x_{n-2})^{-1}] = y = [(x_1\dots x_{n-2})^{-1}x_1\dots x_{n-2}y].$$

(iv) There exists a unary operation $x \mapsto \bar{x}$ on Q such that the following identities are satisfied in Q:

$$[\bar{x}\ x^{n-2}y] = y = [yx^{n-2}\bar{x}].$$

(v) There exists a unary operation $x \mapsto \bar{x}$ on Q such that for some $p: 0 \le p \le n-2$ and for some $s: 0 \le s \le n-2$ the following identity equalities hold:

$$[x^p\bar{x}\ x^{n-p-2}y] = y = [yx^{n-s-2}\bar{x}\ x^s].$$

The notion of free n-group is definied in the class of n-groups in the usual manner, i.e. by a definition of the type (C) of §3. The situation in this case is not the same as in the class of n-semigroups which is evident from the following results 4.3°-4.6°, proved in [10].

<u>4.3°</u>. If Q is a free n-group with a basis B, then $Q^{\wedge}$ is a free group with the same basis B.

<u>4.4°</u>. If $Q^{\wedge}$ is a free group with rank $r \ge 2$, then Q is a free n-group.

<u>4.5°</u>. $Q^{\wedge}$ is an infinite cyclic group iff Q is isomorphic with the following n-subgroup of the additive group of integers:

$$\{(n-1)x + k \mid x \in \mathbb{Z}\} = A_k,$$

where $1 \le k < n-1$ and k is relatively prime with n-1.

The n-group A_k is free iff k = 1 or k = n-2.

<u>4.6°</u>. An n-subgroup of a free n-group is a free n-group or an n-group isomorphic with an n-group of the form A_k.

We note that in the original proofs of the above results are used the corresponding "binary results", and the following property (see 1.1°) of n-groups ([7]):

<u>4.7°</u>. If $\phi : Q \to Q'$ is a monomorphism from an n-group Q into an n-group Q′, then $\phi^\wedge : Q^\wedge \to Q'^\wedge$ is also a monomorphism. (In other words, if P is an n-subgroup of an n-group Q and if P* is the subgroup of $Q^\wedge$ generated by P, then P* is the universal covering group of P, i.e. $P^* = P^\wedge$.)

In the next propositions (4.8°–4.13°) are considered finite n-groups and it will be assumed there that Q is a finite n-group with order $q = |Q|$. First we have:

<u>4.8°</u>. The order of $Q^\wedge$ is $(n-1)\cdot q$.

<u>Proof</u>. By 3.1°, it suffices to prove that the set Q_i has q elements. Let $a_1,\dots,a_{i-1} \in Q$. Since (in $Q^\wedge$)

$$a_1\dots a_{i-1}x = a_1\dots a_{i-1}y \Leftrightarrow x = y \quad (x,y \in Q)$$

it follows that the set $a_1\dots a_{i-1}Q$ (which is a subset of Q_i) has q elements. On the other hand, if $b_1,\dots,b_i \in Q$ and $a_1\dots a_{i-1}x = b_1\dots b_i$, then $x = a_{i-1}^{-1}\dots a_{i-1}^{-1}b_1\dots b_i$ belongs to Q, from what follows that $Q_i = a_1\dots a_{i-1}Q$, i.e. $|Q_i| = q$. □

As an immediate corollary from 4.7° and 4.8° we obtain the following generalization of well known Lagrange Theorem for binary groups ([8; p. 222]):

<u>4.9°</u>. If P is an n-subgroup of Q, and $|P| = p$, then p is a divisor of q.

If $a \in Q$, then the order of the cyclic n-subgroup <a> generated by a is called the order of a in Q, and it is denoted by r(a). Denoting by $\hat{r}(a)$ the order of a in $Q^\wedge$, from 4.7°, 4.8° and 4.9° we obtain:

4.10°. $\hat{r}(a) = (n-1)r(a)$ and $r(a) \mid q$. □

Assume that $a \in Q$ and that $\bar{a}$ is the skew element of a (i.e. $\bar{a}$ is the solution x of the equation $[a^{n-1}x]=a$ in Q; see also 4.2°, (iv).) Then $\bar{a} = a^{2-n}$ (in $Q^\wedge$) and this implies that $\hat{r}(\bar{a})$ is a divisor of $\hat{r}(a)$. Therefore:

4.11°. $r(\bar{a})$ is a divisor of r(a). □

In [9] it is considered the class of finite n-groups Q such that

$$(\forall a \in Q) \qquad r(\bar{a}) = r(a).$$

Since

$$r(a) = r(\bar{a}) \iff \hat{r}(a) = \hat{r}(\bar{a}) = \hat{r}(a^{2-n}) \iff$$

$(\hat{r}(a), n-2) = 1$, it follows that:

4.12°. $r(a) = r(\bar{a})$ iff n-2 and r(a) are relatively prime.

We note that the main result of [9] proved without using the notion of covering group, is the following proposition:

4.13°. If n-2 and q are relatively prime, then $r(a) = r(\bar{a})$ for each $a \in Q$.

It is clear that $4.12^\circ \Rightarrow 4.13^\circ$. Thus, a property of n-groups can be usually easier shown by an indirect method, i.e. by using the notion of the universal covering group, than by a direct method, i.e. by dealing only with n-groups. We shall give another example which support such a conjecture.

Let Q be an n-group (not necessarily finite) and let f and g be defined on $Q^\wedge$ by

$$f(x,y,z) = xy^{-1}z, \quad g(x_1,\ldots,x_{n-2}) = (x_1x_2\ldots x_{n-2})^{-1}.$$

Then

$$x_1x_2\ldots x_n = f(x_1,g(x_2,\ldots,x_{n-1}),x_n)$$

and $(Q;f,g)$ is a subalgebra of $(Q^\wedge;f,g)$.

Therefore:

<u>4.14°.</u> If Q is an n-group, then there is a ternary operation f and an n-2-ary operation g on Q such that

$$[x_1x_2\ldots x_n] = f(x_1,g(x_2,\ldots,x_{n-1}),x_n). \square$$

The proposition 4.14° is the main result of the paper [12], and it is proved there by the "direct method". We also note that by this "indirect method" can be proved some similar results on inverse n-semi-groups which are obtained in [13].

§5 Universal covering semigroups of finite n-semigroups

It will be assumed in this section that Q is a finite n-semigroup and that $|Q| = q$, $n \geq 3$ are given. Clearly, $|Q_i| \leq q^i$ and by 3.1° we obtain:

<u>5.1°.</u> $Q^\wedge$ is finite and $|Q^\wedge| \leq q + q^2 + \ldots + q^{n-1}.$

By 4.8° we have:

<u>5.2°.</u> If Q is an n-group, then $|Q_i| = q$ and $|Q^\wedge| = (n-1)q$.

This proposition sugests the question whether there exist finite n-semigroups with the property $|Q_i| = q$ which are not n-groups. The answer is positive. Namely, let $Q = \{a, a^n, a^{2n-1}, \dots, a^{r(n-1)+1}, \dots, a^{(r+m-1)(n-1)+1}\}$ be a cyclic n-semigroup with index r and period m, where $q = r + m$. Then

$$a^{i(n-1)+1} = a^{j(n-1)+1} \iff i = j \text{ or } (i,j \geq r \text{ and } i \equiv j \pmod m) \text{ and}$$

$$Q^\wedge = \{a, a^2, \dots, a^{n-1}, a^n, \dots, a^{r(n-1)}, a^{r(n-1)+1}, \dots, \dots, a^{(r+m)(n-1)}\}$$

is a cyclic semigroup with index $r(n-1)$ and period $m(n-1)+1$. Therefore:

<u>5.3°</u>. If Q is cyclic, then $|Q_i| = q$, $|Q^\wedge| = (n-1)q$.

Now we shall prove that:

<u>5.4°</u>. If $P = Q \setminus \{[a_1 \dots a_n] \mid a_1, \dots, a_n \in Q\}$ and $|P| = p$, then $|Q_i| \geq p^i + 1$.

<u>Proof</u>. First, if $a_1, \dots, a_i \in P$, then

$$(a_1, \dots, a_i)\, S\ell\, (b_1, \dots, b_i) \iff a_1 = b_1, \dots, a_i = b_i,$$

and this implies that $|Q_i| \geq p^i$. And, since the set $Q^i \setminus P^i$ is a union of ℓ-classes and is nonempty, we have also $|Q_i| \geq p^i + 1$. □

As a corollary of 5.4° we obtain that:

<u>5.5°</u>. If Q is a constant n-semigroup, then

$$|Q_i| = (q-1)^i + 1.$$

<u>Proof</u>. Let $(\forall x_1, \dots, x_n)\ [x_1 \dots x_n] = 0$. By 5.4° we have $|Q_i| \geq (q-1)^i + 1$, for $P = Q \setminus \{0\}$, $p = q-1$. Using the associativity of the operation [ ] and (5.1), it is easy to show that the following equality holds in $Q^\wedge$:

$$a_1 \ldots a_{\nu-1} 0 a_{\nu+1} \ldots a_i = 0^i$$

for any $a_1, \ldots, a_{\nu-1}, a_{\nu+1}, \ldots, a_i \in Q$, $\nu \in \{1, \ldots, i\}$, $i \in \{2, \ldots, n\}$ and this implies that $|Q_i| = (q-1)^i + 1$.□

Let us consider the general case.

<u>5.6°</u>. If $q > 1$, then $1 < |Q_i| < q^i$ for each $i \in \{2, \ldots, n-1\}$.

<u>Proof</u>. We have to prove that $1 \neq |Q_i|$, $|Q_i| \neq q^i$.

Assume that $|Q_i| = q^i$ for some $i \in \{2, \ldots, n-1\}$. Thus, if $a_1, \ldots, a_i, b_1, \ldots, b_i \in Q$ and $a_1 \ldots a_i = b_1 \ldots b_i$, then $a_1 = b_1, \ldots, a_i = b_i$. By the associativity we have:

$$[a_1 \ldots a_n] a_1^{i-1} = a_1 \ldots a_{i-1} [a_i \ldots a_n a_1^{i-1}],$$

which implies that

$$[a_1 \ldots a_n] = a_1$$

for any $a_1, \ldots, a_n \in Q$. By symmetry we also have $[a_1 \ldots a_n] = a_n$ and thus we have $a_1 = [a_1 \ldots a_n] = a_n$ for any $a_1, \ldots, a_n \in Q$. But this is impossible since $q = |Q| > 1$.

Assume now that $|Q_i| = 1$ for some $i \in \{2, \ldots, n-1\}$. If $a, a_1, \ldots, a_i, b, b_1, \ldots, b_i \in Q$, then we have $a_1 \ldots a_i = b_1 \ldots b_i$ and also $ab_1 \ldots b_{i-1} = bb_1 \ldots b_{i-1}$, which implies

$$aa_1 \ldots a_i = ab_1 \ldots b_i = bb_1 \ldots b_i,$$

i.e. $|Q_{i+1}| = 1$. Therefore we have $|Q_n| = 1$, i.e. Q is a constant n-semigroup and, by 5.5°, this implies that $|Q_i| = (q-1)^i + 1 > 1$. □

Denote by $\alpha(q,i,n)$ and $\alpha(q,n)$ the least positive integers such that there exists an n-semigroup Q such that

$|Q| = q$, $|Q_i| = \alpha(q,i,n)$, $|Q^\wedge| = \alpha(q,n)$.

Dually, $\beta(q,i,n)$ and $\beta(q,n)$ are the maximal numbers such that $|Q_i| = \beta(q,i,n)$, $|Q^\wedge| = \beta(q,n)$, for some n-semigroup Q with $|Q| = q$. Clearly:

5.7°. (i) $\alpha(1,i,n) = \beta(1,i,n) = 1$,

(ii) $\alpha(1,n) = \beta(1,n) = n-1$,

(iii) $\alpha(q,1,n) = \beta(q,1,n) = q$,

(iv) $\sum_{i=1}^{n-1}\alpha(q,i,n) \leq \alpha(q,n) \leq \beta(q,n) \leq \sum_{i=1}^{n-1}\beta(q,i,n)$.

Assume that $q > 1$ and $1 < i \leq n-1$. From 4.8° (5.3°) and 5.5° we obtain the following two propositior.

5.8°. $\alpha(q,i,n) \leq q \leq \beta(q,i,n)$,

$\alpha(q,n) \leq (n-1)q \leq \beta(q,n)$. □

5.9°. $\alpha(q,i,n) \leq (q-1)^i + 1 \leq \beta(q,i,n)$,

$\alpha(q,n) \leq \sum_{i=1}^{n-1}(q-1)^i + n - 1 \leq \beta(q,n)$.

5.10°. $2 \leq \alpha(q,i,n) \leq \beta(q,i,n) \leq q^i - 1$,

$q + 2(n-2) \leq \alpha(q,n) \leq \beta(q,n) \leq \sum_{i=1}^{n-1} q^i - (n-2)$.

Clearly (for $q > 2$), 5.8° gives a better approximation for $\alpha(q,n)$, and 5.9° a better approximation for $\beta(q,n)$.

We will make a remark about the decidability of $Q^\wedge$ if Q is a given finite n-semigroup. Namely, the description of $Q^\wedge$ given in §1 and §3 do not give a general procedure for obtaining the semigroup $Q^\wedge$. Namely, if $i \in \{2,\dots,n-1\}$, then there are infinitely many sequences of integers $p_o,\dots,p_i$, $q_o,\dots,q_i$ which satisfy (3.1) in 3.2°. But we can "improve" the description of $S\ell$ by the following proposition.

<u>5.11°</u>. If $a_1,\ldots,a_i, b_1,\ldots,b_i \in Q$ are such that $(a_1,\ldots,a_i)\ s\ell\ (b_1,\ldots,b_i)$, then there exist $(c_1,\ldots ,c_t) = \underline{c} \in Q^t$ and nonnegative integers $p_o,p_1,\ldots, p_i, q_o,q_1,\ldots,q_i$ such that

$$p_{\nu+1} - p_\nu \le n, \quad q_{\nu+1} - q_\nu \le n \tag{5.1}$$

and (3.1) is satisfied.

<u>Proof</u>. Assume that (3.1) holds. If $p_1 \ge q_1,\ldots,q_r$ and $p_1 < q_{r+1}$, then we can put:

$a_1 = [b_1 \ldots b_r\, c_{q_r+1} \ldots c_{p_1}]$, $q_1' = \ldots = q_r' = 1$,

$p_1' = 1$ if $p_1=1$ or $p_1' = n$ if $p_1 > n$,

$p_{\nu+1}' = p_\nu - q_r + r$, $q_{r+\nu}' = q_{r+\nu} - q_r + \nu$.

Thus, we can assume that

$$p_\nu - p_{\nu-1} \le n, \quad q_\lambda - q_{\lambda-1} \le n$$

for all $\nu, \lambda : 1\le\nu\le s, 1\le\lambda\le k$. In the same manner as above we can obtain $0 = p_o^*, p_1^*,\ldots,p_i^*, 0 = q_o^*, q_1^*,\ldots,q_i^*$ such that (3.1) is satisfied and

$$p_{\nu+1}^* - p_\nu^* \le n, \quad q_{\lambda+1}^* - q_\lambda^* \le n$$

for all $\nu, \lambda : 1 \le \nu \le s, 1 \le \lambda \le k$, and this will complete the proof.

From (5.1) it follows that $t \le in$, and thus we can decide if $(a_1,\ldots,a_i)\ s\ell\ (b_1,\ldots,b_i)$. And, as Q^i is finite, this implies a procedure for deciding if $(a_1,\ldots,a_i)\ \ell\ (b_1,\ldots,b_i)$. □

As it concerns finite n-groups, we have a "better" procedure. Namely, $Q_i = \{a^{i-1}x \mid x\in Q\}$, where a is a given element of Q, and if $a_1,\ldots,a_i \in Q$, then the element $x\in Q$, such that $a^{i-1}x = a_1 \ldots a_i$, is the solution of the equation $[a^{n-1}x] = [a^{n-i}a_1 \ldots a_i]$.

§6. Presentation of n-semigroups and n-groups

Let B be a nonempty set and $F = F_B^{(n)}$ be the n-semigroup which is freely generated by B, i.e. F consists of all words $w = b_1 b_2 \ldots b_p$ on B such that $p \equiv 1 \pmod{n-1}$, and the operation [ ] is defined on F by:

$$[w_1 w_2 \ldots w_n] = w_1 w_2 \ldots w_n. \tag{6.1}$$

Let $\Lambda \subseteq F \times F$ and let Λ^* be the congruence on F generated by Λ. Then we say that the n-semigroup $Q = F/\Lambda^*$ "is given by the presentation $\langle B;\Lambda\rangle_n$" and, as usual, if $(u,v) \in \Lambda$, then it will be written $u = v$ instead of (u,v). It can be easily seen that:

6.1°. $\widehat{\langle B;\Lambda\rangle_n} = \langle B;\Lambda\rangle_2$.

(Namely, $\langle B;\Lambda\rangle_2 = \langle B;\Lambda\rangle$ is a presentation in the class of semigroups.) □

We do not known the answer of the following question:

"If the presentation $\langle B;\Lambda\rangle_n$ is decidable, is the presentation $\langle B;\Lambda\rangle$ decidable too?"

This question is equivalent with the following one. Let Λ be as above and assume that the presentation $\langle B;\Lambda\rangle$ has the following property: there exists an effective procedure for deciding whether or not two words $u,v \in F$, such that $u=a_1\ldots a_p$, $v=b_1\ldots b_q$, $p \equiv q \pmod{n-1}$, define the same element in $\langle B;\Lambda\rangle$. "Is the presentation $\langle B;\Lambda\rangle$ decidable?"

Assume now that $B \neq \emptyset$, $B' = B \cup \{b^{-1} \mid b \in B\}$ and $F' = F_{B'}^{(n)}$ be the free n-semigroup generated by B'. Assume also that Σ is a set of words $w = b_1^{i_1} b_2^{i_2} \ldots$ $\ldots b_k^{i_k}$, $b_\nu \in B$, $i_\nu \in \mathbb{Z}$, such that

$$|w| = i_1 + i_2 + \dots + i_k \equiv 0 \pmod{n-1}.$$

Define a relation $\overset{\bullet}{\sim}$ on F' by:

$$u \overset{\bullet}{\sim} v \quad \text{iff} \quad (u=u_1bb^{-1}u_2,\ v=u_1u_2;\ \ b\in B) \ \text{ or}$$
$$(u = u_1b^{-1}bu_2,\ v=u_1u_2;\ \ b\in B) \ \text{ or}$$
$$(u=u_1wu_2,\ v=u_1u_2;\ \ w \in \Sigma).$$

If $\sim$ is the symmetric and transitive extension of $\overset{\bullet}{\sim}$, then it is a conqruence on F' and the factor n-semigroup $F'/\sim$ **is an n-group; we say that** this n-group has a presentation $\langle B;\Sigma\rangle_{n,gp}$. Instead of $\langle B;\Sigma\rangle_{2,gp}$ we shall write $\langle B;\Sigma\rangle_{gp}$ and this is a presentation in the variety of groups. We have the following propositions:

6.2°. $\langle B;\Sigma\rangle_{n,gp} = \langle B;\Sigma\rangle_{gp}$. □

6.3°. The presentation $\langle B;\Sigma\rangle_{n,gp}$ is decidable iff the presentation $\langle B;\Sigma\rangle_{gp}$ is decidable.

Namely, if u and v are two "group words" on B, then we have first that

$$u = v \ \text{ in } \ \langle B;\Sigma\rangle_{gp} \Rightarrow |u| \equiv |v| \pmod{n-1},$$

and if $a\in B$ and k is a nonnegative integer such that $|a^ku| \equiv |a^kv| \equiv 1 \pmod{n-1}$, then

$$u = v \ \text{ in } \ \langle B;\Sigma\rangle_{gp} \ \text{ iff } \ a^ku=a^kv \ \text{ in } \langle B;\Sigma\rangle_{n,gp}. \ \square$$

R E F E R E N C E S

[1] Celakoski N.: On axiom systems for n-groups; Matem.bilt. SDM, 1(XXVII), 1977, 5-14

[2] Celakoski N.: On semigroup associatives MANU, Contributions IX.2, 1977, 5-19

[3] Celakoski N.: On cyclic associatives; MANU Contributions of Sec.Math.tech.Sc. I.1 (1980), in print

[4] Čupona G.: On a representation of algebras in semigroups; MANU, Contributions X.1, 1978, 5-18

[5] Čupona G.: n-subsemigroups of periodic semigroups; God.zbor.Matem.fak.Skopje, 29, 1978, 21-25

[6] Čupona G.: n-subsemigroups of semigroups satisfying the law $x^r = x^{r+m}$; God.zbor.Matem.fak.Skopje 30 (1979), 5-14

[7] Čupona G., Celakoski N.: On representation of associatives into semigroups; MANU, Contributions VI.2 1974, 23-34

[8] Post E.L.: Polyadic groups, Trans.Amer.Math.Soc., Vol. 48, 1940, 208-350

[9] Wanke-Jakubowska M.B.,Wanke-Jerie M.E.: On orders of skew elements in finite n-groups; Dem.math., XII, No 1, 1979, 247-253

[10] Артамонов В.А.: Свободные n-группы; Матем.зам. Т.8, No 4 (1970), 499-507

[11] Колесников О.В.: К теории n-полугрупп; Доклады АН УССР, Физ.мат.тех.науки, No 4 (1979) 302-305

[12] Колесников О.В.: Разложение n-групп; Сборник "Квазигруппы и лупы", Матем.иссл., вып. 51, "Штиинца", 1979, 88-92

[13] Колесников О.В.: Инверсные n-полугруппы ; Ann.Soc. Math.Pol., Series 1, XXI (1979), 101-108

[14] Мальцев А.И.: Алгебраические системы ; Москва 1970

[15] Марковски С.: Подалгебри на групоиди; докторска дисертација, Скопје 1980

[16] Марковски С.: За дистрибутивните полугрупи; Год. збор.Матем.фак.-Скопје, 30(1979) in print

[17] Чупона Ѓ.: За асоцијативните конгруенции; Билтен Друшт.матем.физ. СР Македонија 13 (1962), 5-12

[18] Чупона Ѓ.: Полугрупи генерирани од асоцијативи; Год.збор. ПМФ-Скопје, Сек. А, 15 (1964), 5-25

[19] Чупона Ѓ.: За асоцијативите; МАНУ, Прилози I-1 (1969), 9-20

[20] Чупона Ѓ.: Асоцијативи со кратење; Год.збор. ПМФ-Скопје, 19 (1969), 5-14