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n-DIMENSIONAL SEMINETS AND PARTIAL n-QUASIGROUPS

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It is shown that there is an equivalence between the theories of n-dimensional seminets and regular partial n-quasigroups.

0. First we give some necessary definitions.

Let Q be a non-empty set, $D \subseteq Q^n$, and A be a mapping from D into Q such that:

$$A(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n) = A(a_1,\dots,a_{i-1},y,a_{i+1},\dots,a_n) \Rightarrow x = y \quad (0.1)$$

for each $i \in \{1,2,\dots n\}$. Then (Q,A) is called a partial n-quasigroup with domain D. If $a \in Q$ and $i \in \{1,2,\dots,n+1\}$, then $b_{a,i}$ is a subset of Q^{n+1} defined by:

$$b_{a,i} = \{(x_1,\dots,x_{n+1}) \mid x_i=a,\ A(x_1,\dots,x_n)=x_{n+1}\}. \quad (0.2)$$

A partial n-quasigroup (Q,A) is said to be regular if the following statement is satisfied:

(R) The domain D of (Q,A) is non-empty and if $A(a_1,\dots,a_n) = a_{n+1}$, then the sets $b_{a_1,1},\dots,b_{a_{n+1},n+1}$ are distinct.

Let (Q,A) and (Q',A') be partial n-quasigroups with domains D and D´ respectively and let $\alpha_1,\dots,\alpha_{n+1}$ be partial bijections from Q into Q´. (By a partial bijection α from Q into Q´ we mean a bijection from a subset of Q into a subset of Q´). We say that $(\alpha_1,\dots,\dots,\alpha_{n+1})$ is an isotopy from (Q,A) into (Q´,A´) if:

$$(x_1,\dots,x_n)\in D \iff (\alpha_1(x_1),\dots,\alpha_n(x_n))\in D',$$

$$A(D) \text{ is in the domain of } \alpha_{n+1}, \qquad (0.3$$

$$\alpha_{n+1}(A(x_1,\dots,x_n)) = A'(\alpha_1(x_1),\dots,\alpha_n(x_n)),$$

for every $(x_1,\dots,x_n)\in D$. In that case we say that (Q,A) and (Q´,A´) are isotopic. It is clear that if one of (Q,A), (Q´,A´) is regular, the another one is also regular.

Let P be a nonempty set and let $B_1,\dots,B_{n+1}$ $(n \geq 2)$ be nonempty mutually disjoint collections of subsets of P. The elements of P are called points and the elements of the sets B_ν are called blocks. We say that $(P;\ B_1,\dots,B_{n+1})$ constitutes a structure of n-dimensional seminet (or n-seminet) iff the following two statements are satisfied:

(SN)(i) For every point $p\in P$ there exists a unique sequence of blocks $b_1,\dots,b_{n+1}$ $(b_\nu \in B_\nu)$ such that $\{p\} = b_1\cap\dots\cap b_{n+1}$.

(ii) For every $i\in\{1,2,\dots,n+1\}$ and every sequence of blocks $b_1,\dots,b_{i-1},\ b_{i+1},\dots,b_{n+1}$ such that $b_\nu\in B_\nu$, the set $b_1\cap\dots\cap b_{i-1}\cap b_{i+1}\cap\dots\cap b_{n+1}$ contain at most one point.

<u>1.</u> Here we will show that every regular partial n-quasigroup induces an n-seminet.

Let (Q,A) be a regular partial n-quasigroup, and let P (the set of points) be defined by:

$$P = \{(x_1,\dots,x_{n+1}) \mid A(x_1,\dots,x_n) = x_{n+1}\}. \qquad (1.1)$$

Every non-empty set:

$$b_{x,i} = \{(x_1,\dots,x_{n+1}) \in P \mid x_i = x\} \qquad (1.2)$$

is called a block, and the sequence of blocks $B_1,\dots,$ B_{n+1} is defined by:

$$B_i = \{b_{x,i} \mid b_{x,i} \neq \emptyset,\ x \in Q\} \qquad (1.3)$$

From (R) if follows that $P \neq \emptyset$, and that $B_1,\dots$ $\dots,B_{n+1}$ are nonempty and disjoint.

If $p = (a_1,\dots,a_{n+1}) \in P$, then $b_{a_1,1},\dots,b_{a_{n+1},n+1}$ is the unique sequence such that $b_{a_\nu,\nu} \in B_\nu$ and $\{p\} =$ $= b_{a_1,1} \cap \dots \cap b_{a_{n+1},n+1}$. If $b_{a_\nu,\nu} \in B_\nu$ $(\nu = 1,2,\dots,n)$, then $b_{a_1,1} \cap \dots \cap b_{a_n} = (a_1,\dots,a_{n+1})$, where $a_{n+1} =$ $= A(a_1\dots a_n)$. Also if $b_{a_\nu,\nu} \in B_\nu$ $(\nu = 1,\dots i-1,\ i+1,\dots$ $\dots,n+1,\ i<n+1)$, then

$$b_{a_1,1} \cap \dots b_{a_{i-1},i-1} \cap b_{a_{i+1},i+1} \cap \dots b_{a_{n+1},n+1} = \{p,q\}$$

implies that

$$p = (a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_{n+1}),$$

$$q = (a_1,\dots,a_{i-1}y,a_{i+1},\dots,a_{n+1}),$$

$$a_{n+1} = A(a_1\dots a_{i-1},x,a_{i+1}\dots a_n) =$$

$$= A(a_1\dots a_{i-1}ya_{i+1}\dots a_n);$$

whence by (0.1) we obtain that $x = y$, i.e. $p = q$.

This completes the proof that $(P;B_1,\dots,B_{n+1})$ is an n-seminet.

If (Q,A) and (Q´,A´) are isotopic, then it can be easily shown that the corresponding n-seminets are isomorphic.

Namely, if $(\alpha_1,\ldots,\alpha_{n+1})$ is an isotopy from (Q,A) into (Q´,A´), then the mapping $\phi: P \to P'$ defined by:

$$\phi : (a_1,\ldots,a_{n+1}) \to (\alpha_1(a_1),\ldots,\alpha_{n+1}(a_{n+1}))$$

induces an isomorphism from $(P;\ B_1,\ldots,B_{n+1})$ into $(P';\ B_1',\ldots,B_{n+1}')$.

2. Now we will show that every n-seminet can be coordinatized by a partial regular n-quasigroup.

Let $(P;B_1,\ldots,B_{n+1})$ be an n-seminet such that $m = \max\{|B_i|\,|\,i = 1,2,\ldots,n+1\}$ and let Q be a set with $|Q| = m$. If $f_i:B_i \to Q$ is an injection for each $i = 1,\ldots,n+1$, then a partial n-ary operation A can be defined in Q by:

$$A(q_1,\ldots,q_n) = q_{n+1} \Leftrightarrow (\exists b_1,\ldots,b_{n+1};\ b_\nu \in B_\nu)$$
$$b_1 \cap \ldots \cap b_{n+1} \neq \emptyset,\ \ q_i = f_i(b_i)$$

From (SN) we obtain first that A is a partial n-quasigroup, and it is regular as well, for $B_1,\ldots$ $\ldots,B_{n+1}$ are disjoint.

Certainly, (Q,A) depends on the injections $f_1,\ldots,f_{n+1}$ and the set Q. Assume that the sequence of injections $f_1:B_i \to Q$ induces the partial n-quasigroup (Q,A) and the injections $f_i':B_i \to Q'$ – the n-quasigroup (Q´,A´). If we put $\alpha_\nu = f_\nu' f_\nu^{-1}$, then we get a partial bijection α_ν from Q into Q´. Then $(\alpha_1,\ldots,\alpha_{n+1})$ is an isotopy from (Q,A) into (Q´,A´).

Example. Given a 2-seminet as in Fig. 1 with:

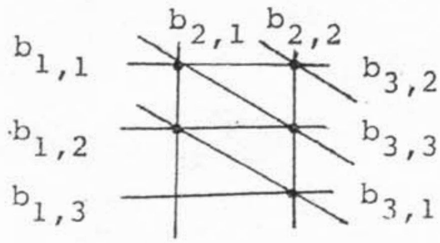

Fig. 1.

$B_1 = \{b_{1,1}, b_{1,2}, b_{1,3}\}$,

$B_2 = \{b_{2,1}, b_{2,2}\}$,

$B_3 = \{b_{3,1}, b_{3,2}, b_{3,3}\}$,

$\max\{|B_i| \,|\, i \in \{1,2,3\} = 3$,

$Q = \{x_1, x_2, x_3\}$.

If

$$f_1 = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ x_3 & x_2 & x_1 \end{pmatrix}; \qquad f_2 = \begin{pmatrix} b_{2,1} & b_{2,2} \\ x_3 & x_2 \end{pmatrix};$$

$$f_3 = \begin{pmatrix} b_{3,1} & b_{3,2} & b_{3,3} \\ x_1 & x_3 & x_2 \end{pmatrix}; \qquad g_1 = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ x_1 & x_3 & x_2 \end{pmatrix};$$

$$g_2 = \begin{pmatrix} b_{2,1} & b_{2,2} \\ x_2 & x_1 \end{pmatrix}; \qquad g_3 = \begin{pmatrix} b_{3,1} & b_{3,2} & b_{3,3} \\ x_2 & x_3 & x_1 \end{pmatrix},$$

then we obtain partial 2-quasigroups (Q,A) and (Q,B):

A	x_1	x_2	x_3
x_1	x_1		
x_2	x_2		x_1
x_3	x_3		x_2

B	x_1	x_2	x_3
x_1	x_3	x_1	
x_2	x_2		
x_3	x_1	x_2	

If

$$\alpha_1 = f_1 g_1^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \end{pmatrix}; \quad \alpha_2 = f_2 g_2^{-1} = \begin{pmatrix} x_1 & x_2 \\ x_1 & x_3 \end{pmatrix};$$

$$\alpha_3 = f_3 g_3^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_3 \end{pmatrix},$$

then the ordered triple $(\alpha_1, \alpha_2, \alpha_3)$ is an isotopy of (Q,B) and (Q,A).

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