ON A REPRESENTATION OF ALGEBRAS IN SEMIGROUPS

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0. *Preliminary definitions and results.* Necessary definitions and main results will be stated first.

0.1. *Definitions.* Let F be a nonempty set of finitary operators, such that $F_0 \cup F_1 = \emptyset$. (F_n is the set of n-ary operators in F). If A is a set and each n-ary operator f is interpreted as an n-ary operation on A, then $\mathbf{A} = (A; F)$ is said to be an F-algebra. Let $\mathbf{S} = (S;.)$ be a semigroup and $\xi : A \to S$ such a mapping that

$$\xi(f x_1 \ldots x_n) = \xi(x_1) \ldots \xi(x_n) \tag{0.1}$$

for each n-ary operator $f \in F_n$ and all $x_1, \ldots, x_n \in A$. Then we say that $\xi : \mathbf{A} \to \mathbf{S}$ is a semigroup homomorphism. The notion of universal semigroup homomorphism is defined in the usual manner. If $\lambda : \mathbf{A} \to \mathbf{A}^\wedge$ is the universal semigroup homomorphism, then $\mathbf{A}^\wedge$ is called the universal semigroup for $\mathbf{A}$. The cardinal number $|\lambda(A)|$ of the set $\lambda(A)$ is called the semigroup order (i.e. s-order) of $\mathbf{A}$, and it is denoted by $\|\mathbf{A}\|$. The algebra $\mathbf{A}$ is said to be s-finite (s-infinite) iff $\|\mathbf{A}\|$ is finite (infinite); if $\|\mathbf{A}\| = 1$ then we say that $\mathbf{A}$ is s-singular. If the universal semigroup homomorphism $\lambda : \mathbf{A} \to \mathbf{A}^\wedge$ is a monomorphism, then $\mathbf{A}$ is said to be a semigroup F-algebra.

An algebra $\mathbf{A} = (A; F)$ is said to be a weak F-associative iff for all $f \in F_n$, $g \in F_m$ and $i \in \{1, 2, \ldots, n\}$ the following identities are satisfied

$$\begin{aligned} fgx_1 \ldots x_{m+n-1} &= gfx_1 \ldots x_{m+n-1} \\ &= fx_1 \ldots x_{i-1} g x_i \ldots x_{m+n-1}\,. \end{aligned} \tag{0.2}$$

A weak F-associative is called an F-associative iff for every pair of sequences $f_1, \ldots, f_r$, $g_1, \ldots, g_s$ such that

$$f_i \in F_{n_i+1},\ g_j \in F_{m_j+1},\ n_1 + \ldots + n_r = m_1 + \ldots + m_s$$

the following identity is satisfied:

$$f_1 \ldots f_r\, x_0 \ldots x_n = g_1 \ldots g_s\, x_0 \ldots x_n. \tag{0.3}$$

Throughout the paper, by d will be denoted the greatest common divisor of the numbers in the set:

$$J = \{ n-1 \mid F_n \neq \emptyset \}. \tag{0.4}$$

0.2 *Main results*

(i) **A** is s-finite iff $\mathbf{A}^\wedge$ is finite.

(ii) If **A** is s-infinite then $\|\mathbf{A}\| = |A^\wedge|$.

(iii) If **A** is s-singular, then $\mathbf{A}^\wedge$ is the cyclic group with order d.

(iv) If $\alpha\ (\neq 0)$ is a given cardinal number, then each F-algebra is a subalgebra of an F-algebra with s-order α.

(v) If A is a set and α a cardinal such that $0 < \alpha \leqslant |A|$, then there is an F-algebra $(A; F)$ with s-order α.

(vi) If $|F| \geqslant 2$, then the class of weak F-associatives satisfies the propositions (v).

(vii) If the direct product of a collection of algebras is s-singular, then all algebras of the collection are s-singular. If $|F| \geqslant 2$ and $|I| \geqslant 2$, then there exist I-collections of s-singular weak F-associatives whose direct products are not s-singular.

(viii) Every semigroup F-algebra is an F-associative, and all F-associatives are semigroup F-algebras iff $d \in J$.

(ix) An associative is s-finite iff it is finite. If an F-associative $\mathbf{A} = (A; F)$ is infinite then $\|\mathbf{A}\| = |A|$.

(x) An associative $(A; F)$ is s-singular iff $|A| = 1$.

1. *Universal semigroup homomorphisms*. Let $\mathbf{A} = (A; F)$ be an F-algebra. A semigroup homomorphism $\lambda : \mathbf{A} \to \mathbf{A}^\wedge$ is said to be a u n i v e r s a l one iff for every semigroup homomorphism $\xi : \mathbf{A} \to \mathbf{S}$ there is a unique homomorphism $\varphi : \mathbf{A}^\wedge \to \mathbf{S}$ such that $\xi = \varphi\lambda$.

Clearly:

1.1. If $\lambda : \mathbf{A} \to \mathbf{A}^\wedge$ and $\lambda_1 : \mathbf{A} \to \mathbf{B}$ are universal semigroup homomorphisms, then there is a unique isomorphism $\varphi : \mathbf{A}^\wedge \to \mathbf{B}$ such tha $\lambda_1 = \varphi\lambda$. ∎

The existence of the universal semigroup homomorphism will be shown now.

Let $\mathbf{T} = (T; \cdot)$ be the semigroup which is freely generated by the carrier A of the algebra **A**. Thus,

$$T = \{a_1 \dots a_n \mid a_1, \dots, a_n \in A, n \geqslant 1\} \tag{1.1}$$

is the set of finite sequences on A, and the operation is the usual concatenation of sequences.

Let

$$\mathbf{a} = a_1 \dots a_r, \quad \mathbf{b} = b_1 \dots b_s, \quad a_i, b_j \in A.$$

We write $\mathbf{a}\,\tau_0\,\mathbf{b}$ iff there exist $\mathbf{c}, \mathbf{c}'_1, \ldots, \mathbf{c}'_r, \mathbf{c}''_1, \ldots, \mathbf{c}''_s \in T$ and continued products $\Pi'_i(\mathbf{c}'_i)$, $\Pi''_j(\mathbf{c}''_j)$ in $\mathbf{A}$ such that

$$\mathbf{c} = \mathbf{c}'_1 \ldots \mathbf{c}'_r = \mathbf{c}''_1 \ldots \mathbf{c}''_s \ , \ a_i = \Pi'_i(\mathbf{c}'_i), \ b_j = \Pi''_j(\mathbf{c}''_j). \quad (1.2)$$

(If, for example, $\mathbf{c}' = c'_1 \in A$, then $a_1 = \Pi'_1(\mathbf{c}'_1) = c'_1$.)

Now we shall prove the following statement:

1.2. The transitive extension τ of τ_0 is a congruence on $\mathbf{T}$, and the canonical mapping:

$$\lambda : \ a \to a^{\tau} \quad (1.3)$$

is a universal semigroup homomorphism from $\mathbf{A}$ into $\mathbf{A}^{\wedge} = \mathbf{T}/\tau$.

Proof. Clearly, τ_0 is a reflexive and symmetric relation in T, such that

$$u\,\tau_0\,v \Rightarrow uw\,\tau_0\,vw, \quad wu\,\tau_0\,wv,$$

and this implies that τ is a congruence on $\mathbf{T}$.

If $f \in F_n$, $a = fa_1 \ldots a_n$ in $\mathbf{A}$, then $a\,\tau_0\,a_1 \ldots a_n$ in $\mathbf{T}$, i.e.

$$\begin{aligned}\lambda(a) &= a^{\tau} = a_1^{\tau} \ldots a_n^{\tau} \\ &= \lambda(a_1) \ldots \lambda(a_n).\end{aligned}$$

Thus, $\lambda\colon \mathbf{A} \to \mathbf{A}^{\wedge}$ is a semigroup homomorphism.

Let $\mathbf{S} = (S; \cdot)$ be a semigroup, and $\xi : \mathbf{A} \to \mathbf{S}$ a semigroup homomorphism. If (1.2) is satisfied, and if $\mathbf{c} = e_1 \ldots e_t$ ($e_i \in A$), then we have:

$$\xi(a_1) \ldots \xi(a_r) = \xi(e_1) \ldots \xi(e_t) = \xi(b_1) \ldots \xi(b_s),$$

and this implies that

$$\varphi : (a_1, \ldots a_r)^{\tau} \to \xi(a_1) \ldots \xi(a_r)$$

is a mapping from $A^{\wedge}$ into S. Clearly, φ is a homomorphism from $\mathbf{A}^{\wedge}$ into $\mathbf{S}$ and it is the unique mapping such that $\xi = \varphi\,\lambda$. ∎

1.3. Let $\mathbf{A} = (A; F)$, $\mathbf{A}' = (A'; F)$ be F-algebras and $\lambda : \mathbf{A} \to \mathbf{A}^{\wedge}$ $\lambda' : \mathbf{A}' \to \mathbf{A}'^{\wedge}$ the corresponding universal semigroup homomorphisms. If $\varphi : \mathbf{A} \to \mathbf{A}'$ is a homomorphism, then there is a unique homomorphism $\varphi^{\wedge} : \mathbf{A}^{\wedge} \to \mathbf{A}'^{\wedge}$ such that $\lambda'\varphi = \varphi^{\wedge}\lambda$.

Proof. Let $\mathbf{T}$, $\mathbf{T}'$, τ and τ' be defined as in the proof of **1.2.** It is easy to show that:

$$a_1 \ldots a_r \tau b_1 \ldots b_s \Rightarrow \varphi(a_1) \ldots \varphi(a_r)\,\tau'\,\varphi(b_1) \ldots \varphi(b_s),$$

and this implies that

$$\varphi^{\wedge} : (a_1 \ldots a_r)^{\tau} \to (\varphi(a_1) \ldots \varphi(a_r))^{\tau'}$$

is a mapping from $A^{\wedge}$ into $A'^{\wedge}$. Clearly, φ is a homomorphism from $\mathbf{A}^{\wedge}$ into $\mathbf{A}'^{\wedge}$, and it is the unique mapping which satisfies the equation $\lambda' \varphi = \varphi^{\wedge} \lambda$. ∎

It is easy to show that:

1.4. $^{\wedge}$ is a covariant functor from the category of F-algebras into the category of semigroups. ∎

1.5. If $\varphi : \mathbf{A} \to \mathbf{A}'$ is an epimorphism (isomorphism), then $\varphi^{\wedge}$ is an epimorphism (isomorphism) too.

Proof. If φ is an epimorphism, and $(a'_1 \ldots a'_r)^{\tau'}$, then

$$(a'_1 \ldots a'_r)^{\tau'} = \varphi^{\wedge}((a_1 \ldots a_r)^{\tau})$$

where $a'_k = \varphi(a_k)$.

If φ is an isomorphism then:

$$\varphi^{\wedge}(\varphi^{-1})^{\wedge} = (\varphi\varphi^{-1})^{\wedge} = (1_{A'})^{\wedge}, (\varphi^{-1})^{\wedge}\varphi^{\wedge} = (1_A)^{\wedge}. \ ∎$$

We notice that the statement for monomorphisms is not true. Namely the proposition **2.8** states that each algebra $\mathbf{A}$ is a subalgebra of an s-singular algebra $\mathbf{B}$. Thus, if $\mathbf{A}$ is not singular, then the embedding homomorphism. $\varepsilon : \mathbf{A} \to \mathbf{B}$ is a monomorphism, but $\varepsilon^{\wedge} : A^{\wedge} \to B^{\wedge}$ is not a monomorphism.

In the following, $\mathbf{T}$, $\mathbf{A}$, λ and τ will have the same meaning as in the proof of **2.1**. If k is a positive integer, then A_k is a subset of $A^{\wedge}$ defined by:

$$A_k = \{(a_1 \ldots a_k)^{\tau} \mid a_1, \ldots, a_k \in A\}. \tag{1.4}$$

1.6. If $n \in J$ then $A^{\wedge} = A_1 \cup \ldots \cup A_n$.

Proof. This is a consequence from the following relations:

$$n \in J \Rightarrow A_{n+1} \subseteq A_1$$

$$A^{\wedge} = A_1 \cup \ldots \cup A_k \cup A_{k+1} \cup \ldots$$

1.7. $\mathbf{A}$ is s-finite iff $\mathbf{A}^{\wedge}$ is finite. If $\mathbf{A}$ is s-infinite then $\|\mathbf{A}\| = |A^{\wedge}|$.

Proof. From (1.4) and **1.6** follows that if $n \in J$ then

$$\|\mathbf{A}\| \leqslant |A^{\wedge}| \leqslant \|\mathbf{A}\| + \|\mathbf{A}\|^2 + \ldots + \|\mathbf{A}\|^n. \ ∎$$

The following propositions are obvious.

1.8. $\|\mathbf{A}\| = \alpha$ iff $|\xi(A)| \leqslant \alpha$ for every semigroup homomorphism $\xi : \mathbf{A} \to \mathbf{S}$, and $|\eta(A)| = \alpha$ for a semigroup homomorphism $\eta : \mathbf{A} \to \mathbf{S}$.

1.9. If $\mathbf{A}'$ is a homomorphic image of $\mathbf{A}$, then $\|\mathbf{A}'\| \leqslant \|\mathbf{A}\|$.

1.10. If $F' \subseteq F$ then $\|(A; F)\| \leqslant \|(A; F')\|$.

2. *Semigroup-singular algebras.* Some example of s-singular algebras will be given first.

2.1. Let A be a nonempty set, 0 a fixed element of A, and $\varphi : x \to x'$ an injection from A into A. If f is an n-ary operation on A such that

$$fx'^n = x, \quad fx'' x'^{n-1} = fx''^{n-1}\, 0 = 0, \tag{2.1}$$

then the algebra $\mathbf{A} = (A; f)$ is s-singular.

Proof. If $\xi : \mathbf{A} \to \mathbf{S}$ is a semigroup homomorphism, and $a \in A$, then

$$\xi(a) = \xi(fa'^n) = \xi(f^2 a''^n a'^{n-1})$$

$$= \xi(a'')^n \xi(a')^{n-1} = \xi(fa''^{n-1} fa'' a'^{n-1})$$

$$= \xi(fa''^{n-1} 0) = \xi(0). \blacksquare$$

2.2. Let f, g be two distinct elements of F, such that $f \in F_n$, $g \in F_m$. If there is an element $e \in A$ such that

$$fxe^{n-1} = x, \quad gxe^{m-1} = e, \tag{2.2}$$

for every $x \in A$, then the algebra $(A; F)$ is s-singular.

Proof. If $\xi : \mathbf{A} \to \mathbf{S}$ is a semigroup homomorphism and $a \in A$ then we have:

$$\xi(a) = \xi(gfae^{m+n-2}) = \xi(a)\, \xi(e)^{m+n-2}$$

$$= \xi(fgae^{m+n-2}) = \xi(fe^n)$$

$$= \xi(e). \blacksquare$$

2.3. Let $F = F' \cup F''$, $F' \cap F'' = \emptyset$, $F' \neq \emptyset$, $F'' \neq \emptyset$. If $\mathbf{G} = (G; \cdot)$ is a semigroup with a zero 0 and an identity e, and if an F-algebra $\mathbf{G}(F', F'')$ is defined by:

$$f \in F' \cap F_n \Rightarrow fx_1 \dots x_n = 0 \tag{2.3}$$

$$g \in F'' \cap F_m \Rightarrow gx_1 \dots x_m = x_1 \dots x_m,$$

then $\mathbf{G}(F', F'')$ is an s-singular weak F-associative.

Proof. It is easy to see that $\mathbf{G}(F', F'')$ is a weak F-associative. Let $\xi : \mathbf{G}(F', F'') \to \mathbf{S}$ be a semigroup homomorphism. If $f \in F_n \cap F'$, $g \in F_m \cap F''$ and $a \in G$, then we have:

$$\xi(a) = \xi(g^{n-1} a\, e^{(n-1)(m-1)}) = \xi(a)\, \xi(e)^{(n-1)(m-1)}$$

$$= \xi(f^{m-1}\, a\, e^{(n-1)(m-1)}) = \xi(0),$$

and this implies that $\mathbf{G}(F', F'')$ is s-singular. ∎

2.4. If A is a nonempty set, and f an n-ary operator, then there is an s-singular algebra $(A; f)$.

Proof. Let $\varphi : x \to x'$ be an injective transformation of A and 0 a fixed element of A such that

$$x' = x \ \text{ or } \ x' = 0 \Rightarrow x = 0. \tag{2.4}$$

Clearly, there is an n-ary operation on f such that (2.1) is satisfied, and if f is such an operation, then by **2.1** $(A; f)$ is s-singular. ∎

2.5. If $|F| \geqslant 2$ and if A is a nonempty set, then there is an s-singular weak F-associative.

Proof. First, a semigroup $(A; \cdot)$ with a zero and an identity can be built, and then an s-singular weak F-associative can be obtained as in **2.3.** ∎

If $F = F_n = \{f\}$, then a weak associative $(A; f)$ is called an n-semigroup. It is well known that:

2.6. If $(A; f)$ is an n-semigroup, then there is a semigroup $\mathbf{B} = (B; \cdot)$ such that $A \subseteq B$ and

$$f x_1 \ldots x_n = x_1 \ldots x_n\,,$$

for all $x_1, \ldots, x_n \in A$. And, if $\mathbf{B}$ is generated by A, then $\mathbf{B}$ is said to be a c o v e r i n g s e m i g r o u p of $(A; f)$. ([4], p. 25). ∎

As a consequence of **2.6**, we obtain that the assumption $|F| \geqslant 2$ is essential in **2.5**. Namely, we have:

2.7. An n-semigroup $(A; f)$ is singular iff $|A| = 1$. ∎

Now, we shall show that the class of s-singular F-algebras is not hereditary.

2.8. Every algebra is a subalgebra of an s-singular algebra.

Proof. Let $A = (A; F)$ be an F-algebra.

(i) Let $f \in F_n$, $g \in F_m$ be two distinct elements of F, and $e \notin A$. An algebra $\mathbf{B} = (A \cup \{e\}; F)$ can be defined such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$, and (2.2) is satisfied for all $x \in B$. Then, by 2.2, $\mathbf{B}$ is s-singular.

(ii) Let $F = F_n = \{f\}$, and let B be a set and $\varphi : x \to x'$ an injective transformation of B, such that $A \subseteq B \setminus \varphi(B)$, $0 \in B$, and (2.4) is satisfied. Then, an algebra $\mathbf{B} = (B; f)$ can be defined in which $\mathbf{A}$ is a subalgebra and (2.1) is satisfied. By **2.1**, $\mathbf{B}$ is s-singular.

2.9 The class of subalgebras of s-singular weak F-associatives is a proper subclass of the class of weak F-associatives.

Proof. 1) Let $\mathbf{A}$ be a weak F-associative and $\lambda : \mathbf{A} \to \mathbf{A}^{\wedge}$ the universal semigroup homomorphism. In a similar way as in the proof of **4.6**, it can be shown that if $a, b \in A$ and $\lambda(a) = \lambda(b)$, then there exist two sequences of operators $f_1 \dots f_r$, $g_1 \dots g_s$ such that:

$$f_1 \dots f_r \; a x_1 \dots x_n = g_1 \dots g_s \; b x_1 \dots x_n, \tag{2.5}$$

for all $x_1, \dots, x_n \in A$.

2) Let B be a nonempty set and A the weak F-associative which is freely generated by B. It is easy to see that if $a \in C$ and $b \in A$ are two distinct elements of A, then no equation of the form (2.5) is satisfied in $\mathbf{A}$. Therefore $\mathbf{A}$ can not be embedded in an s-singular weak F-associative.

2.10. If $\mathbf{A}$ is an s-singular F-algebra, then $\mathbf{A}^{\wedge}$ is the cyclic group with d elements.

Proof. Let $\mathbf{C}_d$ be the cyclic group with a generator c and order d. It is easy to see that the mapping $\xi : \mathbf{A} \to \mathbf{C}_d$ defined by: $(\forall x)\, \xi(x) = c$ is a universal semigroup homorphism. ∎

3. *Algebras with arbitrary semigroup orders.* The main results in this part are statements **3.6** and **3.7** which are generalizations of **2.4**, **2.5**, **2.8**, and **2.9**.

3.1. If $F(A) = \bigcup_{n=0}^{\infty} \bigcup_{f \in F_n} f A^n$ and $A^* = A \setminus F(A)$,

then

$$\|\mathbf{A}\| \geqslant |A^*| + 1.$$

Proof. If $0 \notin A^*$, $S = A^* \cup \{0\}$, $(\forall x, y \in S)\ x \cdot y = 0$ and

$$\xi(x) = \begin{cases} x & \text{if } x \in A^* \\ 0 & \text{if } x \in A \setminus A^*, \end{cases}$$

then $\xi : \mathbf{A} \to (S; \cdot)$ is a semigroup homomorphism such that $|\xi(A)| = 1 + |A^*|$. ∎

As a consequence of **3.1** we obtain that:

3.2. If $\mathbf{A}$ is s-singular, then it is surjective, i.e. $F(A) = A$. ∎

3.3. If $\mathbf{A} = (A; F)$ is a subalgebra of $\mathbf{B} = (B; F)$, then

$$\|\mathbf{B}\| \leqslant \|\mathbf{A}\| + |B \setminus A|.$$

Proof. Let $\lambda_B : \mathbf{B} \to \mathbf{B}^\wedge$ be the universal semigroup homomorphism. The restriction $\xi = \lambda_B | A : \mathbf{A} \to \mathbf{B}^\wedge$ is a semigroup homomorphism and this implies that:

$$\|\mathbf{B}\| = |\lambda_B(B)| \leqslant |\lambda_B(A)| + |\lambda_B(B \setminus A)|$$

$$\leqslant |\xi(A)| + |B \setminus A| \leqslant \|\mathbf{A}\| + |B \setminus A|. \blacksquare$$

3.4. Let $\mathbf{A} = (A; F)$ be an algebra, C a set disjoint with A and 0 a fixed element of C. If $B = A \cup C$ and if an algebra $\mathbf{B} = (B; F)$ is defined by:

(i) $\mathbf{A}$ is a subalgebra of $\mathbf{B}$;

(ii) $f \in F_n, (b_1, \ldots, b_n) \in B^n \setminus A^n \Rightarrow fb_1 \ldots b_n = 0$,

then

$$\|\mathbf{B}\| = \|\mathbf{A}\| + |C|.$$

(The algebra $\mathbf{B}$ will be denoted by $\mathbf{A}(C)$.)

Proof. Let $\xi : \mathbf{A} \to \mathbf{S}$ be a semigroup homomorphism such that $|\xi(A)| = \|\mathbf{A}\|$, and $S \cap C = \varnothing$. Define a groupoid $\mathbf{D} = (S \cup C; *)$ such that $\mathbf{S}$ is a subgroupoid of $\mathbf{D}$ and

$$(x, y) \in D \times D \setminus S \times S \Rightarrow x * y = 0.$$

Clearly, $\mathbf{D}$ is a semigroup and the mapping $\eta : B \to D$ defined by

$$\eta(x) = \begin{cases} \xi(x) & \text{if } x \in A \\ x & \text{if } x \in C \end{cases}$$

is a semigroup homomorphism $\eta : \mathbf{B} \to \mathbf{D}$. We also have:

$$|\eta(B)| = |\eta(A)| + |\eta(C)| = |\xi(A)| + |C| =$$

$$= \|\mathbf{A}\| + |C|,$$

and this implies that $\|\mathbf{A}\| + |C| \leqslant \|\mathbf{B}\|$, whence by **3.3** we obtain that the equality holds. $\blacksquare$

It is obvious that:

3.5. If $\mathbf{A}$ is a weak F-associative, then $\mathbf{A}(C)$ is also a weak F-associative. $\blacksquare$

3.6. Let $\alpha\ (\neq 0)$ be a cardinal number. Every algebra is a subalgebra of an algebra with semigroup order α.

Proof. Let **A** be an F-algebra. By **2.8**, **A** is a subalgebra of an s-singular algebra **B**. Let C be a nonempty set such that $B \cap C = \emptyset$, and $\alpha = 1 + |C|$. Then **A** is a subalgebra of $\mathbf{B}(C)$. By **3.4**, we have:

$$\|\mathbf{B}(C)\| = \|\mathbf{B}\| + |C| = 1 + |C| = \alpha.$$

3.7. Let A be a set and $\alpha (\neq 0)$ a cardinal number.

(i) There exists an F-algebra $\mathbf{A} = (A; F)$ with semigroup order α iff $\alpha \leqslant |A|$.

(ii) If $|F| \geqslant 2$ and $\alpha \leqslant |A|$, then there is a weak F-associative $\mathbf{A} = (A; F)$ with semigroup order α.

Proof. (i) If $\mathbf{A} = (A; F)$ is an algebra then $\|\mathbf{A}\| = |\lambda(A)| \leqslant |A|$.

Assume that $\alpha \leqslant |A|$, and $A = B \cup C$, $B \cap C = \emptyset$, $1 + |C| = \alpha$. By **2.4** and **2.5** there exists an s-singular algebra $\mathbf{B} = (B; F)$, and by **3.4** if $\mathbf{A} = \mathbf{B}(C) = (A; F)$, we have

$$\|\mathbf{A}\| = \|\mathbf{B}\| + |C| = 1 + |C| = \alpha$$

(ii) By **2.5**, **3.4** and **3.5.** ∎

Some properties of semigroup orders of direct products will be shown now.

3.8. If $(\mathbf{A}_i)_{i \in I}$ is a collection of F-algebras, then:

$$\prod_{i \in I} \|\mathbf{A}_i\| \leqslant \|\prod_{i \in I} \mathbf{A}_i\| \leqslant \prod_{i \in I} |A_i|.$$

Proof. If $(\lambda_i : \mathbf{A}_i \to \mathbf{A}_i{}^{\wedge})_{i \in I}$ is the corresponding collection of universal homomorphisms, then

$$\xi = (\lambda_i)_{i \in I} : \prod_{i \in I} \mathbf{A}_i \to \prod_{i \in I} \mathbf{A}_i{}^{\wedge}$$

is a semigroup homomorphism such that

$$\xi (\prod_{i \in I} A_i) = \prod_{i \in I} \lambda_i (A_i). \quad ∎$$

As corollary of **3.8** we obtain the following statement.

3.9. If the direct product of a collection of algebras is s-singular, then all algebras of the collection are s-singular.

The following proposition shows that the converse is not true.

3.10. If $|F| \geqslant 2$ and $|I| \geqslant 2$, then there exists a collection $(\mathbf{A}_i)_{i \in I}$ of -singular weak F-associatives whose direct product is not s-singular.

Proof. Let F', F'', I' and I'' be nonempty, and

$$F = F' \cup F'', \quad I = I' \cup I'', \quad F' \cap F'' = I' \cap I'' = \emptyset.$$

If **G** is a semigroup with a zero 0 and an identity e ($\neq 0$), and if:

$$i \in I' \Rightarrow \mathbf{A}_i = \mathbf{G}(F', F''),$$

$$i \in I'' \Rightarrow \mathbf{A}_i = \mathbf{G}(F'', F'),$$

then $(\mathbf{A}_i)_{i \in F}$ is a collection of s-singular weak F-associatives. The direct product $A = \prod_{i \in I} \mathbf{A}_i$ is a weak F-associative which is not surjective, and thus (by **3.2**) it is not s-singular. ∎

4. *Associatives.* Here, it will be shown that nontrivial s-singular associatives do not exist (Theorem **4.8**) and that the semigroup order of an infinite associative is the ordinary order of the associative (Theorem **4.11**).

In the following it will always be assumed that $\mathbf{A} = (A; F)$ is an F-associative and that K is the additive semigroup of nonnegative integers generated by $J \cup \{0\}$.

The following general associative law can easily be shown.

4.1. Let $n_0, \ldots, n_r, m_0, \ldots, m_s \in J$ and $n \in K$ be such that

$$n_0 + \ldots + n_r = n = m_0 + \ldots + m_s,$$

and $f_i \in F_{n_i+1}, g_j \in F_{m_j+1}$. If $p_1, \ldots, p_r$ is a sequence of nonnegative integers such that $p_\nu \leqslant p_{\nu+1} < n_0 + \ldots + n_\nu$, then the following identity is satisfied:

$$g_0 \ldots g_s\, x_0 \ldots x_n = f_0 x_0 \ldots\ f_1\, x_{p_1} \ldots f_r\, x_{p_r} \ldots x_n.$$

In the following, every continued product $\Pi(x_0, \ldots, x_n)$ will be denoted by $(x_0 \ldots x_n)$; if $n = 0$ then $\Pi\,(x_0) = x_0$.

The following results (in slighty different formulations) are known.

4.2. The class of semigroup F-algebras is equal to the class of F-associatives iff $d \in J$. [3]

4.3. The class of semigroup F-associatives admits homomorphic images iff $d \in J$. [2].

4.4. **A** is surjective iff $(A^{n+1}) = A$, for each $n \in K$. [2]. ∎

4.5. (i) There exist $n_1, \ldots, n_k \in J$ such that every $n \in K$ has a form $n = \nu_1 n_1 + \ldots + \nu_k n_k$, where $\nu_i \geqslant 0$. If k is the minimal number with the above property, then $\{n_1, \ldots, n_k\}$ is uniquely determined and it is called the basis of K.

(ii) There is a positive integer q such that

$$K^* = \{\nu d \mid \nu \geqslant q\} \subseteq K,$$

and if q is the least such a positive number, then K^* is said to be the regular part of K. [1]

4.6. If $n \in K^*$, $\nu \in \{1, \dots, n+1\}$ $a, b \in A$ and $a \tau b$, then the following identity is satisfied in **A**:

$$(x_1 \dots x_{\nu-1} a x_\nu \dots x_n) = (x_1 \dots x_{\nu-1} b x_\nu \dots x_n). \qquad (4.1)$$

||

Proof. If $\mathbf{a} = a_0 \dots a_r$, $\mathbf{b} = b_0 \dots b_s$, $a_i, b_j \in A$ and $\mathbf{a} \tau_0 \mathbf{b}$, then by **1.2**, there exists an $e_0 e_1 \dots e_t \in T$ $(e_\nu \in A)$ such that the following equations are satisfied:

$$a_0 = (e_0 \dots e_{m_0}, a_1 = (e_{m_0+1} \dots e_{m_0+m_1+1}), \dots, a_r = (\dots e_t)$$

$$(4.2)$$

$$b_0 = (e_0 \dots e_{p_0}), b_1 = (e_{p_0+1} \dots e_{p_0+p_1+1}, \dots, b_s = (\dots e_t),$$

and

$$p_0 + \dots + p_r + r = t = m_0 + \dots + m_s + s \qquad (4.3)$$

$$p_0, \dots, p_r, m_0, \dots, m_s \in K.$$

From (4.3) it follows that $d \mid r \Leftrightarrow d \mid s$, and if this is satisfied, then $n+r$, $n+s \in K^*$, for $n+r$, $n+s \geqslant qd$ and $d \mid n+r$, $d \mid n+s$.

This implies that:

$$(x_1 \dots x_{\nu-1} \mathbf{a} x_\nu \dots x_n) = (x_1 \dots x_{\nu-1} e_0 \dots e_t x_\nu \dots x_n) \qquad (4.4)$$

$$= (x_1 \dots x_{\nu-1} \mathbf{b} x_\nu \dots x_n),$$

for all $x_1, \dots, x_n \in A$.

Assume now that $a, b \in A$ and $a \tau b$. Then there exist

$$\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r \in T,$$

such that

$$a \tau_0 \mathbf{c}_1, \mathbf{c}_1 \tau_0 \mathbf{c}_2, \dots, \mathbf{c}_r \tau_0 b,$$

and this, by (4.4), implies that:

$$(x_1 \ldots x_{\nu-1}\, a\, x_\nu \ldots x_n) = (x_1 \ldots x_{\nu-1}\, \mathbf{c}_1\, x_\nu \ldots x_n)$$

$$— — — — — — — —$$

$$= (x_1 \ldots x_{\nu-1}\, \mathbf{c}_r\, x_\nu \ldots x_n)$$

$$= (x_1 \ldots x_{\nu-1}\, b\, x_\nu \ldots x_n). \blacksquare$$

4.7. If **A** is surjective, and $a, b \in A$, $a \tau b$, then (4.1) is an identity in **A**, for each $n \in K$, and $\nu \in \{1, \ldots, n+1\}$.

Proof. This is a consequence from **4.4** and **4.6**. ∎

4.8. If **A** is s-singular then $|A| = 1$.

Proof. By **3.2**, **A** is surjective, and by **4.7** the following identity is satisfied

$$(x_1 \ldots x_n) = (y_1 \ldots y_n) = c_n$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ and $n \in K$.

If $m, n \in K$, then

$$c_n = ((y_0 \ldots y_m)\, x_1 \ldots x_n) = (y_0 \ldots y_{m-1}\, (y_m x_1 \ldots x_n)) = c_m \, (= c).$$

If we define a binary operation on A by $(\forall x, y)\ x \cdot y = c$, then we obtain a semigroup $(A; \cdot)$ and the identity mapping 1_A is a semigroup homomorphism from **A** in $(A; \cdot)$. This implies that **A** is both s-singular and semigroup F-associative, and this is possible iff $|A| = 1$. ∎

4.9. Let $\{m_1, \ldots, m_s\} = Q$ be the basis of the regular part K^* of K, and $\{n_1, \ldots, n_p\} = P = K \setminus K^*$.

If $||\mathbf{A}|| = \alpha$ then

$$|A| \leqslant (1-\alpha)\,[1 + (\alpha - 1)^{n_1} + \ldots + (\alpha - 1)^{n_p}] + \alpha(\alpha^{m_1} + \ldots + \alpha^{m_s}). \quad (4.5)$$

Proof. By **4.6**, if $m \in K^*$, $\nu \in \{1, \ldots, m+1\}$, and if $a, b \in A$ are such that $a \tau b$, then

$$(x_1 \ldots x_{\nu-1}\, a\, x_\nu \ldots x_m) = (x_1 \ldots x_{\nu-1}\, b\, x_\nu \ldots x_m)$$

This implies that $|(A^{m+1})| \leqslant \alpha^{m+1}$ and thus

$$B = \bigcup_{m \in K^*} (A^{m+1}) \Rightarrow |B| \leqslant \alpha \sum_{m \in Q} \alpha^m . \quad (4.6)$$

By **3.1**, we have:

$$A^* = A \setminus F(A) \Rightarrow |A^*| \leqslant \alpha - 1. \qquad (4.7)$$

Let $C = F(A) \setminus B$. If $c \in C$, and if n is the maximal number of P such that $c \in (A^{n+1})$, then there exist $a_0, \ldots, a_n \in A^*$, such that $c = (a_0 \ldots a_n)$, and this, by (4.7), implies that

$$|C| \leqslant (\alpha - 1) \sum_{n \in P} (\alpha - 1)^n. \qquad (4.8)$$

Finally, (4.6), (4.7), (4.8), and $A = A^* \cup B \cup C$ imply that (4.5) is satisfied. ∎

The following two statements are direct consequences from **4.9**.

4.10. **A** is s-finite iff it is finite.

4.11. If A is infinite then $||\mathbf{A}|| = |A|$.

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РЕЗИМЕ

ЕДНО ПРЕТСТАВУВАЊЕ НА АЛГЕБРИ ВО ПОЛУГРУПИ

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Нека $\mathbf{A} = (A; F)$ е алгебра со носител A и множество оператори F и нека $\mathbf{S} = (S;\cdot)$ е полугрупа. За пресликувањето $\xi : A \to S$ велиме дека е полугрупен хомоморфизам, ако (0.1) е идентитет за секој n-арен оператор $f \in F$. Поимот за универзален полугрупен хомоморфизам $\lambda : \mathbf{A} \to A^\wedge$ се воведува на обичен начин, и притоа за $\mathbf{A}^\wedge$ велиме дека е универзална полугрупа придружена на **A**. Кардиналниот број $|\lambda(A)| = ||\mathbf{A}||$ се вика полугрупен ред на **A**; ако е $||\mathbf{A}|| = 1$, тогаш за алгебрата **A** велиме дека е сингуларна. Алгебрата **A** се вика слаб асоцијатив ако еден сложен производ не зависи од распоредот на операторите; **A** се вика асоцијатив ако сложените производи зависат само од низите елементи од носителот на алгебрата, но не и од низите оператори.

Ќе формулираме неколку резултати што се докажани во работава. (i) $||\mathbf{A}||$ е конечен ако $|\mathbf{A}^\wedge|$ е конечен; (ii) ако $||\mathbf{A}||$ е бесконечен, тогаш $||\mathbf{A}|| = |A^\wedge|$; (iii) ако $\mathbf{A}$ е сингуларна алгебра, тогаш $\mathbf{A}^\wedge$ е циклична група со ред d, каде што d е најголемиот заеднички делител на броевите од множеството (0.4); (iv) ако $\alpha\,(\neq 0)$ е даден кардинален број, тогаш секоја алгебра е подалгебра на некоја алгебра со полугрупен ред α; (v) ако е $|A| = \alpha$ и $0 < \beta \leqslant \alpha$, тогаш постои алгебра $(A; F)$ со полугрупен ред β; (vi) ако е $|F| \geqslant 2$, тогаш (iv) и (v) важат за класата слаби F-асоцијативи; (vii) ако директниот производ на една колекција алгебри е сингуларен, тогаш секоја алгебра од колекцијата е сингуларна, но ако е $|F| \geqslant 2$, обратното не мора да важи; (viii) еден асоцијатив има конечен полугрупен ред ако е конечен; (ix) полугрупниот ред на еден бесконечен асоцијатив е еднаков со редот на асоцијативот; (x) не постои нетривијален сингуларен асоцијатив.

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