ON SEMIGROUP ASSOCIATIVES*

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1. PRELIMINARIES

Let $A=(A;F)$ be a universal algebra with the carrier A and the set F of finitary operations. We shall consider in this paper that $F_0 \cup F_1 = \emptyset$, where $F_n = \{f \in F \mid f$ is an n-ary operation on $A\}$ and that there exists at least one $n \geqslant 2$ such that $F_n \neq \emptyset$. If $f \in F_{n+1}$ and $f:(x_0, x_1, \ldots, x_n) \to y$, then we write $y = fx_0x_1 \ldots x_n$. We call the integer n the assigned number to f and we denote it by n_f.

An algebra $A=(A;F)$ is called an *associative* or, more precisely, an F-associative ([1]) iff[1]:

(i) for any $f, g \in F$ and $r \in \mathbf{N}$[2], $r \leqslant n_f$, the following identity equality holds

$$fgx_0x_1 \ldots x_n = fx_0x_1 \ldots x_{r-1}gx_r \ldots x_n,$$

where $n = n_f + n_g$;

(ii) if $f_1, \ldots, f_k, g_1, \ldots, g_s \in F$ and $n_{f_1} + \ldots + n_{f_k} = n_{g_1} + \ldots + n_{g_s}$, then the following identity equality holds:

$$f_1 \ldots f_k \, x_0x_1 \ldots x_n = g_1 \ldots g_s \, x_0x_1 \ldots x_n.$$

Let $f_0, f_1, \ldots, f_k$ be elements of F with the assigned numbers $n_0, n_1, \ldots, n_k$ respectively and let $i_1, \ldots, i_k$ be a sequence of integers such that $0 \leqslant i_\upsilon \leqslant i_{\upsilon+1}$, $i_\upsilon \leqslant n_0 + n_1 + \ldots + n_{\upsilon-1}$ $(\upsilon = 1, 2, \ldots, k)$. Then the following "continued product" with the "length" $1+n = 1+n_0+n_1+ + \ldots + n_k$,

$$\Pi x_0x_1 \ldots x_n = f_0x_0 \ldots x_{i_1-1}f_1x_{i_1} \ldots x_{i_k-1}f_kx_{i_k} \ldots x_n \qquad (1.1)$$

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[1] „iff" stands for „if and only if".

[2] **N** is the set of all positive integers.

can be defined in an obvious way. If A is an F-associative, then by (i) and (ii) it follows that the continued product (1.1) does not depend on the sequence $i_1, \ldots, i_k$ and on the sequence of operations $f_0, f_1, \ldots, f_k$, but it depends only on the sequence of elements $x_0, x_1, \ldots, x_n \in A$. Therefore, any two products of the same sequence of elements are equal in an F-associative, i.e. the "general associative law" holds in A. The converse is obviously true.

If $f, g \in F_n$, then by (ii) it follows that f and g, as mappings, are equal and so we may assume that in an F-associative, for any positive integer $n \geqslant 2$, there exists at most one operation $f \in F_n$. Therefore it is convenient to consider the set

$$J = \{n \mid n \in \mathbf{N},\ F_{n+1} \neq \emptyset\}$$

and call $A = (A;F)$ a *J-associative*; we shall denote it by $A(J)$ or, simply, by A.

If $\langle J \rangle$ is the additive subsemigroup of $\mathbf{N}(+)$ generated by the set J, then for any $m \in \langle J \rangle$ it is possible to define in an obvious way an $(m+1)$-ary operation on A and the new algebra (with the same carrier A) will also be an associative.

The associatives $A(J)$ and $A(\langle J \rangle)$ are distinct as algebras, but the distinction is not essential for the questions which will be considered in this paper. For example, if $F_{n+1} \neq \emptyset$ for some $n \in \mathbf{N}$ and $F_{i+1} = \emptyset$ for all $i \neq n$, then the J-associative A with $J = \{n\}$ is, in fact, an n-semigroup which can be considered as a $\langle J \rangle$-associative with $\langle J \rangle = \{n, 2n, \ldots\}$ if one takes the "extensions" of the given $(n+1)$-ary operation; in particular, any semigroup S is an $\{1\}$-associative or an $\mathbf{N}$-associative and also a J-associative for any non-empty subset $J \subseteq \mathbf{N}$ Therefore, we may always consider J as the subsemigroup of $\mathbf{N}(+)$ generated by the set of the assigned numbers to the operations of F.

The following result ([6], [9]) will be used in the next section of this paper:

1.1. Lemma. *If J is a subsemigroup of $\mathbf{N}(+)$ and $d = G.C.D.\ (J)$*[3]*, then there exists $r \in \mathbf{N}$, such that*

$$(x \in J \wedge x \geqslant r) \Leftrightarrow (x \geqslant r \wedge d \mid x) \cdot \square$$

The set $J_* = \{r,\ r+d,\ r+2d, \ldots\}$ will be called the *regular part* of J.

[3] Throughout the paper d will mean "the greatest common divisor of th numbers of J".

No confusion will result if we use the same symbol for all the operations in an associative $A(J)$. Therefore, for any $n \in J$, we shall write $[x_0 x_1 \dots x_n]$ and sometimes $x_0 x_1 \dots x_n$ (without any operation symbol) instead of the product $f x_0 x_1 \dots x_n$.

If $B_0, B_1, \dots, B_n, B\ (n \in J)$ are non-empty subsets of A and $a \in A$, then the symbols $B_0 B_1 \dots B_n$, $B_0 \dots B_{i-1} a B_{i+1} \dots B_n$ and B^{n+1} have the usual meanings. The symbol a^k will substitute the sequence $a \dots a$ (k elements a).

Further on we shall often assume that the identity operation on A is defined, i.e. we shall put $[x] = x$ for all $x \in A$. (It is clear that A remains an associative after adjoining this unary operation to the operations of $A(J)$.)

Let A be a J-associative. If B is a non-empty subset of A such that

$$n \in J \wedge b_0, b_1, \dots, b_n \in B \Rightarrow [b_0 b_1 \cdot \dots \cdot b_n] \in B,$$

then B is called a *J-subassociative of the associative A*. If $J \subseteq \mathbf{N}$ and B is a subset of a semigroup $S(\cdot)$, such that

$$n \in J \wedge b_0, b_1, \dots, b_n \in B \Rightarrow b_0 \cdot b_1 \dots b_n \in B,$$

then B is called a *J-subassociative of the semigroup S*. More generally, let A be a J-associative and S a semigroup. A mapping $\xi : A \to S$ is said to be a *semigroup homomorphism* from A to S iff ξ is a homomorphism of the J-assocciative A to the semigroup S (S is considered as a J-associative), i.e. iff for any $n \in J$ and $x_0, x_1, \dots, x_n \in A$

$$\xi([x_0 x_1 \dots x_n]) = \xi(x_0)\, \xi(x_1) \dots \xi(x_n).$$

If there exists at least one semigroup S and at least one semigroup homomorphism from $A(J)$ to $S(\cdot)$ which is a monomorphism, then $A(J)$ can be (isomorphically) embedded in S and it is called a *semigroup associative*. Any n-semigroup is a semigroup associative ([3], [5]), but there exist associatives which are not semigroup associatives ([1], p. 11).

2. THE UNIVERSAL SEMIGROUP FOR A J-ASSOCIATIVE

A semigroup can be associated, in a natural manner, to any associative ([1], Theorem on p. 11) and, more generally, to any algebra $(A;F)$ ([4], 1.2). Namely, let A be a J-associative and let $U = U_A$ be the semigroup which is freely generated by the set A (i.e. U consists of all finite sequences $\mathbf{a}=(a_1,\dots,a_k)$, $k\in\mathbf{N}$, $a_\upsilon\in A$ and the operation is the usual concatenation of sequences). We say that $\mathbf{a}=(a_1,\dots,a_p)$ and $\mathbf{b}=(b_1,\dots,b_q)$ $(a_\upsilon, b_\mu\in A)$ are *strongly linked* in A and we write $\mathbf{a}\ sl_A\ \mathbf{b}$ iff there exists an element $\mathbf{e}=(e_0,e_1,\dots,e_t)$ of U $(e_\upsilon\in A)$ and two sequences $k_1,\dots,k_p\in J$, $m_1,\dots,m_q\in J$, such that

$$\begin{aligned}&t+1=k_1+\dots+k_p+p=m_1+\dots+m_q+q,\\&a_1=[e_0\dots e_{k_1}],\ a_2=[e_{k_1+1}\dots e_{k_1+k_2+1}],\dots,\ a_p=[\dots e_t],\\&b_1=[e_0\dots e_{m_1}],\ b_2=[e_{m_1+1}\dots e_{m_1+m_2+1}],\dots,\ b_q=[\dots e_t].\end{aligned}\tag{2.1}$$

Denote by l_A the transitive extension of $sl(=sl_A)$, i.e. put

$$\mathbf{a}\ l_A\ \mathbf{b}\Leftrightarrow(\exists\,\mathbf{c}_1,\dots,\mathbf{c}_s\in U)\ \mathbf{a}\ sl\ \mathbf{c}_1\ sl\dots sl\ \mathbf{c}_s\ sl\ \mathbf{b}.\tag{2.2}$$

(In that case we say that **a** and **b** are *linked* in A.)

It can be shown ([1], [2], [4] and [5]) that:

1°. *l is a congruence on U,*

2°. *$A^l=\{a^l\mid a\in A\}$ is a generating set for the factor semigroup U/l and a J-subassociative of U/l.*

3°. *The canonical mapping $\lambda: a\to a^l$ is a semigroup homomorphism from the J-associative A to the semigroup U/l, and its restriction $\lambda_1: A\to A^l$ is an epimorphism.*

4°. *The canonical homomorphism $\lambda: A\to U/l$ has a universal property, i.e. for any semigroup homomorphism $\xi: A\to S$, there exists unique homomorphism $\varphi: U/l\to S$ such that $\xi=\varphi\lambda$.*

The next theorem ([1], Corollary on p. 13) will often be used in this paper.

2.1. Theorem. *A J-associative A is a semigroup associative iff the restriction l° of l on the set A is the equality relation on A* □.

According to 4°, the mapping $\lambda: A\to U/l$ is called a *universal* semigroup homomorphism and the factor semigroup U/l a *universal semigroup* for the associative A. If U' and U'' are two universal semigroups for an associative A, then it is easy to see that $U'\cong U''$. So there exists a unique (up to isomorphism) üniversal semigroup for a J-associative A and it will be denoted by $A^\frown$ or, in some circumstances, by $A(J)^\frown$.

If an associative A is a semigroup associative, then the universal semigroup $A^\frown$ will be called the *free covering* (*semigroup*) of A or the *maximal covering of A*.

The next theorem gives a better insight into the structure of the universal semigroup for a J-associative.

2.2. Theorem. *Let A be a J-associative and $d = G.\,C.\,D.\,(J)$. If $n \in J$, then*

$$A^{\frown} = A_1 \cup A_2 \cup \ldots \cup A_n, \tag{2.3}$$

where $A_k = \{(a_1, \ldots, a_k)^l \mid a_\nu \in A\}$. If $p,\ q \leqslant n$, then

$$A_p \cap A_q \neq \emptyset \Leftrightarrow d \mid q - p. \tag{2.4}$$

In particular, if $p,\ q \leqslant d$ and $p \neq q$, then $A_p \cap A_q = \emptyset$.

Proof. By the way of constructing the set $A^{\frown} = U/l$ it follows that $A^{\frown} = A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots$ The equality (2.3) follows from the fact that

$$A_{n+i} \subseteq A_i \tag{2.5}$$

for any $n \in J$ and $i: 1 \leqslant i \leqslant n$.

Let $(a_1, \ldots, a_p)\, sl\, (b_1, \ldots, b_q)$, where $a_i,\ b_j \in A,\ p,\ q \leqslant n$ and $d = G.\,C.\,D.\,(J)$. By (2.1),

$$k_1 + \ldots + k_p + p = m_1 + \ldots + m_q + q$$

for some $k_i,\ m_j \in J$ and thus $d \mid k_i,\ d \mid m_j$, from which it follows that

$$d \mid (k_1 + \ldots + k_p) - (m_1 + \ldots + m_q) = q - p.$$

Hence $A_p \cap A_q \neq \emptyset$ implies $d \mid q - p$.

Conversely, let $d \mid q - p$, i.e. $q - p = rd$. Then there exist (large enough) integers $m,\ n \in J$, such that $m = n + rd = n + q - p$ (it is sufficient to take n to be the first element of the regular part of the semigroup J; see **1.1**), i.e. $m + p = n + q$. Let a be any element of $A(J)$. Consider the sequence $\mathbf{e} = (a, \ldots, a)$ (with $m + p = n + q$ elements a). The elements

$$\mathbf{a} = (a_1,\ a_2, \ldots, a_p) = (a^{m+1}, \underbrace{a, \ldots, a}_{p-1}),$$

$$\mathbf{b} = (b_1,\ b_2, \ldots, b_q) = (a^{n+1}, \underbrace{a, \ldots, a}_{q-1})$$

are strongly linked (in A) since $a_1 = [a^{m+1}],\ a_2 = [a], \ldots, a_p = [a]$ and $b_1 = [a^{n+1}],\ b_2 = [a], \ldots, b_q = [a]$, which means that $A_p \cap A_q \neq \emptyset$. This proves (2.4).

Since $d \mid q - p$ for any $p,\ q \leqslant d$ and $p \neq q$, it follows that the first d members A_i in (2.3) are pairwise disjonit. □

The relationship between the universal semigroup for a J-associative A and the universal semigroup for the J-associative A_1 (i.e. A^l) is considered in the next theorem.

2.3. Theorem. *Let A be a J-associative, $A^{\frown}$ the universal semigroup for A and $A_1^{\frown}$ the universal semigroup for A_1. Then $A_1^{\frown} \cong A^{\frown}$.*

Proof. Let $\lambda: A \to A^\frown$ be the universal semigroup homomorphism $\lambda_1: A \to A_1$ its restriction on A_1, $\varepsilon: A_1 \to A^\frown$ the inclusion monomorphism and let $\xi: A_1 \to S$ be any semigroup homomorphism. We shall show that there exists a (unique) homomorphism $\varphi: A^\frown \to S$ such that $\varphi\varepsilon = \xi$, i.e. that $\varepsilon: A_1 \to A^\frown$ is the universal semigroup homomorphism for A_1. Putting $\eta(x) = \xi\lambda_1(x)$ $(= \xi(a^l))$, we obtain a mapping $\eta: A \to S$ which is a semigroup homomorphism. Since λ is the universal one for A, there exists a homomorphism $\varphi: A^\frown \to S$, such that $\varphi\lambda = \eta$. Therefore

$$\varphi\varepsilon(x^l) = \varphi\varepsilon\lambda_1(x) = \varphi\lambda(x) = \eta(x) = \xi(x^l)$$

for any $x^l \in A_1$, i.e. $\varphi\varepsilon = \xi$. If $\psi: A^\frown \to S$ is a homomorphism with the property $\psi\varepsilon = \xi$, then

$$\psi(x^l) = \psi(\varepsilon(x^l)) = (\psi\varepsilon)(x^l) = \xi(x^l) = (\varphi\varepsilon)(x^l) = \varphi(x^l),$$

which means that $\psi = \varphi$. Thus $\varepsilon: A_1 \to A^\frown$ is the universal semigroup homomorphism for A_1, i.e. $A^\frown$ is a universal semigroup for A_1. Since the universal semigroup for an associative is unique (up to isomorphism), it follows that $A_1^\frown \cong A^\frown$. □

3. SURJECTIVE ASSOCIATIVES

A J-associative A is called *surjective* iff every element of A can be represented as a non-trivial J-product in A, i.e.

$$(\forall a \in A)(\exists n \in J)(\exists a_0, a_1, \ldots, a_n \in A)\ a = [a_0 a_1 \ldots a_n]. \tag{3.1}$$

In short, $A(J)$ is surjective iff

$$A = \cup \{A^{n+1} \mid n \in J\}. \tag{3.2}$$

3.1. Theorem. *A J-associative A is surjective iff*

$$(\forall n \in J)\ A^{n+1} = A. \tag{3.3}$$

Proof. The part "if" is obvious. Now let $A(J)$ be surjective. Since every subsemigroup of $\mathbf{N}(+)$ is finitely generated ([6]), it follows that $J = \langle n_1, \ldots, n_k \rangle$ and so it suffices to show that $A^{n_i+1} = A$ for every $i = 1, 2, \ldots, n$. Note first that by (3.2)

$$A = A^{n_1+1} \cup \ldots \cup A^{n_k+1}. \tag{3.4}$$

Let n be any fixed element of $\{n_1, \ldots, n_k\}$ and let $a \in A$. By (3.4) $a \in A^{n_i+1}$ for some $i \in \{1, \ldots, k\}$, i.e. a can be represented as a product

$$a = [a_0 a_1 \ldots a_{n_i}].$$

The element a_0 can be represented in the same manner: $a_0 = [a_{00} a_{01} \ldots a_{0n_j}]$ and so

$$a \in A^{n_i + n_j + 1}.$$

Continuing this procedure one obtains

$$a \in A^{\upsilon_1 n_1 + \upsilon_2 n_2 + \ldots + \upsilon_k n_k + 1}$$

where $\upsilon_1, \ldots, \upsilon_k \geqslant 0$ and $\upsilon_1 + \ldots + \upsilon_k > 0$.

Note that the integer $\upsilon_1 + \ldots + \upsilon_k$ increases for 1 in any further step. Hence some of the integers $\upsilon_1, \ldots, \upsilon_k$ will be just the chosen element n (of $\{n_1, \ldots, n_k\}$) after finitely many steps. It can be assumed that $\upsilon_1 = n$ and so

$$a \in A^{nn_1 + \upsilon_2 n_2 + \ldots + \upsilon_k n_k + 1} =$$

$$= A^n A^{n(n_1 - 1) + \upsilon_2 n_2 + \ldots + \upsilon_k n_k + 1} \subseteq A^n A = A^{n+1},$$

i.e. $A \subseteq A^{n+1}$. Since $A^{n+1} \subseteq A$ too, it follows that $A^{n+1} = A$ for any $n \in \{n_1, \ldots, n_k\}$. □

Some properties of surjective associatives will be stated here. It is clear that:

3.2. Proposition. *Any homomorphic image of a surjective J-associative is a surjective J-associative.* □

3.3. Proposition. *The direct product of any non-empty family of surjective J-associatives is a surjective J-associative.*

Proof. Let $\{A_i \mid i \in I\}$ be any (non-empty) family of surjective J-associatives and let P be its direct product. If (a_i) is any element of P, then every component a_i can be represented as a J-product in A_i, i.e.

$$(\exists\, n \in J)\ a_i = [a_{i0} a_{i1} \ldots a_{in_i}]$$

for some $a_{i\upsilon} \in A_i$. By Theorem **3.1,** $n_i = n \in J$ can be taken for all $i \in I$, and so

$$(a_i) = ([a_{i0}\, a_{i1}) \ldots a_{in}]) = [(a_{i0})\,(a_{i1}) \ldots (a_{in})]$$

which means that the J-associative P is surjective. □

The next proposition is a direct generalisation of a result for semigroups ([8], p. 67, Ex. 3).

3.4. Proposition. *A nontrivial J-associative A is surjective iff no nontrivial homomorphic image of A is a zero J-associative.*

Proof. Let A be a nontrivial surjective J-associative and let $A' = \varphi(A)$ be a nontrivial homomorphic image of A (i.e. A' has at least two distinct elements). If A' were a zero associative, then

$$(\exists o' \in A')\,(\forall n \in J)(\forall x'_0, \ldots, x'_n \in A')\ [x'_0 \ldots x'_n] = o'.$$

Let $a' \neq o'$. Since A is surjective, it follows that for some $n \in J$ and $a_0, \ldots, a_n \in A$ would be

$$a' = \varphi(a) = \varphi([a_0 \ldots a_n]) = [\varphi(a_0) \ldots \varphi(a_n)] = o',$$

which contradicts the assumption $a' \neq o'$. Therefore A' is not a zero associative.

Conversely, let no nontrivial homomorphic image of a nontrivial J-associative A be a zero J-associative. Put $A^* = \cup\{A^{n+1} \mid n \in J\}$. If A were not surjective, then we would have $A^* \neq A$, i.e. $A \setminus A^* = R \neq \emptyset$. Define J-operations $[\,]_1$ on the set $A = R \cup \{o\}$, where $o \notin A$, by

$$[x_0 x_1 \ldots x_n]_1 = o$$

for all $n \in J$ and $x_0, \ldots, x_n \in B$, then B becomes a nontrivial J-associative. Define a mapping $\varphi : A \rightarrow B$ by

$$\varphi(x) = o \text{ if } x \in A^*, \ \varphi(x) = x \text{ if } x \in R.$$

If $x_0, \ldots, x_n \in A \, (n \in J)$, then $z = [x_0 \ldots x_n] \in A^*$ and

$$\varphi([x_0 \ldots x_n]) = \varphi(z) = o = [\varphi(x_0) \ldots \varphi(x_n)]_1,$$

i.e. φ is a homomorphism and $B(J)$ is a homomorphic image of $A(J)$. Thus, if $A^* \neq A$, then there exists a nontrivial homomorphic image of A which is a zero J-associative.

Hence, $A(J)$ is surjective. □

Let A be a J-associative and K a subsemigroup of $\mathbf{N}(+)$ which contains the subsemigroup J. If it is possible to define a $(k+1)$-operation on A for any $k \in K \setminus J$ in a such way that A, together with the operations of the given J-associative, becomes a K-associative, then $A(J)$ is said to be *K-reducible* ([1]).

3.5. Theorem. *If $A(J)$ is surjective, then it is reducible to a d-semigroup $A(d)$, where $d = G.C.D.(J)$.*

Proof. Define a $(d+1)$-operation on A by

$$[x_0 x_1 \ldots x_d]' = [x_0 x_1 \ldots x_{d-1} y_0 y_1 \ldots y_r], \tag{3.5}$$

where $x_d = [y_0 \ldots y_r]$ and r is any (ex. the smallest) element of J (Lemma **1.1**). (Instead of x_d, any x_i $(i = 0, 1, \ldots, d-1)$ can be substituted in (3.5) by some r-product.) If $x_d = [z_0 z_1 \ldots z_r]$ too, then putting $x_0 = [u_0 \ldots u_r]$ one obtains

$$\begin{aligned}&[x_0 \ldots x_{d-1} y_0 \ldots y_r] = [u_0 \ldots u_r x_1 \ldots x_{d-1} y_0 \ldots y_r] = \\ &= [u_0 \ldots u_r x_1 \ldots x_{d-1} x_d] = [u_0 \ldots u_r x_1 \ldots x_{d-1} z_0 \ldots z_r] = \\ &= [x_0 x_1 \ldots x_{d-1} z_0 \ldots z_r],\end{aligned}$$

which means that the $(d+1)$-operation on A with (3.5) is well-defined. It is easy to show that this $(d+1)$-operation is associative, i.e.

$$[[x_1 \dots x_d y]' x_{d+1} \dots x_{2d}]' = [x_1 [x_2 \dots x_d y x_{d+1}]' x_{d+2} \dots x_{2d}]' = \\ = \dots = [x_1 \dots x_d [y x_{d+1} \dots x_{2d}]']'.$$

(Namely, using the surjectivity of $A(J)$, we can represent y and x_{2d} by some r-products in $A(J)$, say $[y_0 \dots y_r]$ and $[u_0 \dots u_r]$ respectively, and replacing them in the above expressions, we shall obtain some products in $A(J)$, where the "brackets can be removed".)

We note that any $(sd+1)$-operation on A $(s \in \mathbf{N})$ which is an extension of the $(d+1)$-operation can be represented by

$$[x_0 x_1 \dots x_{sd}]' = [y_0 \dots y_r x_1 \dots x_{sd}], \tag{3.6}$$

where $x_0 = [y_0 \dots y_r]$.

Let $[x_0 \dots x_n]$ be any product in $A(J)$ and let $x_0 = [y_0 \dots y_r]$. Since $n = kd$, by (3.6) it follows that

$$[x_0 x_1 \dots x_n] = [y_0 \dots y_r x_1 \dots x_{kd}] = [x_0 x_1 \dots x_{kd}]',$$

which means that any J-operation on A is some extension of the $(d+1)$-operation.

Therefore $A(J)$ is K-reducible, where $K = \langle d \rangle \supseteq J$, i.e. it is reducible to a d-semigroup. □

3.6. Theorem. *If a J-asscoiative A is surjective, then A is a semigroup associative.*

Proof. Let $A(J)$ be surjective and let it be reduced to a d-semigroup $A(d)$. By **3.5**, $A(J)$ is a J-subassociative of $A(d)$ which can be considered as a K-associative $A(K)$, $K = \langle d \rangle$. If a, $b \in A$ are linked in $A(J)$, i.e. $a^l = b^l$, then they are linked in $A(K)$ too, i.e. $a^{l'} = b^{l'}$, where l' is the relation of linking in $A(K)$. Since $A(K)$ is a semigroup associative, it follows by **2.1** that $a = b$. Thus a, $b \in A$ and $a^l = b^l$ implies $a = b$ and so $A(J)$ is a semigroup associative. □

(We note that **3.5** is proved in [1] with an additional assumption — that A is a semigroup associative, but this assumption can be omitted as **3.6** shows. Also, in contrast with [1], we define the $(d+1)$-operation „in one step". On the other hand, using the mentioned result of [1], the $(d+1)$-operation $[\,]'$ can be found immediately.)

As a consequence of **2.2** and **2.3** we obtain the following.

3.6. Proposition. *If $A(J)$ is a surjective J-associative, $d = G.C.D.(J)$ and $A^\frown$ is the universal semigroup for $A(J)$, then:*

a) $A^\frown = A_1 \cup \ldots \cup A_d$, *where* $A_i \cap A_j = \emptyset$ *for* $i, j \leqslant d, i \neq j$ *and* $A_i = \{(a_1, \ldots, a_i) \mid a_\nu \in A\}$.

b) A_1 *is a d-subsemigroup of the semigroup* $A^\frown$ *and* $A^\frown$ *is the free covering of* $A_1(d)$.

c) $A(J)^\frown \cong A(d)^\frown$.

Proof. a) Let $(a_1, \ldots, a_{d+1})^l \in A_{d+1}$. Since

$$(a_1, \ldots, a_{d+1})^l = (a_1)^l \ldots (a_{d+1})^l = [a_1 \ldots a_{d+1}]^l \in A_1,$$

it follows that $A_{d+1} \subseteq A_1$ and moreover $A_{d+i} \subseteq A_i$ $(1 \leqslant i \leqslant d)$. Hence a) follows by **2.2.**

b) Let $a^l{}_0, \ldots, a^l{}_d \in A_1$. If $a_0 = [b_0 \ldots b_r]$ and $[b_0 \ldots b_r a_1 \ldots a_d] = c$, where r is as in **1.1,** then

$$a^l{}_0 \ldots a^l{}_d = [b_0 \ldots b_r]^l a_1 \ldots a^l{}_d \ldots = b^l{}_0 \ldots b^l{}_r a^l{}_1 \ldots a^l{}_d = c^l \in A_1,$$

i.e. A_1 is a d-subsemigroup of $A^\frown$. By **2.3,** $A(J)^\frown \cong A_1(J)^\frown$ and this implies that $A^\frown$ is the free covering for $A_1(d)$ too.

c) Since $A(J)^\frown$ is the free covering of $A(J)$, it follows by b) that $A(J)^\frown \cong A(d)^\frown$.

Note that the epimorphism $\lambda_1 : A \to A_1$ (see 3° in section 2) in this case is an isomorphism, i.e. for any surjective associative $A(J)$, $A \cong A_1$. □

Note that the assumption of surjectivity in a) can not be omitted. For example, the J-associative $A = \{a, b, c\}$ with $J = \{2,3\}$ and $[x_0 x_1 x_2 x_3] = a$ if some $x_i \neq c$, $[cccc] = b$ and $[x_0 x_1 x_2] = a$ is not surjective. It is clear that $d = G.C.D.(J) = 1$, $A^\frown = A_1 \cup A_2$ and $A_1 \neq A^\frown$ (since c is not a J-product and $(c, c) \in U_A$ is not linked with any element of U_A which is not equal to (c, c), and so $(c, c)^l \notin A_1$) Since (a) and (a, c) of U_A are linked via the sequence $\mathbf{e} = (a, a, a, c)$ (namely, $a = [aaac]$ and $a = [aaa]$, $c = [c]$ which means that $(a)\,l\,(a, c)$), it follows that $A_1 \cap A_2 \neq \emptyset$.

4. SOME PROPERTIES OF SEMIGROUP ASSOCIATIVES

The following "local theorem for semigroup associatives" holds.

4.1. Proposition. *If every finitely generated J-subassociative of a J-associative A is a semigroup associative, then A is a semigroup associative.*

Proof. If A were not a semigroup associative, then by **2.1** there exist at least two distinct elements a, $b \in A$, such that $(a)\,l\,(b)$, i.e. $(a)\,sl\,\mathbf{c}_1\,sl \ldots sl\,\mathbf{c}_r\,sl\,(b)$ for some $\mathbf{c}_i \in U_A$, $\mathbf{c}_i = (c_{i1}, \ldots, c_{ik_i})$. By (2.1) in the definition of sl every $c_{i\nu_i}$ is a product of elements of A and the set E

of all these elements is finite. The J-subassociative B which is generated by the set $E \cup \{a, b\}$ is finitely generated and is not a semigroup associative. This proves the proposition. □

4.2. Proposition. *If $\mathcal{B} = \{B_i \mid i \in I\}$ is a chain of semigroup J-subassociatives of a J-associative A, then its union is a semigroup J-subassociative too*.

Proof. Clearly, $M = \cup\{B_i \mid i \in I\}$ is a J-subassociative of A. Suppose that M is not a semigroup associative. Then by **2.1** there exist at least two distinct elements $b_1, b_2 \in M$ which are linked in M, $(b_1)\, l_M\, (b_2)$, i.e. $(b_1)\, sl_M\, \mathbf{c}_1\, sl_M \ldots sl_M\, \mathbf{c}_r\, sl_M\, (b_2)$ for some $\mathbf{c}_1, \ldots, \mathbf{c}_r \in U_A$, $\mathbf{c}_j = (c_{j1}, \ldots, c_{jk_j})$, which components $c_{j\nu_j}$ are in M. Since $\mathcal{B}$ is a chain, there exists a member B_s of $\mathcal{B}$ such that all the elements $c_{j\nu_j}$, b_1, b_2 belong to B_s. Thus (b_1) and (b_2) are linked in B_s which (by **2.1** and $b_1 \neq b_2$) is a contradiction.

Therefore M is a semigroup associative. □

4.3. Corollary. *Every semigroup J-subassociative of a J-associative A is contained in a maximal J-subassociative of A (where the set of all J-subassociatives of A is ordered by inclusion).* □

Let A be the free J-associative which is generated by some set X (i.e. A consists of all J-sequences in X, where the J-operations are all admissible concatenations of J-sequences). This means that any element of A can be uniquely represented as a sequence $(x_0, x_1, \ldots, x_n)$, where $n \in J$ and $x_0, \ldots, x_n \in X$. Hence, A is a J-subassociative of the free semigroup U_X which is freely generated by the set X. Moreover, $A^{\frown} = U_X$. Thus:

4.4. Proposition. *If A is a free J-associative, then A is a semigroup associative.* □

4.5. Corollary. *Every J-associative is a homomorphic image of a semigroup J-associative.* □

5. A SYSTEM OF AXIOMS FOR SEMIGROUP ASSOCIATIVES

As we deduced in the first section, the identities of the form

$$\Pi' x_0 x_1 \ldots x_n = \Pi'' x_0 x_1 \ldots x_n, \tag{5.1}$$

where Π' and Π'' are continued products, hold in any J-associative A. If A is a J-associative which is freely generated by two elements, then it is clear that no other identities besides (5.1) hold in A. Thus no other identities besides (5.1) hold in the class $\mathcal{D}$ of semigroup J-associatives. It is easy to show that $\mathcal{D}$, for a fixed J, is hereditary and admits direct products. Since there exist associatives which are not semigroup associatives ([1], p. 11), it follows by **4.5** that $\mathcal{D}$ does not admit homomorphic images (unless $J=\langle n\rangle$, for some $n\in\mathbf{N}$). Thus, by the Birkhof theorem ([7], p. 337), if $J\neq\langle n\rangle$, $\mathcal{D}$ is not a variety.

However, $\mathcal{D}$ is a quasivariety ([7], Corollary 5 on p. 274). This implies that the conditions for embedding of an associative into a semigroup can be expressed by quasiidentities i.e. by formulae of the form

$$(\forall x_1,\dots,x_k)\,(f_1=g_1\,\&\dots\&\,f_s=g_s\Rightarrow f_{s+1}=g_{s+1}),\tag{5.1}$$

where f_i, g_i are terms with a signature F of the variables $x_1, x_2,\dots,x_k$.

We are going to state a system of axioms for the semigroup associatives, i.e. to find effectively those quasiidentities, using the theorem **2.1.** The definition of the relation l (section 2) will be repeated in detail.

Let A be a semigroup J-associative and let $(x)\,l_A\,(y)$, where $x, y\in A$. Then there exist elements $\mathbf{u}_1,\dots,\mathbf{u}_s$ of $U=U_A$ such that

$$(x)\,sl_A\,\mathbf{u}_1\,sl_A\dots sl_A\,\mathbf{u}_s\,sl_A\,(y).$$

Let $\mathbf{u}_i=(u_{i1},\dots,u_{ik_i})$, $i=1,\dots,s$. Then $(x)\,sl_A\,\mathbf{u}_1$ implies that there exist integers n', $n_{11},\dots,n_{1k_1}$ of J and an element $\mathbf{a}_0=(a_{00},a_{01},\dots,a_{on'})$ of U $(a_{oi}\in A)$, such that

$$n'+1=n_{11}+\dots+n_{1k_1}+k_1,\ x=[a_{00}\,a_{01}\dots a_{on'}]$$
$$u_{11}=[a_{00}\dots a_{on_{11}}],\ u_{12}=[a_{on_{11}+1}\dots a_{on_{11}+n_{12}+1}],$$
$$\dots,u_{1k_1}=[\dots a_{on'}].$$

Further, $\mathbf{u}_1\,sl_A\,\mathbf{u}_2$ means that there exist integers $n'_{11},\dots,n'_{1k_1}$, $n_{21},\dots,n_{2k_2}$ of J and a sequence $\mathbf{a}_1=(a_{10},a_{11},\dots,a_{1t_1})$ of U such that

$$n'_{11}+\dots+n'_{1k_1}+k_1=n_{21}+\dots+n_{2k_2}+k_2=t_1+1,$$
$$u_{11}=a_{10}\dots a_{1n'_{11}},\dots,u_{1k_1}=\dots a_{1t_1},$$
$$u_{21}=a_{10}\dots a_{1n_{21}},\dots,u_{2k_2}=\dots a_{1t_1}.\ \text{etc.}$$

Continuing in that way, $\mathbf{u}_s\,sl_A\,(y)$ implies that there exist integers n'_{s1}, $n'_{s2},\dots,n'_{sk_s}\in J$, $n\in J$ and $\mathbf{a}_s=(a_{so},\ a_{s1},\dots,a_{sn})\in U$, such that

$$n'_{s1}+\dots+n'_{sk_s}+k_s=n+1,$$
$$u_{s1}=[a_{so}a_{s1}\dots a_{sn'_{s1}}],\dots,u_{sk_s}=[\dots a_{sn}],$$
$$y=[a_{so}\,a_{s1}\dots a_{sn}].$$

Since the restriction of l_A on A is the equality on A, it follows by **2.1** that $x=y$.

The above procedure can be effected by a scheme, using only the indices. Namely, let n', $n'_{i\upsilon_i}$, $n'_{i\upsilon_i}$, $n\,(i=1,2,\dots,s;\ \ \upsilon_i=1,2,\dots,k_i)$ be integers of J such that:

$$
\begin{aligned}
&n' + 1 = n_{11} + \ldots + n_{1k_1} + k_1 \\
&n'_{11} + \ldots + n'_{1k_1} + k_1 = n_{21} + \ldots + n_{2k_2} + k_2 \; (= t_1 + 1), \\
&n'_{21} + \ldots + n'_{2k_2} + k_2 = n_{31} + \ldots + n_{3k_3} + k_3 \; (= t_2 + 1), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&n'_{s1} + \ldots + n'_{sk_s} + k_s = n + 1
\end{aligned}
\tag{5.2}
$$

The integers $n_{i\upsilon_i}$, $n'_{i\upsilon_i}$, taken pairwise in a convenient way, can be arranged in the following scheme:

$(0, n')$	(n_{11}, n'_{11})	(n_{21}, n'_{21})	$\ldots$	(n_{s1}, n'_{s1})	$(n, 0)$
	(n_{12}, n'_{12})	(n_{22}, n'_{22})	$\ldots$	(n_{s2}, n_{s2})	
	$\vdots$				
	(n_{1k_1}, n'_{1k_1})	(n_{2k_2}, n'_{2k_2})	$\ldots$	(n_{sk_s}, n'_{sk_s})	

(5.3)

Every scheme of this form which is constructed by integers of J and satisfies the conditions (5.2) will be called a *J-configuration*. The following system of equalities (further on referred to as equalities (SE)) can be assigned to the *J*-configuration (5.3):

$$
\begin{aligned}
x = {} & [x_{00}x_{01}\ldots x_{on'}], \\
& [x_{00}x_{01}\ldots x_{on_{11}}] = [x_{10}x_{11}\ldots x_{1n'_{11}}], \\
& [x_{0n_{11}+1}\ldots x_{0n_{11}+n_{12}+1}] = [x_{1n'_{11}+1}\ldots x_{1n'_{11}+n'_{12}+1}], \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& [\ldots x_{0n'}] = [\ldots x_{1t_1}] \;\; (t_1 + 1 = n'_{11} + \ldots + n'_{1k_1} + k_1); \\
& [x_{10}x_{11}\ldots x_{1n_{21}}] = [x_{20}x_{21}\ldots x_{2n'_{21}}], \\
& [x_{1n_{21}+1}\ldots x_{1n_{21}+n_{22}+1}] = [x_{2n'_{21}+1}\ldots x_{2n'_{21}+n'_{22}+1}], \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& [\ldots x_{1t_1}] = [\ldots x_{2t_2}] \;\; (t_2 + 1 = n'_{21} + \ldots + n'_{2k_2} + k_2); \\
& \vdots \\
& [x_{s-1,0}\, x_{s-11}\ldots x_{s-1,n_{s1}}][= [x_{s0}x_{s1}\ldots x_{sn'_{s1}}]
\end{aligned}
$$

$$[x_{s-1,\, n_{s-1}+1} \dots x_{s-1,\, n_{s2}+1}] = [x_{sn'_{s1}+1} \dots x_{sn'_{s1}+n'_{s2}+1}],$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$[\dots x_{s-1,\, t_{s-1}}] = [\dots x_{sn}] \quad (n+1 = n'_{s1} + \dots + n'_{sk_s} + k_s);$$
$$[x_{s0} x_{s1} \dots x_{sn}] = y.$$

Thus, any system of equalities of the form (SE) which is obtained by some J-configuration, gives an axiom:

$$(SE) \Rightarrow x = y,$$

which we call a J-configuration theorem. By theorem **2.1** it follows that:

5.1. Theorem. *A J-associative A is a semigroup associative iff every configuration theorem holds in A.* □

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РЕЗИМЕ

ЗА ПОЛУГРУПНИТЕ АСОЦИЈАТИВИ

Наум ЦЕЛАКОСКИ

Една алгебра $A = A(F)$, со носител A и фамилија F од финитарни операции меѓу кои нема нуларни и унарни, а има барем една n-арна ($n \geqslant 2$), се вика *асоцијатив* ([1]) или, попрецизно, *F-асоцијатив* акко[1] во A важи „општиот асоцијативен закон". Наместо фамилијата F се разгледува множеството

$$J = \{n-1 \mid n \in \mathbf{N},\ F_n \neq \varnothing\},$$

каде што F_n е множеството n-арни операции од F, па во таа смисла се вели *J-асоцијатив* наместо F-асоцијатив. Притоа, наместо J-асоцијативот A може да се разгледува $\langle J\rangle$-асоцијативот A, каде што $\langle J\rangle$ е потполугрупата од $\mathbf{N}(+)$, генерирана

[1] „акко" стои наместо „ако и само ако".

од J. Еден J-асоцијатив се вика *полугрупен асоцијатив* акко тој може да се смести во некоја полугрупа S, т.е. акко A е J-подасоцијатив од некоја полугрупа S. (Да забележиме дека секоја полугрупа може да се смета за J-асоцијатив за кое било непразно подмножество J од $\mathbf{N}$.)

Нека $U = U_A$ е полугрупата што е слободно генерирана од носачот A на J-асоцијативот A, т. е. $U=\{(a_1, \dots, a_k) | k \in \mathbf{N},\ a_\upsilon \in A\}$, а операцијата е обично надоврзување на низи. Нека $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{b} = (b_1, \dots, b_q)$, $a_i,\ b_j \in A$. Дефинираме релација sl во U со: $\mathbf{a}\, sl\, \mathbf{b}$ акко постојат $\mathbf{c},\ \mathbf{c}'_1, \dots, \mathbf{c}'_p,\ \mathbf{c}''_1, \dots, \mathbf{c}''_q \in U$ и сложени производи $\Pi'_i \mathbf{c}_i$, $\Pi''_j \mathbf{c}_j$ во A, такви што

$$\mathbf{c} = \mathbf{c}'_1 \dots \mathbf{c}'_p = \mathbf{c}''_1 \dots \mathbf{c}''_q,\ a_i = \Pi'_i\ \mathbf{c}'_i,\ b_j = \Pi''_j\ \mathbf{c}_j''.$$

Се покажува дека транзитивното проширување l на релацијата sl е конгруенција на U, $A^l = \{a^l \mid a \in A\}$ е генераторно множество за фактор-полугрупата U/l, каноничното пресликување $\lambda : a \to a^l$ е хомоморфизам од J-асоцијативот A во полугрупата U/l и го има универзалното својство, т.е. за кој било хомоморфизам ξ од J-асоцијативот A во некоја полугрупа S постои единствен хомоморфизам $\varphi: U/l \to S$, таков што $\xi = \varphi\lambda$. Поради ова својство, фактор-полугрупата U/l се вика *универзална полугрупа* за асоцијативот A и, бидејќи таа е единствената (до изоморфизам), се означува со $A^\frown$ или со $A(J)^\frown$. Во случај кога J-асоцијативот е полугрупен, $A^\frown$ се вика *слободна покривка* на A.

Нека A е J-асоцијатив и $d =$ нзд (J). Се покажува дека: на универзалната полугрупа $A^\frown$ може да ѝ се даде следнава форма:

$$A^\frown = A_1 \cup A_2 \cup \dots \cup A_n,$$

каде што $n \in J$ и $A_k = \{(a_1, \dots, a_k)^l \mid a_\upsilon \in A\}$; ако $p,\ q \leqslant n$, тогаш $(A_p \cap A_q \neq \neq \emptyset \Leftrightarrow d \mid q - p)$; ако A_1 е универзалната полугрупа за J-асоцијативот $A_1 (= A^l)$ тогаш $A_1^\frown \cong A^\frown$.

Еден J-асоцијатив A се вика *сурјективен асоцијатив* акко секој елемент од A може да се претстави како нетривијален τ-производ во A, т. е. акко $A = \cup \{A^{n+1} \mid n \in J\}$. Се покажува дека $A(J)$ е сурјективен акко, за секој $n \in J$, $A^{n+1} = A$. Класата сурјективни J-асоцијативи допушта хомоморфни слики и директни производи, но не е наследсвена. Ако $A\ (J)$ е сурјективен, тогаш на A може да се изгради d-полугрупа, каде што $d =$ нзд (J) и секоја J-операција претставува проширување на дефинираната $(d + 1)$-операција; притоа: $A^\frown = A_1 \cup \dots \cup A_d$, каде што $A_i \cap A_j = \emptyset$ за $i,\ j \leqslant d,\ i \neq j$ и A_1 е d-потполугрупа од полугрупата $A^\frown$, којашто е слободна покривка на d-полугрупата A_1. Тука се докажува и главниот резултат за сурјективните асоцијативи: секој сурјективен асоцијатив е полугрупен асоцијатив.

Натаму се изнесуваат неколку резултати за полугрупните асоцијативи. Имено, се докажува дека: ако секој конечногенериран J-подасоцијатив од еден J-асоцијатив A е полугрупен асоцијатив, тогаш и A е полугрупен; секој полугрупен J-подасоцијатив од еден J-асоцијатив A се содржи во некој максимален полугрупен J-подасоцијатив на A; секој слободен асоцијатив е полугрупен; секој J-асоцијатив е хомоморфна слика на полугрупен J-асоцијатив.

На крајот се добива подесен систем аксиоми за полугрупните асоцијативи; имено, се покажува дека класата на полугрупните J-асоцијативи е квазивариетет, при што се добива и системот квазиидентитети со кој е дефиниран тој квазивариетет.

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