ON SEMILATTICE DECOMPOSITIONS OF TERNARY SEMIGROUPS

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The notion of a ternary semilattice is introduced, as wel. as that of a semilattice congruence in a ternary semigroup. It is then shown that almost all known properties relating semilattice decompositions of binary semigroups generalise to the ternary case.

1. *Semilattice congruences.* Let S be a ternary semigroup, i.e. an algebra S with an associative ternary operation

$$(x, y, z) \to xyz.$$

S is called a ternary semilattice if S is commutative. idempotent and satisfies

$$x^2 y = xy^2.$$

A congruence α on a ternary semigroup S is called a semilattice congruence if S/α is a ternary semilattice.

A subset I of S is called a completely simple ideal if

$$xyz \in I \Leftrightarrow x \in I \text{ or } y \in I \text{ or } z \in I.$$

$F (\subseteq S)$ is called a filtre in S if $I = S \setminus F$ is a completely simple ideal.

We shall first show that the semilattice congruences can be characterised by the class of completely simple ideals.

1.1. Let $\sum$ be the set of completely simple ideals in S. Then the relation defined by

$$x \alpha y \Leftrightarrow (\forall I \in \textstyle\sum) (x, y, \in I \text{ or } x, y \notin I)$$

is a congruence on S.

Proof. Clearly, α is an equivalence. Since the elements of $\sum$ are completely simple ideals, one obtains easily that α is a congruence on S, so it remians to show that α is a semilatice congruence.

Let $I \in \Sigma$ and $x, y, z \in S$. Since I is a completely simple ideal, we have:

$$x^3 \in I \Leftrightarrow x \in I; \quad xyz \in I \Leftrightarrow yxz \in I \Leftrightarrow xzy \in I; \quad x^2y \in I \Leftrightarrow xy^2 \in I.$$

It follows that

$$x^3 \,\alpha\, x,\ xyz \,\alpha\, yxz \ \ \alpha\, xzy,\ x^2y \,\alpha\, xy^2$$

i.e. α is a semilattice congruence. ■

Denote the congruence α of 1.1 by α_Σ.

We shall now show that the converse of 1.1. is also true, i.e. that:

1.2. If α is a semilattice congruence, then there is a family Σ of completely simple ideals in S such that $\alpha = \alpha_\Sigma$.

Proof. Let α be a semilattice congruence on S and let associate to each element $x \in S$ the subset F_x of S defined by

$$F_x = \{y \in S \mid x \,\alpha\, x^2y\}.$$

We shall show that F_x is a nonepty filtre in S. Firstly, it is clear that $x \in F_x$. Then, if $u, v, w \in F_x$, we have

$$x \,\alpha\, x^2\, w \,\alpha\, x\,(x^2u)\, w,\ x^2(x^2u)\, vw \,\alpha\, x^2uvw$$

from which it follows that $uvw \in F_x$.

Conversely, let $uvw \in F_x$. Then we have

$$x \,\alpha\, x^2uvw \,\alpha\, x^2uvw^3 = x^2uvww^2 \,\alpha\, xw^2,$$

i.e. $w \in F_x$. We obtain similarly that $u, v \in F_x$. Thus, we proved that F_x is a filtre. Put $I_x = S \setminus F_x$ and let $\Sigma_\alpha = \{I_x \mid x \in S\}$. So Σ_α is a set of completely simple ideals in S. We shall show that $\alpha = \alpha_{\Sigma_\alpha}$.

Let $y \,\alpha\, z$, $I_x \in \Sigma_\alpha$ and $y \notin I_x$. Therefore $y \in F_x$, i.e. $x \,\alpha\, x^2y$, from which, since $x^2y \,\alpha\, x^2z$, we get $z \in F_x$, i.e. $z \notin I_x$. We have thus shown that $\alpha \subseteq \alpha_{\Sigma_\alpha}$ Conversely, let $x \,\alpha_{\Sigma_\alpha}\, y$; then by $x \in F_x$, we have $y \in F_x$, i.e. $x \,\alpha\, x^2y$. For the same reason. $y \in F_y$ implies $x \in F_y$, i.e. $y \,\alpha\, y^2x$. But $x^2y \,\alpha\, y^2x$, so $x \,\alpha\, y$.

Therefore $\alpha = \alpha_{\Sigma_\alpha}$ ■

Let us note that:

1.3. If $S \notin \Sigma_1 S \notin \Sigma_2$, then $\alpha_{\Sigma_1} = \alpha_{\Sigma_2} \Leftrightarrow \Sigma_1 = \Sigma_2$. ■

2. *Least semilittice congruence.*

Clearly, the intersection n of all semilittice congruences is a semilittice congruence. So:

2.1. $x\, n\, y$ if and only if for every completely simple ideal I in S we have $x,\ y \in I$ or $x,\ y \notin I$.

We shall give another description of n.

Let us first denote by $N(x)$ the minimal filtre in S in which x is contained, i.e. $N(x)$ is the filtre generated by x.

A direct consequence of. 2.1. and the definition of $N(x)$ is:

2.2. $x\, n\, y \Leftrightarrow N(x) = N(y)$. ■

The classes of the congruence n are called n-c l a s s e s. If $x \in S$, then the n-class which contains x is denoteed by N_x. Therefore we have:

2.3. i) $N_{xyz} = N_{yxz} = N_{xzy}$; ii) $N_x 3 = \mathrm{N}_x$;

iii) $N_{xy^2} = N_{yx^2}$; iv) is a subsemigroup of S.

As in the binary case, we say that S is n-s i m p l e if S has no proper completely simple ideals.

Properties analogues to those in the binary case are vilid in ternary semigroups. These, in the author's opinion, justiyfy the difinition of the notion of a ternary semigroup. Governed by the fact that the proofs of these properties are similar (as was the case with the previous ones) to those in the binary case, we shall only formulate some of them.

The folloving theorem gives a constructive way of obtaining $N(x)$ which has an inductive nature.

2.4. Let x be any element in S. Let $N_1(x) = \{x^{2k+1} \mid k = 0, 1, 2, \ldots\}$ and let $N_{n+1}(x)$ be the ternary semigroup generated by all elements y in S such that $N_n(x) \cap J(y) \neq \emptyset$, where $J(y) = y \cup S^2 y \cup SyS \cup yS^2 \cup S^2yS^2$, Then

$$N_x = \bigcup_{n=1}^{\infty} N_x(x). \ ■$$

2.5. If I is an ideal of some n-class of a ternary semigroup S, then I has no proper completely simple ideals.

Proof. Let S be a ternary semigroup, $z \in S$ and I an ideal of N_z. It is enough to show that I is the only filtre of I. Let F be a filtre of I, a any element of F and let

$$T = \{x \in S \mid a^3x^2 \in F\}.$$

We shall show that T is a filtre of S. Let $u, v,\ w \in T$. By the inclusion $F \subseteq I \subseteq N_z$ we have

$$N_{a^3u^2} = N_{au^2} = N_{a^2u} = N_z.$$

Since F is a filtre, we easily conclude that $a^2uvw, uvwa^2 \in F$. But then

$$a[a^2(uvw)]\,[(uvw)a^2] = [a^3(uvw)^2]aa \in F$$

so $a^3(uvw)^2 \in F$. Therefore uvw $\in T$.

Conversely, let $uvw \in T$; this implies that $a^2(uvw)^3 \in F$. From

$$N_{a^3(uvw)^2} = N_{a(uvw)^2} = N_{a^2(uvw)} = N_z$$

we get $a^2(uvw) \in N_z$. Since

$$a[a^2(uvw)]\,[(uvw)a^2] = [a^3(uvw)^2]aa \in F$$

it follows that $(uvw)a^2 \in F$. From there we easily get that $u, v, w \in T$ We have thus proved that T is a filtre.

It is clear that $F \subseteq T \cap I$. Let $x \in T \cap I$. Then $a^3x^2 \in F$. Since F is a filtre, it follows that $x \in F$. But from $a \in N_z \cap T$ we have $N_z \subseteq T$. So $T \cap I = I$ and finally $F = I$.

2.6. If I is a completely simple ideal in S and if $I \cap N_x = \emptyset$, then $I \cap N_x$ is completely simple. ∎

As a conceqence of 2.6. we get:

2.7. Every completely simple ideal of a ternary semigroup S is a union of n — classes. ∎

If Y_s denotes the set of all n — classes of a ternary semigroup S we have.

2.8. If I is a completely simple ideal of a ternary semigroup S, then $J = \{N_x \in Y_s \mid x \in I\}$ is a completely simple ideal in Y_s. Conversely, of J is a completely simple ideal in Y_s, then $\mathbf{I} = \{x \in S \mid N_x \in J\}$ is a completely simple ideal in S. ∎

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ПОЛУМРЕЖНИ ДЕКОМПОЗИЦИИ НА ТЕРНАРНИТЕ ПОЛУГРУПИ

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Резиме

Нека S е тернарна полугрупа, т.е. алгебра со една тернарна асоцијацтивна операција $(x, y, z) \rightarrow xyz$. За S велиме дека е тернарна полумрежа ако S е идемпотентна и комутативна тернарна полугрупа, во која важи идентитетот $x^2y = xy^2$. За едно непразно подмножество I од S велиме дека е комплетно прост идеал на S ако

$$x, y, z \in S, \quad xyz \in I \Rightarrow x \in I, \quad y \in I, \quad z \in I$$

Една тернарна потполугрупа F од S се вика филтер, ако $S \setminus F$ е комплетно прост идеал на S или $S \setminus F = \varnothing$.

Со **1.1**. и **1.2**. е дадена карактеризација на тернарните полумрежни декомпозиции од S со помош на комплетно простите идеали од S.

За $x \in S$, нека $N(x)$ е најмалиот филтер што го содржи x и нека

$$N_x = \{y \in S \mid N(x) = N(y)\}$$

Ако со Y_s го означиме множеството од сите различни множества N_x и ако на Y_s дефинираме операција со: $N_x N_y N_z = N_{xyz}$, тогаш Y_s станува тернарна полумрежа (**2.3**). Натаму, со 2.4. е даден еден конструктивен начин за добивање на $N(x)$. Во **2.5**. покажваме дека нема идеал на N_x којшто содржи вистински комплетно прост идеал. На крајот, со **2.8** е покажано дека постои обратно еднозначна кореспонденција меѓу парцијално подреденото множество од сите комплетно прости идеали од S и парцијално подреденото множество од сите комплетно прсти идеали од Y_s.