

## ON GROUPOIDS WITH THE IDENTITY $x^2y^2 = xy$

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**Abstract:** The variety  $\mathcal{U}$  of groupoids which satisfy the identity  $x^2y^2 = xy$  is considered.

**Key words:** free groupoid, injective groupoid, word problem

### 0. Introduction

The main object of the paper, which consists of four sections, are free groupoids in the variety  $\mathcal{U}$ .

A description of free groupoids in  $\mathcal{U}$  is given in the first section. In fact, first we show that the reduction  $x^2y^2 \rightarrow xy$  is not convenient for obtaining corresponding canonical free groupoids in  $\mathcal{U}$ , and afterwards we show that the pair of reductions:  $xy^2 \rightarrow xy$ ,  $x^2y \rightarrow xy$  has the desired properties.<sup>1</sup>

The class of  $\mathcal{U}$ -injective groupoids (which is a proper subclass of  $\mathcal{U}$ , and properly contains the class of  $\mathcal{U}$ -free<sup>2</sup> groupoids) is considered in Section 2. Some characterizations of  $\mathcal{U}$ -free groupoids within the class of  $\mathcal{U}$ -injective groupoids are also given. It is shown that neither of the classes of  $\mathcal{U}$ -injective groupoids,  $\mathcal{U}$ -free groupoids is hereditary. Namely, a subgroupoid  $Q$  of a  $\mathcal{U}$ -injective ( $\mathcal{U}$ -free) groupoid  $H$  is  $\mathcal{U}$ -injective ( $\mathcal{U}$ -free) if each idempotent in  $Q$  is a square of a nonidempotent in  $Q$ .

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<sup>1</sup> Here we use the following usual abbreviations:  $x^2y^2 = (xx)(yy)$ ,  $xy^2 = x(yy)$ ,  $x^2y = (xx)y$ .

<sup>2</sup> „ $\mathcal{U}$ -free groupoid“ means „groupoid free in  $\mathcal{U}$ “.

Some characterizations of  $\mathcal{U}$ -free subgroupoids of a  $\mathcal{U}$ -free groupoid are given in Section 3. It is also shown that a  $\mathcal{U}$ -free groupoid with rank 2 admits subgroupoids with infinite rank.

In Section 4, as a corollary of Evans' results ([3] and [4]) we obtain that the word problem is solvable in  $\mathcal{U}$ .

We note that the results of this paper have corresponding analogies in the variety of commutative  $\mathcal{U}$ -groupoids, but we do not translate them explicitly.

### 1. Canonical $\mathcal{U}$ -groupoids

Throughout the paper we denote by  $F = (F, \cdot)$  a free groupoid (in the variety of all groupoids) with a given basis  $B$ . It is well-known (for example [1; I.1]) that the following two properties characterise  $F$ .

a)  $ab = cd \Rightarrow a = c, b = d$ .

(Any groupoid with this property is said to be *injective*.)

b)  $B$  is the set of primes in  $F$  and it generates  $F$ .

(An element  $c \in G$  is *prime* in a groupoid  $G = (G, \cdot)$  iff  $c \neq xy$ , for all  $x, y \in G$ .)

The *norm* in  $F$  is the homomorphism  $x \mapsto |x|$  from  $F$  into the additive groupoid  $(\mathbb{N}, +)$  of positive integers which extends the mapping  $B \rightarrow \{1\}$ . Thus:

$$|b| = 1, \quad |uv| = |u| + |v|,$$

for  $b \in B, u, v \in F$ .

Let  $\mathcal{V}$  be a variety of groupoids such that there exists a groupoid  $R = (R, *)$  with the following properties:

c)  $B \subseteq R \subseteq F$  &  $(\forall t, u \in F) \{tu \in R \Rightarrow t, u \in R \text{ \& } t * u = tu\}$ .

d)  $R$  is free in  $\mathcal{V}$  with a basis  $B$ .

Then we say that  $R$  is a  $\mathcal{V}$ -canonical groupoid.

Below we are looking for a  $\mathcal{U}$ -canonical groupoid with a given basis  $B$ .

The identity

$$x^2 y^2 = xy \tag{1.1}$$

(which is the axiom of  $\mathcal{U}$ ) suggests to consider the following groupoid  $S = (S, \bullet)$  as a candidate for a  $\mathcal{U}$ -canonical groupoid with the basis  $B$ . As the carrier of  $S$ ,

choose the subset  $S$  of all the elements of  $F$  without subwords of the form  $\alpha^2 \beta^2$ , and define the operation  $\bullet$  on  $S$  in the following way:

$$(\forall t, u \in S) \quad t \bullet u = \begin{cases} tu & \text{if } tu \in S \\ \alpha\beta & \text{if } t = \alpha^2, u = \beta^2 \end{cases}$$

However, although  $S$  is a well-defined groupoid (with the least generating subset  $B$ ), which satisfies the condition c),  $S$  does not belong to  $\mathcal{U}$ . (Namely, if  $b \in B$ , then  $b, b^2 \in S$ , but  $(b^2 \bullet b^2) \bullet (b \bullet b) = (b \bullet b) \bullet (b \bullet b) = b^2 \neq b^2 b = b^2 \bullet b$ .)

The following property of  $\mathcal{U}$  will provide a system of axioms of  $\mathcal{U}$  which will enable us to obtain a  $\mathcal{U}$ -canonical groupoid.

**1.1.** A groupoid  $G = (G, \cdot)$  belongs to  $\mathcal{U}$  iff it satisfies the following identities:

$$x^2 y = x y, \quad x y^2 = x y. \quad (1.2)$$

**Proof.** If  $G$  satisfies (1.2), then for any  $a, b \in G$  :  $a b = a b^2 = a^2 b^2$ , i.e.  $G \in \mathcal{U}$ . Conversely, if  $G \in \mathcal{U}$ , then:

$$a^2 b^2 = a^2 (b b) = a^2 (b^2 b^2) = a^2 (b^2)^2 = a b^2.$$

Similarly:  $a^2 b^2 = a^2 b$ .  $\square$

**1.2.** If  $G = (G, \cdot) \in \mathcal{U}$  and  $a \in G$ , then:

$$(a^2)^2 = a^2 = a a^2 = a^2 a.$$

(Therefore, for each  $a \in G$ ,  $a^2$  is an idempotent.)  $\square$

**Remark.** Two varieties of groupoids,  $\mathcal{U}_r$  and  $\mathcal{U}_l$ , defined by the axioms  $x^2 y = x y$  and  $x y^2 = x y$  respectively, are investigated in [2]. By 1.1, it follows that  $\mathcal{U} = \mathcal{U}_r \cap \mathcal{U}_l$ . As a corollary, one obtains that, for any  $k, m, s \in \mathbb{N}$ , where  $k \geq 2$ , the following identities are true in  $\mathcal{U}$ :

$$x^k = x^2, \quad x^m y = x y^s = x y \quad (1.3)$$

Here, for a given  $n \in \mathbb{N}$ ,  $x^n$  denotes any of  $\frac{(2n-2)!}{n!(n-1)!}$  possible  $n$ -th powers.

Now denote by  $R$  the set of all elements  $u \in F$  such that  $u$  has not a subword of the form  $\alpha \beta^2$  neither of the form  $\alpha^2 \beta$ .

Using the transformation  $t \mapsto \underline{t}$  in  $F$ , defined as follows:

$$\underline{t} = \begin{cases} t & \text{if } (\forall \alpha \in F) t \neq \alpha^2 \\ \alpha & \text{if } t = \alpha^2 \end{cases}$$

we obtain the following characterization of  $R$  :

**1.3.**  $R$  is the least subset of  $F$  such that  $B \subseteq R$  and satisfies the following statement:

$$(\forall t, u \in F) \{ tu \in R \Leftrightarrow [t, u \in R \ \& \ (\underline{t} = t, \underline{u} = u)] \}. \quad \square$$

The following implications are clear:

$$t \in R \Rightarrow u = \underline{t} \in R \ \& \ \underline{u} = u \quad (1.5)$$

$$t, u \in R \Rightarrow \underline{t}\underline{u} \in R \quad (1.6)$$

Now we will show the following

**Theorem 1.4.** If the operation  $*$  on  $R$  is defined by

$$(\forall t, u \in R) t * u = \underline{t}\underline{u}, \quad (1.7)$$

then  $R = (R, *)$  is a  $\mathcal{U}$ -canonical groupoid with the basis  $B$ .

**Proof.** 1) From (1.5) and (1.6) it follows that  $*$  is a well-defined operation on  $R$ , i.e. that  $R$  is a groupoid, and it is clear that  $B$  is the least generating set of  $R$ .

2) By (1.4), 1.3 and (1.7), the condition c) is satisfied.

3) Let  $t, u \in R$  and  $v = \underline{t}$ ,  $w = \underline{u}$ . Then:

$$(t * t) * (u * u) = (\underline{t})^2 * (\underline{u})^2 = v^2 * w^2 = v * w = t * u, \text{ i.e. } R \in \mathcal{U}.$$

4) Let  $G = (G, \cdot) \in \mathcal{U}$ ,  $\lambda : B \rightarrow G$  be any mapping and  $\varphi : F \rightarrow G$  be the homomorphism from  $F$  into  $G$  which is an extension of  $\lambda$ . If  $t, u \in R$ , then:

$$\varphi(t * u) = \varphi(\underline{t}\underline{u}) = \varphi(\underline{t}) \varphi(\underline{u}) = \varphi(\underline{t})^2 \varphi(\underline{u})^2 = \varphi(tu).$$

Therefore, the restriction  $\psi$  of  $\varphi$  on  $R$  is a homomorphism from  $R$  into  $G$ , and  $\psi$  extends  $\lambda$ .  $\square$

Below we assume that  $H = (H, \cdot)$  is a  $\mathcal{U}$ -free groupoid with a given basis  $B$ , and therefore it is isomorphic to the above obtained  $\mathcal{U}$ -canonical groupoid, which implies that the following statements hold.

**1.5.** The basis of  $H$  coincides with the set of primes in  $H$ .  $\square$

1.6. If  $a \in HH$ , then there exists a unique pair  $(b, c) \in H^2$ , such that  $a = bc$ , and neither of  $b, c$  is idempotent. Then,  $a$  is idempotent iff  $b = c$ .  $\square$

1.7. There is a mapping  $a \mapsto |a|$  from  $H$  into  $|N$  such that:  $|bc| \leq |b| + |c|$ , where the equality holds iff neither of  $b, c$  is idempotent.  $\square$

1.8. If  $B = \{b\}$  is a singleton, then  $H = \{e, b\}$ , where  $e \neq b$ , and  $xy = b$  in  $H$ , for any  $x, y \in H$ .  $\square$

1.9. If  $B$  contains at least two distinct elements, then  $H$  is infinite.

**Proof.** Assume that  $B = \{a, b\}$ , where  $a \neq b$ . Consider the  $\mathcal{U}$ -canonical groupoid  $R$  with basis  $B$ , and define the subset  $C = \{c_k \mid k \in N\}$ :

$$c_1 = ab, \quad c_{k+1} = c_k b. \quad (1.8)$$

Then  $B \subset R$  and  $c_m = c_n \Leftrightarrow m = n$ .  $\square$

## 2. $\mathcal{U}$ -injective groupoids

A  $\mathcal{U}$ -groupoid  $H = (H, \cdot)$  is said to be  $\mathcal{U}$ -injective iff it satisfies the following condition:

(inj) For every element  $a \in HH$ , there is a unique pair of nonidempotents  $b, c \in H$  (i.e.  $b^2 \neq b, c^2 \neq c$ ), such that  $a = bc$ . In that case,  $b = c$  iff  $a$  is an idempotent. (Then we say that  $(b, c)$  is the pair of divisors of  $a$  in  $H$ , and write  $b \mid a, c \mid a$ .)

From 1.3 and 1.6 we obtain:

2.1. If  $H = (H, \cdot)$  is a  $\mathcal{U}$ -free groupoid, then  $H$  is  $\mathcal{U}$ -injective and there is a mapping  $a \mapsto |a|$  from  $H$  into  $|N$  such that:

$(b, c)$  is the pair of divisors of  $a$  implies that  $|a| = |b| + |c|$ .  $\square$

Below we assume that  $H = (H, \cdot)$  is a given  $\mathcal{U}$ -injective groupoid, and  $N, I$  are the sets of nonidempotents, idempotents in  $H$ , respectively. Also the fact that  $(a^2)^2 = a^2$ , for all  $a \in H$  (see 1.2), implies the following property which gives a convenient description of  $\mathcal{U}$ -injective groupoids.

2.2. If  $H = (H, \cdot)$  is a groupoid, then the following two conditions are equivalent.

(i)  $H$  is  $\mathcal{U}$ -injective.

(ii) There exist two nonempty disjoint subsets  $N, I$  of  $H$ , a bijection  $\varphi: N \rightarrow I$ , and an injection  $\psi: N^2 \setminus \{(a, a) \mid a \in N\} \rightarrow N$ , such that  $H = N \cup I$  and:

$$\begin{aligned}\varphi(a)\varphi(a) &= a\varphi(a) = \varphi(a)a = aa = \varphi(a), \\ \varphi(a)\varphi(b) &= a\varphi(b) = \varphi(a)b = ab = \varphi(a,b),\end{aligned}\quad (2.1)$$

for any  $a, b \in N$ , and  $a \neq b$ .

**Proof.** If  $H$  is  $\mathcal{U}$ -injective, and  $I, N$  are the sets of idempotents, non-idempotents in  $H$ , respectively, then the mapping  $\varphi: a \mapsto a^2$  is a bijection from  $N$  onto  $I$ , and  $\psi: (a, b) \mapsto ab$  is an injection from  $N^2 \setminus \{(a, a) \mid a \in N\}$  into  $N$  such that the equations (2.1) hold.

Conversely, if  $H = (H, \cdot)$  satisfies the conditions stated in (ii), then  $I, N$  are the sets of idempotents, nonidempotents in  $H$ , respectively. From (2.1), it follows that  $H \in \mathcal{U}$ . Let  $a \in HH$ ; if  $a \in I$  and  $b = \varphi^{-1}(a)$ , then  $(b, b)$  is the pair of divisors of  $a$  in  $H$ ; and if  $a \in N$ , there exists a unique pair  $(b, c) \in N^2$  and  $a = bc = \psi(b, c)$ , and thus  $(b, c)$  is the pair of divisors of  $a$  in  $H$ .  $\square$

(Below we denote by  $(N, I; \varphi, \psi)$  the  $\mathcal{U}$ -injective groupoid defined in 2.2 (ii). Namely, from 2.2 it follows that  $N, I, \varphi$  and  $\psi$  are uniquely determined by the given  $\mathcal{U}$ -injective groupoid  $H$ .)

As a corollary from 2.2 it follows that, if  $H$  is a finite  $\mathcal{U}$ -groupoid, and  $|N| = n$ , then  $n^2 - n \leq n$ , i.e.  $1 \leq n \leq 2$ . Therefore, we obtain the following description of finite  $\mathcal{U}$ -injective groupoids.

**2.3.** A  $\mathcal{U}$ -injective groupoid  $H$  is finite iff it is isomorphic to one of the groupoids defined by the following multiplication tables:

$\cdot$	$i$	$a$
$i$	$i$	$i$
$a$	$i$	$i$

$\cdot$	$i$	$j$	$a$	$b$
$i$	$i$	$b$	$i$	$b$
$j$	$a$	$j$	$a$	$j$
$a$	$i$	$b$	$i$	$b$
$b$	$a$	$j$	$a$	$j$

As a corollary of 1.6 and 2.3 we obtain:

**2.4.** The class of  $\mathcal{U}$ -free groupoids is a proper subclass of the class of  $\mathcal{U}$ -injective groupoids.  $\square$

The following statement is a description of  $\mathcal{U}$ -free groupoids within the class of  $\mathcal{U}$ -injective groupoids.

**Theorem 2.5.** If  $H = (H, \cdot)$  is a  $\mathcal{U}$ -injective groupoid, then the following conditions are equivalent.

- (i)  $H$  is  $\mathcal{U}$ -free.  
 (ii) There is a mapping  $a \mapsto |a|$  from  $H$  into  $\mathbb{N}$ , such that:  
 if  $(b, c)$  is the pair of divisors of  $a$ , then  $|a| = |b| + |c|$ .  
 (iii) The set of primes in  $H$  generates  $H$ .

**Proof.** By 1.6 and 1.7, (i)  $\Rightarrow$  (ii).

Assuming that (ii) holds, by induction, it can be easily shown that  $B$  generates  $H$ , i.e. we obtain: (ii)  $\Rightarrow$  (iii).

Finally, assume that the set  $B$  of primes in  $H$  generates  $H$ .  $B$  is non-empty, for if  $a \in H$ , then  $\{a^2\}$  is a proper subgroupoid of  $H$ . Let  $R = (R, *)$  be the  $\mathcal{U}$ -canonical groupoid with the basis  $B$ , and let  $\varphi: R \rightarrow H$  be the homomorphism from  $R$  into  $H$  which extends the embedding  $B \rightarrow H$ . Then  $\varphi$  is surjective, for  $B$  generates the both groupoids  $R$  and  $H$ . Thus we have to show that  $\varphi$  is injective as well, and this can be easily shown by induction using the mapping  $u \mapsto |u|$ .  $\square$

The fact that, for any  $a \in H$ , where  $H = (H, \cdot)$  is a  $\mathcal{U}$ -injective groupoid, the set  $\{a^2\}$  is a subgroupoid (of  $H$ ) which is not  $\mathcal{U}$ -injective, by 2.4, implies the following proposition:

**2.6.** Neither of the classes of  $\mathcal{U}$ -injective,  $\mathcal{U}$ -free groupoids is hereditary.  $\square$

In the next section we will use the following description of the  $\mathcal{U}$ -injective subgroupoids, which is a corollary from the proof of 2.2.

**2.7.** Let  $H = (N, I; \varphi, \psi)$  be a  $\mathcal{U}$ -injective groupoid and  $Q = (Q, \cdot)$  a subgroupoid of  $H$ .  $Q$  is  $\mathcal{U}$ -injective iff  $a \in Q \cap I \Rightarrow \varphi^{-1}(a) \in Q$ .  $\square$

### 3. Subgroupoids of $\mathcal{U}$ -free groupoids

Below we assume that  $H$  is a  $\mathcal{U}$ -free groupoid with the basis  $B$ , and  $Q$  a subgroupoid of  $H$ . Therefore (by 2.5 and 2.1),  $H$  is  $\mathcal{U}$ -injective, and there exists a mapping  $a \mapsto |a|$  from  $H$  into  $\mathbb{N}$ , such that: if  $(b, c)$  is the pair of divisors of  $a \in HH$ , then  $|a| = |b| + |c|$ . Thus, for any  $a \in Q$ ,  $|a| \in \mathbb{N}$ , but  $Q$  is not necessarily  $\mathcal{U}$ -injective, as we saw in 2.6. We will divide the set of idempotents in  $Q$  into two parts: regular and singular. Namely, an idempotent  $a \in Q$  is said to be *regular* in  $Q$  iff the divisor  $b$  belongs to  $Q$ ; in the contrary,  $a$  is said to be *singular* in  $Q$ . The set of singular idempotents in  $Q$  will be denoted by  $S$ , and the set of primes in  $Q$  by  $P$ .

We will prove the following lemma:

**3.1.**  $P \cup S$  is the least generating set for  $Q$ .

**Proof.** If  $T \subseteq Q$ , then by  $\langle T \rangle$  will be denoted the subgroupoid of  $Q$  generated by  $T$ . The following relations are clear:

$$p \in P \Rightarrow p \notin \langle Q \setminus \{p\} \rangle \quad (3.1)$$

$$a \in S \Rightarrow a \notin \langle Q \setminus \{a\} \rangle \quad (3.2)$$

From (3.1) and (3.2) it follows that  $P \cup S$  is a subset of every generating set of  $Q$ . So, it remains to show that  $P \cup S$  generates  $Q$ .

Let  $\alpha \in Q$ . We will show that  $\alpha \in \langle P \cup S \rangle$  by induction on  $|\alpha|$ . First, if  $|\alpha| = \min \{ |a| \mid a \in Q \}$ , then  $\alpha \in P \cup S$ . Assume that

$$\alpha \in Q \ \& \ |\alpha| \leq k \Rightarrow \alpha \in \langle P \cup S \rangle.$$

Let  $\beta \in Q$  be such that  $|\beta| = k+1$ ,  $\beta \notin P \cup S$ . Then there exist  $\gamma, \delta \in Q$  such that  $\beta = \gamma\delta$ . If  $\gamma$  and  $\delta$  are nonidempotents, then  $|\gamma| + |\delta| = |\beta|$ , and so  $\gamma, \delta \in \langle P \cup S \rangle$ , which implies that  $\beta \in \langle P \cup S \rangle$ . The same conclusion arises also in the case  $\gamma, \delta \in S$ . It is clear that:  $\gamma, \delta \in S \Rightarrow \beta = \gamma\delta \in \langle P \cup S \rangle$ . Therefore we can assume that, for example,  $\gamma \in Q \setminus S$ ,  $\delta \in S$ . Then we may consider that  $\gamma$  is a nonidempotent, and this implies  $|\gamma| < k$ , i.e.  $\gamma \in Q$ .  $\square$

As a corollary of the above lemma (and also 2.7), we obtain the following characterization of  $\mathcal{U}$ -free subgroupoids of a  $\mathcal{U}$ -free groupoid.

**Theorem 3.2.** Let  $H$  be a  $\mathcal{U}$ -free groupoid. If  $Q$  is a subgroupoid of  $H$ , then the following conditions are equivalent:

- a)  $Q$  is  $\mathcal{U}$ -injective.
- a') Every idempotent in  $Q$  is regular.
- b)  $Q$  is  $\mathcal{U}$ -free.
- c) The set  $P$  of primes in  $Q$  generates  $Q$ .

**Proof.** a)  $\Leftrightarrow$  a'), by 2.7 and the definition of regular idempotents.

Let  $Q$  be  $\mathcal{U}$ -injective. For any  $a \in QQ$ , there exists a uniquely determined pair  $(b, c)$  of divisors of  $a$  in  $Q$ . Then,  $(b, c)$  is the pair of divisors of  $a$  in  $H$ , and  $|b| + |c| = |a|$ . Therefore, by Th.2.5,  $Q$  is  $\mathcal{U}$ -free. Thus: a)  $\Rightarrow$  b). Also b)  $\Rightarrow$  c) by Th.2.5. Finally, from c) and 3.1, we obtain that  $S = \emptyset$ , i.e. a') holds.  $\square$

The situation with the ranks of  $\mathcal{U}$ -free subgroupoids of  $\mathcal{U}$ -free groupoids is similar as for the varieties  $\mathcal{U}_r$  and  $\mathcal{U}_1$ . Namely, if  $H$  is a  $\mathcal{U}$ -free groupoid with



rank 1, then  $H$  has only two elements, and  $H$  is the unique  $\mathcal{U}$ -free subgroupoid of  $H$ .  $\mathcal{U}$ -free groupoids with rank greater than 1, according to 1.9, have the following property:

**3.3.** *If  $H$  is a  $\mathcal{U}$ -free groupoid with rank 2, then there is a  $\mathcal{U}$ -free subgroupoid  $Q$  of  $H$  with infinite rank.  $\square$*

#### 4. Word problem for $\mathcal{U}$

As a corollary of an Evans' result in [3], below we will obtain the following

**Theorem 4.1.** *The word problem for the variety  $\mathcal{U}$  is solvable.*

First recall ([1; I.1]) that  $G = (G, \cdot)$  is a *multigroupoid* if  $(a, b) \mapsto a \cdot b$  is a mapping from  $G^2$  into the collection of the subsets (including the empty one) of  $G$ . If, for any  $a, b \in G$ ,  $a \cdot b$  contains at most one element, then  $G$  is called a *halfgroupoid*. Thus, a *groupoid* may be considered as a special kind of a halfgroupoid which satisfies the condition:

$$(\forall a, b \in G) a \cdot b \neq \emptyset,$$

where the equation  $c = \{c\}$  is assumed. A halfgroupoid  $G = (G, \circ)$  is called a *subhalfgroupoid* of a groupoid  $H = (H, \cdot)$  iff  $G \subseteq H$  and:

$$(\forall a, b \in G) [a \cdot b \neq \emptyset \Rightarrow a \circ b = a \cdot b].$$

Finally, we say that a halfgroupoid  $G$  is a  *$\mathcal{U}$ -halfgroupoid* iff the following condition holds:

$$(\forall a, b, c \in G) [a^2 \neq \emptyset \Rightarrow ab = a^2b \ \& \ ca = ca^2]. \quad (4.1)$$

A special case of the Evans' Theorem in [3; p. 68] is the following proposition.

**4.2.** *If each  $\mathcal{U}$ -halfgroupoid is a subhalfgroupoid of a  $\mathcal{U}$ -groupoid, then the word problem is solvable for the variety  $\mathcal{U}$ .  $\square$*

Thus, Th.4.1 is a corollary of 4.2 and the following:

**Lemma 4.3.** *Let  $(G, \cdot)$  be a  $\mathcal{U}$ -halfgroupoid and  $e$  a fixed element in  $G$ . Define a groupoid  $(G, \circ)$  as follows:*

$$x \cdot y \neq \emptyset \Rightarrow x \circ y = x \cdot y, \quad (4.2)$$

$$x^2 = \emptyset \Rightarrow x \circ x = x, \quad (4.3)$$

$$x \neq y \ \& \ x \cdot y = \emptyset \Rightarrow x \circ y = e. \quad (4.4)$$

Then,  $(G, \circ)$  is a  $\mathcal{U}$ -groupoid and  $(G, \cdot)$  is its  $\mathcal{U}$ -subhalfgroupoid.

**Proof.** Let  $a, b \in G$ . If  $a^2 = \emptyset$ , then by (4.3):

$$(a \circ a) \circ b = a \circ b.$$

If  $a^2 \neq \emptyset$ , then by (4.1) and (4.2):

$$(a \circ a) \circ b = a^2 \circ b = a^2 b = ab = a \circ b, \text{ in the case } a \cdot b \neq \emptyset, \text{ and}$$

$$(a \circ a) \circ b = e = a \circ b, \text{ if } ab = \emptyset.$$

Similarly, one shows that  $b \circ (a \circ a) = b \circ a$ .

Thus,  $(G, \circ)$  is a  $\mathcal{U}$ -groupoid.  $\square$

The notion of a  $\mathcal{U}$ -halfgroupoid depends essentially on the set of axioms of  $\mathcal{U}$  (which follows also by the Evans' general note in [3; p. 66]). Namely, the definition of a  $\mathcal{U}$ -halfgroupoid given by (4.1) assumes the axioms (1.2). If one chooses (1.1) as the axiom of  $\mathcal{U}$ , then a halfgroupoid  $G = (G, \cdot)$  would be called a  $(\mathcal{U}, 1)$ -halfgroupoid if:

$$(\forall a, b \in G) [ a^2 \neq \emptyset, b^2 \neq \emptyset \Rightarrow a^2 b^2 = ab ]. \quad (4.1')$$

There are  $(\mathcal{U}, 1)$ -halfgroupoids which are not subhalfgroupoids of  $\mathcal{U}$ -groupoids. Such a subhalfgroupoid is defined by the following multiplication table:

·	a	b	c
a	b	b	a
b	b	b	b
c	$\emptyset$	$\emptyset$	$\emptyset$

Therefore, L. 4.3 would not be true, and Th.4.1 would not be a corollary of [3].

However, it is easy to show that the problem: whether a given finite  $(\mathcal{U}, 1)$ -halfgroupoid  $G$  is a subhalfgroupoid of a  $\mathcal{U}$ -groupoid is solvable. Therefore, if one chooses (1.1) as the axiom of  $\mathcal{U}$ , Th.4.1 is a corollary of the main Evans' result in [4; p. 76].

Finally, we note that an algorithm for solving the word problem for the varieties of more general kind is formulated explicitly in [3]. This algorithm is

simpler for the variety  $\mathcal{U}$ , because here it is possible to use the canonical  $\mathcal{U}$ -groupoid  $R$  instead of  $F$ .

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## Резиме

ЗА ГРУПОИДИТЕ СО ИДЕНТИТЕТОТ  $x^2y^2 = xy$ 

Предмет на статијава е многуобразието  $\mathcal{U}$  од групоици што го задоволуваат идентитетот  $x^2y^2 = xy$ . Прво се дава опис на  $\mathcal{U}$ -слободни (т.е. слободни во  $\mathcal{U}$ ) групоици (Th. 1.4), а потоа се наведуваат некои нивни карактеризации во рамките на класата  $\mathcal{U}$ -инјективни групоици (Th. 2.5). Притоа се покажува дека ни една од споменатите класи ( $\mathcal{U}$ -слободни,  $\mathcal{U}$ -инјективни групоици) не е наследна (Prop. 2.6). Натаму се дава карактеризација на  $\mathcal{U}$ -слободни подгрупоици од  $\mathcal{U}$ -слободни групоици (Th. 3.2), а на крајот, како последица од главниот резултат на Еванс од [3], се добива дека проблемот на зборови е решлив во  $\mathcal{U}$ .

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