Прилози, Одд. мат. тех. науки. МАНУ, 16, 1-2, с. 11-21 (1995)
Contributions, Sec. Math. Tech. Sci. p. MAHY, 16, 1-2, 11-21 (1995)
ISSN 0351-3246
UDC: 512. 001

A GENERALIZATION OF THE NOTION OF ALGEBRA

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A b s t r a c t: In this paper the notion of φ-algebra is introduced which is a generalization of the notion of algebra. As in [2], there are two kinds of substructures, usual and strong ones, and one kind of homomorphism. As a consequence we obtain two kinds of free objects (free φ-algebras and weakly free φ-algebras) in the class of φ-algebras of a given type. Both kinds of substructure have some of the usual properties of subuniverses of usual algebras but not all of them, and it seems that every one of them has exactly those properties which the other has not. For example, the homomorphic image of substructure is always substructure, but this doesn't hold for strong substructures; the intersection of strong substructures is a strong substructure, which is not always true for the intersection of substructures. A complete desctription of the free objects of both kinds is given here. Weakly free structures have some unusual properties. For example, there are nonisomorphic weakly free structures with the same weak basis. Most of the results are similar to the results in [2], although the objects which are examined are of very different types.

1. φ – Algebras. Subalgebras, Strong Subalgebras

Def. 1.1. Let **B** be an algebra of type Ω, A be a (nonempty) set and φ: A→B be a surjective mapping. The ordered triple $\boldsymbol{A}$ = (A, φ, **B**) is called *φ-algebra of type* Ω. The set A is called the *outer carrier* of φ-algebra $\boldsymbol{A}$.

It should be mentioned that the name φ-algebra does not depend on the name of defined surjective mapping. For example, if C is a set, **D** is an algebra of type Ω and ψ: C → D is a subjective mapping, then the ordered triple **C** = (C, ψ, **D**) is also called φ-algebra of type Ω. Further, φ-algebras (A, φ, **B**) and (C, ψ, **D**) of type Ω will be denoted by $\boldsymbol{A}$ and $\boldsymbol{C}$, respectively. φ-algebras $\boldsymbol{A}$ and **C** of type Ω are considered equal if and only if $\boldsymbol{A} = \boldsymbol{C}$, $\varphi = \psi$ and **B** = **D**.

If the mapping φ is bijective, then there is a unique algebra **A** of type Ω with carrier A, such that **A** ≅ **B** and φ: **A** → **B** is on isomorphic.

If **B** is an algebra of type Ω, then **B** could be considered as φ-algebra, where A = B and $\varphi = 1_B$. Whence, the notion of φ-algebra is a generalization of the notion of algebra. From now on, we do not distinguish algebra **B** from its corresponding φ-algebra $\boldsymbol{B} = (B, 1, \mathbf{B})$.

Def. 1.2. A subset $C \subseteq A$ is *φ-subalgebra* of $\boldsymbol{A}$ if φC is subalgebra[1] of **B**.

C is *strong φ-subalgebra* of $\boldsymbol{A}$ if there is some subalgebra D of **B** such that $C = \varphi^{-1}D$.

The set of all φ-subalgebras of $\boldsymbol{A}$ will be denoted by Sub $(\boldsymbol{A})$ and the set of all strong φ-subalgebras of $\boldsymbol{A}$ will be denoted by $\mathrm{Sub}_s(\boldsymbol{A})$.

Considering the definition of φ-subalgebras and strong φ-subalgebras of φ-algebras and the definition of subalgebras of algebras, we obtain the following characterization of φ-subalgebras and strong φ-subalgebras:

Proposition 1.3. Let C be a nonempty subset of A. Then, the following statements are equivalent:

i) C is a φ-subalgebra of $\boldsymbol{A}$.

ii) $(\forall n \in N)\,(\forall \omega \in \Omega_n)\,(\forall a_1, \ldots, a_n \in C)\ \omega^B \varphi a_1 \ldots \varphi a_n \in \varphi C$.

iii) $(\forall n \in N)\,(\forall \omega \in \Omega_n)\,(\forall a_1, \ldots, a_n \in C)\,(\exists a \in C)\ \omega^B \varphi a_1 \ldots \varphi a_n = \varphi a$.

Also, the following statements are equivalent:

iv) C is a strong φ-subalgebra of A.

v) $(\forall n \in N)(\forall \omega \in \Omega_n)\,(\forall a_1, \ldots, a_n \in C)\ \varphi^{-1}\omega^B \varphi a_1 \ldots \varphi a_n \subseteq C$.

vi) $(\forall n \in N)\,(\forall \omega \in \Omega_n)\,(\forall a_1, \ldots, a_n \in C)\,(\forall a \in A)\ [\omega^B \varphi a_1 \ldots \varphi a_n = \varphi a \Rightarrow a \in C]$. ■

If the mapping φ is bijective, there is no distinction between φ-subalgebras and strong φ-subalgebras. Especially, if A = B and $\varphi = 1_B$, then the subalgebras of **B**, and only they, are φ-subalgebras and strong φ-subalgebras of $\boldsymbol{A}$.

From now on, we will call φ-subalgebras and strong φ-subalgebras just subalgebras and strong subalgebras, keeping in mind that when we talk about subalgebra or strong subalgebra of φ-algebra we are actually tolking about φ-subalgebra or strong φ-subalgebra.

The following proposition clearly explains the motivation for naming strong subalgebras as strong:

Proposition 1.4. Every strong subalgebra of $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{A}$.

Proof. If $C = \varphi^{-1}D$, where D is a subalgebra of **B**, then $\varphi C = \varphi\varphi^{-1}D = D$. Thus, C is a subalgebra of $\boldsymbol{A}$. ■

Example 1.5. Let A = {1, 2} and be the trivial algebra. Then, {1} is a subalgebra of $\boldsymbol{A}$, but it is not a strong subalgebra of $\boldsymbol{A}$. This example shows that the converse of the previous proposition does not hold. ■

There is a natural connection between strong subalgebras of $\boldsymbol{A}$ and subalgebras of **B**. This connection is described by the following:

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[1] Nonempty subset closed for fundamental operations on **B**.

Proposition 1.6. The set $\mathrm{Sub}_s(\mathcal{A})$ of strong subalgebras of $\mathcal{A}$, ordered by $\subseteq$, is a lattice isomorphic with the lattice **Sub(B)** of subalgebras of **B.**

Proof. The mappings φ: $\mathrm{Sub}_s(\mathcal{A}) \to \mathrm{Sub}(\mathbf{B})$ and φ^{-1}: $\mathrm{Sub}(\mathbf{B}) \to \mathrm{Sub}_s(\mathcal{A})$ are monotonic. If $D \in \mathrm{Sub}(B)$, then $\varphi\varphi^{-1}D = D$, so $\varphi\varphi^{-1} = 1$. If $C \in \mathrm{Sub}_s(\mathcal{A})$, then $C=\varphi^{-1}D$ for some subalgebra $D \in \mathrm{Sub}(\mathbf{B})$, so

$$\varphi^{-1}\varphi C = \varphi^{-1}\varphi\varphi^{-1}D = \varphi^{-1}D = C,$$

which means that $\varphi^{-1}\varphi = 1$.

Thus, $\mathbf{Sub}_s(\mathcal{A})$ is a lattice isomorphic with the lattice **Sub(B)**. ■

By the previous proposition, the lattice of strong subalgebras of φ-algebra is an algebraic lattice (since the lattice of subalgebras of an algebra is algebraic) whose compact elements are finitely generated strong subalgebras (the notion of strong subalgebra generated by a set is defined in a natural way as the smallest strong subalgebra which contains that set and it is equal to the intersection of all strong subalgebras that contain that set). Also, it is clear that the intersection of strong subalgebras as well as the union of a chain of strong subalgebras is a strong subalgebra.

Proposition 1.7.

i) The intersection of strong subalgebras is a strong subalgebra.

ii) The union of a chain of strong subalgebras is a strong subalgebra.

iii) The union of a chain of subalgebras is a subalgebra.

Proof. *iii)* Let $\{A_i \mid i \in I\}$ be a chain of subalgebras of $\mathcal{A}$. Then, $\{\varphi A_i\ i \in I\}$ is a chain of subalgebras of **B,** so $\cup\{\varphi A_i \mid i \in I\}$ is a subalgebra of. **B.** But, $\varphi(\cup\{A_i \mid i \in I\}) = \cup\{\varphi A_i \mid i \in I\}$, which shows that $\cup\{A_i\ i \in I\}$ is a subalgebra of $\mathcal{A}$. ■

Example 1.8. It is well known that there are algebras where the union of two subalgebras is not a subalgebra. Those algebras are examples of φ-algebras where the union of two subalgebras (strong subalgebras) is not a subalgebra (strong subalgebra).

Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$ and $\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$. Then, $\{1, 2\}$ and $\{1, 3\}$ are subalgebras of $\mathcal{A}$, regardless of the fundamental operations of **B.** It is easy to define operations on B such that $\{1\}$ is not a subalgebra of the algebra **B.** In such a case we have

$$\varphi(\{1, 2\} \cap \{1, 3\}) = \varphi\{1\} = \{1\},$$

which shows that there are subalgebras whose intersection is not a subalgebra. ■

Example 1.9. Let **A** and **B** be algebras of the same type and φ: $A \to B$ be a surjective homomorphism. Then $\mathcal{A} = (A, \varphi, \mathbf{B})$ is φ-algebra.

If $C \subseteq A$ is a strong subalgebra of $\mathcal{A}$, then $C = \varphi^{-1}D$ for some subalgebra D of **B**, and then C is a subalgebra of the algebra **A** (the inverse homomorphic image of a subalgebra is a subalgebra). Further, if $C \subseteq A$ is a subalgebra of the algebra **A**, then φC is a subalgebra of **B** (the homomorphic image of a subalgebra is a subalgebra), so C is a subalgebra of the φ-algebra $\mathcal{A}$. Whence,

$$\mathrm{Sub}_s(\mathcal{A}) \subseteq \mathrm{Sub}(\mathcal{A}) \subseteq \mathrm{Sub}(\mathcal{A}).$$

Both inclusions may be strict as may be seen in the following example. Let us consider the Example 1.5., again. Let **A** be a monounary algebra (algebra with one unary operation) with operation ω^A defined by $\omega^A(1) = \omega^A(2) = 1$. Then, $\{1\}$ is a subalgebra of the algebra **A** which is not a strong subalgebra of the φ-algebra $\mathcal{A}$, while $\{2\}$ is a subalgebra of the φ-algebra $\mathcal{A}$, which is not a subalgebra of the algebra **A.** ■

Let C be a subalgebra of $\mathcal{A}$, **D** be the corresponding subalgebra of **B** with carrier $D = \varphi C$ and $\varphi' : C \to D$ be the corresponding restriction of φ. Then the triple $\mathcal{C} = (C, \varphi', \mathbf{D})$ is a φ-algebra. Also, if $\mathcal{G} = (C, \xi, \mathbf{H})$ is a φ-algebra such that **H** is a subalgebra of **B** and $\xi : C \to H$ is a restriction of φ, then $\mathbf{G} = \mathbf{C}$.

The same is true for strong subalgebras of $\mathcal{A}$.

2. Generating and Weakly Generating Subsets

Since we have defined two kinds of substructures (subalgebras and strong subalgebras), there are two kinds of generating subsets.

Def. 2.1. Let C be a subset of A. We say that C is a *generating (weakly generating)* subset of $\mathcal{A}$ if $\mathcal{A}$ is the only subalgebra (strong subalgebra) which contains C. In that case we say that $\mathcal{A}$ is *generated (weakly generated)* by C *and that* C *(weakly generates)* the φ-algebra $\mathcal{A}$.

If φ is a bijective mapping, there is no disinction between generating and weakly generating subsets. Especially, if $A = B$ and $\varphi = 1_A$, then the generating subsets of **B** (and only they) are the generating and weakly generating subsets of $\mathcal{A}$.

Since every strong subalgebra is a subalgebra it is obvious that:

Proposition 2.2. Every generating subset of $\mathcal{A}$ is a weakly generating subset. ■

Example 2.3. In order to show that the converse of the previous proposition does not hold, let us consider, again, Example 1.5. Namely, $C = \{1\}$ is a weakly generating subset of $\mathcal{A}$ which is not a generating subset. ■

Def. 2.4. Let C be a subset of A. The smallest strong subalgebra of $\mathcal{A}$ which contains C is called a strong subalgebra of $\mathcal{A}$ *generated* by C and it is denoted by $< C >_w$. We also say that C *weakly generates* $< C >_w$.

Considering Proposition 1.7., $< C >_w$ always exists and it is equal to the intersection of all strong subalgebras of $\boldsymbol{A}$ that contain C. At the same time, the given definition is compatible with the definition of the weakly generating subset. Example 1.8. shows that a similar definition is not possible for a subalgebra generated by a subset, because, for example, {1} is contained in subalgebras {1, 2}, {1, 3} and {1, 2, 3} of $\boldsymbol{A}$, but there is the smallest among them.

As there is a natural connection between the strong subalgebras of $\boldsymbol{A}$ and subalgebras of **B**, there exists a nátural connection between weakly generating subsets of $\boldsymbol{A}$ and generating subsets of **B**. In fact, if $C \subseteq A$ and $D \subseteq B$, then

$$\varphi < C >_w = < \varphi C > ; \quad < \varphi^{-1} D >_w = \varphi^{-1} < D >.$$

According to this, it is easy to see that:

Proposition 2.5. If $C \subseteq A$ is a weakly generating subset of $\boldsymbol{A}$, then φC is a generating subset of **B**. If $D \subseteq B$ is a generating subset of **B**, then $\varphi^{-1} D$ is a weakly generating subset of $\boldsymbol{A}$. ■

Actually, $D \subseteq B$ is a generating subset of **B** if and only if it is an image under φ of some weakly generating subset of $\boldsymbol{A}$. Of course, since every generating subset C of $\boldsymbol{A}$ is a weakly generating subset, φC is a generating subset of **B** for every generating subset C of $\boldsymbol{A}$.

Example 2.6. Let us consider, again, Example 1.8. Let **B** be an algebra such that {1} is a generating subset. Then, $\varphi^{-1}\{1\} = \{1\}$, but {1} is not generating subset of $\boldsymbol{A}$, because it is contained in {1, 2} which subset is a subalgebra of $\boldsymbol{A}$. This example shows that $\varphi^{-1} D$ is not always a generating subset of $\boldsymbol{A}$ when D is a generating subset of **B**.

Considering Example 1.5. we may notice that not every weakly generating subset of $\boldsymbol{A}$ is an inverse image under φ of some generating subset of **B**, and that some generating subset of **B** could be an image under φ of several distinct weakly generating subsets of $\boldsymbol{A}$. Indeed, {1} and {2} are weakly generating subsets of $\boldsymbol{A}$, none of them is an inverse image under φ of some subset of B and its image under φ is {1} which is a generating subset of B. ■

3. *Homomorphisms*

Def. 3.1. Let $\boldsymbol{A} = (A, \varphi, B)$ and $\boldsymbol{C} = (C, \psi, \mathbf{D})$ be two φ-algebras of the same type Ω. A mapping $\alpha : A \to C$ is *homomorphism* from $\boldsymbol{A}$ to **C** if for every $n \in N$, n-ary function symbol $\omega \in \Omega$, $a_1, \ldots, a_n, a \in A$ the following implication holds:

$$\omega^B(\varphi a_1, \ldots, \varphi a_n) = \varphi a \Rightarrow \omega^D(\psi\alpha a_1, \ldots, \psi\alpha a_n) = \psi\alpha a.$$

$\alpha : A \to C$ is *isomorphism* from $\boldsymbol{A}$ to $\boldsymbol{C}$ if α is bijective mapping such that α and α^{-1} are homomorphisms.

If A and C are algebras, then homomorphisms defined in this way are the usual ones.

Also, it is obvious that φ is a surjective homomorphism from A to the φ-algebra (B, 1_B, **B**).

If **D** is the trivial algebra, then every mapping $\alpha : A \to C$ is a homomorphism.

The identity mapping $1 : A \to A$ is an isomorphism from A to A.

If α is an isomorphism from A to C, then α^{-1} is an isomorphism from C to A.

Example 3.2. Let $A = C = D \{1, 2, 3\}$, $B = \{1, 2\}$, $\varphi 1 = \varphi 2 = 1$, $\varphi 3 = 2$, $\alpha = \psi$ be the identity mapping and **B** and **D** be monounary algebras with fundamental operations defined by

$$\omega^B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{i} \quad \omega^D = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Then, α is an isomorphism from A to C, but $\psi\alpha 1 \neq \psi\alpha 2$ although $\varphi 1 = \varphi 2$. Even more, the algebras **B** and **D** are not isomorphic (they have a different number of elements). This example shows that it is possible for two elements of A, which have same image under φ, to have images in C under homomorphism $\alpha : A \to C$ such that these images under α do not have the same images under ψ, and it is possible even if α is an isomorphism. We could say that homomorphism are not necessarily compatible with the given mappings φ i ψ.

But, if the elements a and b of A have the same image under φ which is an image under any fundamental operation of **B**, then the images of a and b (under any homomorphism) have the same image under ψ. Indeed, if $\varphi a = \varphi b = \omega^B(b_1, \ldots, b_n)$, then $\varphi a = \varphi b = \omega^B(\varphi a_1, \ldots, \varphi a_n)$ for some $a_1, \ldots, a_n \in A$. Thus, for every homomorphism $\alpha : A \to C$

$$\psi\alpha a = \psi\alpha b = \omega^B(\psi\alpha a_1, \ldots, \psi\alpha a_n). \blacksquare$$

Proposition 3.3. The composition of homomorphisms (isomorphisms) is a homomorphism (isomorphism).

Proof. Let $\alpha : A \to C$ and $\beta : C \to G$ be homomorphisms from A to C and from C to G, respectively, where $G = (G, \xi, \mathbf{H})$. If $\omega \in \Omega_n$ and $a_1, \ldots, a_n, a \in A$, then,

$$\omega^B(\varphi a_1, \ldots, \varphi a_n) = \varphi a \Rightarrow \omega^D(\psi\alpha a_1, \ldots, \psi\alpha a_n) = \psi\alpha a \Rightarrow$$
$$\Rightarrow (\omega^H(\xi\beta\alpha a_1, \ldots, \xi\beta\alpha a_n) = \xi\beta\alpha a. \blacksquare$$

Now, we will see what happens with subalgebras and strong subalgebras under homomorphisms.

Proposition 3.4. Homomorphic images of subalgebras are subalgebras.

Inverse homomorphic images of strong subalgebras are strong subalgebras.

Proof. Let A' be a subalgebra of ***A***, $\alpha : \boldsymbol{A} \to \boldsymbol{C}$ be homomorphism, $\omega \in \Omega_n$ and $c_1, \ldots, c_n \in \alpha A'$. Then, there are elements $a_1, \ldots, a_n \in A'$ such that $\alpha a_1 = c_1, \ldots, \alpha a_n = c_n$. A' is a subalgebra of ***A***, so there is an element $a \in A'$ such that $\omega^B(\varphi a_1, \ldots, \varphi a_n) = \varphi a$. α is homomorphism, so $\omega^D(\psi\alpha a_1, \ldots, \psi\alpha a_n) = \psi\alpha a$, and then $\omega^D(\psi c_1, \ldots, \psi c_n) = \psi\alpha a$. But, $a \in A'$, from which we conclude that $\alpha a \in \alpha A'$.

Let C' be a strong subalgebra of ***C***, $\alpha : \boldsymbol{A} \to \boldsymbol{C}$ be homomorphism, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in \alpha^{-1}C'$, $a \in A$ and $\omega^B(\varphi a_1, \ldots, \varphi a_n) = \varphi a$. Then

$$\omega^D(\psi\alpha a_1, \ldots, \psi\alpha a_n) = \psi\alpha a.$$

But, $\alpha a_1, \ldots, \alpha a_n \in C'$, so $\alpha a \in C'$ and $a \in \alpha^{-1}C'$. ■

Example 3.5. Let $A = B = C = \{1, 2\}$, $D = \{1\}$, $\varphi = \alpha$ be the identity mapping and **B** be an algebra such that $\{1\}$ is not its subalgebra. Then, α is a homomorphism from ***A*** to ***C*** and $\{1\}$ is a subalgebra of ***C***, but $\alpha^{-1}\{1\}=\{1\}$ is not a subalgebra of ***A***.

Let $A = B = D = \{1\}$, $C = \{1, 2\}$ and $\alpha(1) = 1$. Then, α is a homomorphism from ***A*** to ***C*** and $\{1\}$ is a strong subalgebra of ***A***, but $\alpha\{1\} = \{1\}$ is not a strong subalgebra of ***C***.

These two examples show that the inverse homomorphic image of a subalgebra is not always a subalgebra and that the homomorphic image of a strong subalgebra is not always a strong subalgebra. ■

4. Freee φ-algebras

There are two kinds of subalgebras, so there are two kinds of free φ-algebras.

Def. 4.1. ***A*** is a *free (weakly free)* φ-algebra with the *basis (weak basis)* $A' \subseteq A$ if A' is a generating (weakly generating) subset of ***A*** and for every φ-algebra ***C*** of the same type and every mapping $\alpha' : A \to C$, α' could be extended to homomorphism α from ***A*** to ***C***.

Since every generating subset of ***A*** is a weakly generating subset of ***A***, we obtain the following:

Propostition 4.2. If ***A*** is a free φ-algebra with the basis A', then ***A*** is a weakly free φ-algebra with the weak basis A'. ■

It is natural to ask if every set A' is a (weak) basis of some (weakly) free φ-algebra. First, we will consider the existence of a weakly free φ-algebra with the given weak basis A'.

Def. 4.3. A φ-algebra $\mathcal{A}$ is called a *pumped free algebra* with the basis $A' \subseteq A$ if **B** is a free algebra with the basis $B' = \varphi A'$ and $(\forall a \in A')\ \varphi^{-1}\varphi a = a$.

The condition $\varphi^{-1}\varphi a = a$ should be written as $\varphi^{-1}\varphi\{a\} = \{a\}$, but we will keep the first notation.

Proposition 4.4. If $\mathcal{A}$ is a pumped free algebra with the basis A', then **A** is a weakly free φ-algebra with weak basis A'.

Proof. Since $A' = \varphi^{-1}\varphi A'$, and $\varphi A'$ is a generating subset of **B**, according to Proposition 2.5., A' is weakly generating subset of $\mathcal{A}$.

Let **C** be a φ-algebra of the same type and $\alpha' : A' \to C$ be a mapping. We define a mapping $\beta' : B' \to D$ by $\beta'\varphi a = \psi\alpha' a$, for $a \in A'$. This mapping is well defined, because $\varphi A' = B'$ and $(\forall a \in A')\ \varphi^{-1}\varphi a = a$. But B' is a basis of **B**, so there is a homomorphism $\beta : \mathbf{B} \to \mathbf{D}$ which is an extension of β'. The set $\psi^{-1}\beta\varphi a$ is nonempty, for $a \in A'$. Further, $\alpha' a \in \psi^{-1}\beta\varphi a$, for $a \in A'$. So, there is a mapping $\alpha : A \to C$ such that $\alpha a \in \psi^{-1}\beta\varphi a$ for $a \in A$ and $\alpha a = \alpha' a$ for $a \in A'$. Thus, α is an extension of α' and $\psi\alpha = \beta\varphi$. For every $n \in N$, $\omega \in \Omega_n$, $a_1, \dots, a_n, a \in A$, we have

$$\omega^B(\varphi a_1, \dots, \varphi a_n) = \varphi a \Rightarrow$$
$$\omega^D(\psi\alpha a_1, \dots, \psi\alpha a_n) = \omega^D(\beta\varphi a_1, \dots, \beta\varphi a_n) = \beta\omega^B(\varphi a_1, \dots, \varphi a_n) = \beta\varphi a = \psi\alpha a.$$

So, α is an extension of α' to homomorphism. ■

Proposition 4.5. Let $\mathcal{A}$ be a φ-algebra, $A' \subseteq A$, φ be a bijective mapping and **B** be a free algebra with basis $\varphi A'$. Then **A** is a free φ-algebra with basis A'.

Proof. $\mathcal{A}$ is a pumped free algebra with basis A', so $\mathcal{A}$ is a weakly free φ-algebra with a weak basis A'. It suffices to show that A' is a generating subset of. $\mathcal{A}$. But, φ is a bijective mapping, and since A' is a weakly generating subset it is a generating subset as well. ■

We already have examples of (weakly) free φ-algebras with a given (weak) basis. We will show that if **A** is a weakly free φ-algebra with some weak basis then the basis is unique. Then, we will show that if the type Ω includes at least one function symbol with arity at least one (the type Ω is nontrivial), then there are no other examples of (weakly) free φ-algebras with a given (weak) basis.

Proposition 4.6. Let $\mathcal{A}$ be a weakly free φ-algebra with weak basis A'. Then, $a \in A'$ if and only if φa is not an image (under any fundamental operation) in the algebra **B**.

Proof. Let $a \in A'$, $n \in N$, $\omega \in \Omega_n$, $b_1, \dots, b_n \in B$ and $\omega^B(b_1, \dots, b_n) = \varphi a$. Then, for some elements $a_1, \dots, a_n \in A$ we have $\varphi a_1 = b_1, \dots, \varphi a_n = b_n$. Further, let $\mathcal{C} = (C, 1_c, \mathbf{C})$ be a φ-algebra, where the algebra **C** has at least two elements and the image of any elements of C under any fundamental operation in **C** is some fixed element c ∈ C. Then, since $\omega^B(\varphi a_1, \ldots, \varphi a_n) = \varphi a$, we obtain that

$$c = \omega^C(\alpha a_1, \ldots, \alpha a_n) = \alpha a,$$

for every homomorphism $\alpha : \boldsymbol{A} \to \boldsymbol{C}$, which contradicts the fact that a is an element of a weak basis of ***A***.

Let a ∉ A'. Then, the set A" = A\{a} contains A'. The set A" is not a strong subalgebra of ***A*** (because A' is a weakly generating subset of ***A***), so there are n ∈ N, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in$ A", a" ∈ A such that

$$\omega^B(\varphi a_1, \ldots, \varphi a_n) = \varphi a'' \text{ and } a'' \notin A''.$$

But, a" ∉ A" means that a" = a, so $\omega^B(\varphi a_1, \ldots, \varphi a_n) = \varphi a$. ■

The second part of the given proof actually shows that every weakly generating (generating) subset of ***A*** includes all elements a ∈ A such that φa is not an image (under any fundamental operation) in the algebra. **B**.

We may define A' to be a weakly free subset of ***A*** if for every φ-algebra ***C*** of the same type and every mapping $\alpha' :$ A' → C, α' could be extended to homomorphism α from $<$ **A'** $>_w$ to ***C***. According to this, the first part of the previous proof shows that every weakly free subset of ***A*** is included in the set of all elements a ∈ A such that φa is not an image (under any fundamental operation) in the algebra **B.**

It is obvious that the basis of free φ-algebras is also unique.

Proposition 4.7. Let ***A*** be a weakly free φ-algebra with the weak basis A' and let the type Ω be nontrivial. Then, $(\forall a \in A')\ \varphi^{-1}\varphi a = a$.

Proof. Let $a' \in \varphi^{-1}\varphi a$, i.e., $\varphi a' = \varphi a$. Since a ∈ A' where A' is a weak basis of ***A***, $\varphi a' = \varphi a$ is not an image in **B,** and so a' ∈ A'.

Let ***C*** = (F(A'), 1, **F**(A')), where **F**(A') is the free algebra with basis A'. Further, let $\alpha : \boldsymbol{A} \to \boldsymbol{C}$ be the homomorphism which is an extension of the mapping $\alpha :$ A' → C given by $\alpha' a = a$, for a ∈ A', and let ω be a nontrivial function symbol. Since $\varphi a' = \varphi a$, we have $\omega^B(\varphi a', \ldots, \varphi a') = \omega^B(\varphi a, \ldots, \varphi a)$, and then

$$\omega^{F(A')}(\alpha a', \ldots, \alpha a') = \omega^{F(A')}(\alpha a, \ldots, \alpha a).$$

But, a, a' ∈ A', so $\omega^{F(A')}(a', \ldots, a') = \omega^{F(A')}(a, \ldots, a)$.

The last equation holds in **F**(A') only for a = a'. ■

Now, we are ready to give the complete description of weakly free φ-algebras with a given weak basis when the type Ω is nontrivial.

Theorem 4.8. Let Ω be a nontrivial type. A φ-algebra ***A*** of type Ω is a weakly free φ-algebra with the weak basis A' if and only if it is a pumped free algebra with basis A'.

Proof. One direction is already proved.

Let $\mathcal{A}$ be a weakly free φ-algebra with weak basis A'.

Since A' is a weakly generating subset of $\mathcal{A}$, φA' = B' is a generating a subset of **B.** The previous proposition shows that $(\forall a \in A')\ \varphi^{-1}\varphi a = a$. We will show that B' is a free subset of **B.**

Let **D** be an algebra of type Ω $\beta' : B' \to D$ be a mapping and $\mathcal{C} = (C, \psi, \mathbf{D})$ be a φ-algebra. Since $\psi^{-1}\beta'\varphi a$ is a nonempty set for every $a \in A'$, there is a mapping α' : $A' \to C$ such that $\psi\alpha' a = \beta'\varphi a$ for $a \in A'$. Let $\alpha : \mathbf{A} \to \mathbf{C}$ be a homomorphism which is an extension of α'.

We will show that if $\varphi a = \varphi a'$ for $a, a' \in A$, then $\psi\alpha a = \psi\alpha a'$. It is obvious when $a \in A'$ or $a' \in A'$, because in that case a = a' (previous proposition). Let $a, a' \notin A'$. Then $\varphi a = \varphi a'$ is an image under some fundamental operation in **B,** and then, according the discussion in Example 3.2., $\psi\alpha a = \psi\alpha a'$. This shows that the mapping $\beta : B \to D$ defined below is well defined.

The set $\varphi^{-1}b$ is nonempty for $b \in B$. For every $b \in B$, we choose an element $a \in \varphi^{-1}b$ and define $\beta b = \psi\alpha a$. It is clear that $\beta\varphi = \psi\alpha$. Further, let $b \in B'$ and a be that element of A' such that $\varphi a = b$. Then,

$$\beta b = \psi\alpha a = \psi\alpha' a = \beta'\varphi a = \beta' b.$$

Whence, β is an extension of β'.

Let $n \in N$, $\omega \in \Omega_n$, $b_1, \ldots, b_n \in B$. Then, there are $a_1, \ldots, a_n$, $a \in A$ such that $\varphi a_1 = b_1, \ldots, \varphi a_n = b_n$ and $\varphi a = \omega^B(\varphi a_1, \ldots, \varphi a_n)$. We obtain $\beta\omega^B(b_1, \ldots, b_n = \beta\omega^B(\varphi a, \ldots, \varphi a_n) = \beta\varphi a = \psi\alpha a = \omega^D(\psi\alpha a_1, \ldots, \psi\alpha a_n) = \omega^D(\beta\varphi a_1, \ldots, \beta\varphi a_n) = \omega^D(\beta b_1, \ldots, \beta b_n)$.

Whence, β is a homomorphism from **B** to **D** which is an extension of β. ■

The previous theorem gives a complete dectription of the weakly free φ-algebras when the type Ω is nontrivial. But, every free φ-algebra is weakly free, so everything that was said for weakly free φ-algebras holds for free φ-algebras as well.

Theorem 4.9. Let Ω be a nontrivial type. A φ-algebra $\mathcal{A}$ of type Ω is a free φ-algebra with basis A' if and only if **B** is a free algebra with basis B' = φA' and φ is a bijective mapping.

Proof. One direction is already proved.

Let $\mathcal{A}$ be a free φ-algebra with basis A'. Then, $\mathcal{A}$ is a weakly free φ-algebra with weak basis A', which means that **B** is a free algebra with basis φA'. We will show that φ is a bijective mapping.

Let $a, a' \in A$ and $\varphi a = \varphi a'$. If $a \in A'$ then, according to Proposition 7., we obtain a = a'. Let $a \notin A'$ and $a \neq a'$. Then, the set $A'' = A\backslash\{a\}$ contains A' and it is a subalgebra of $\boldsymbol{A}$ (because φA" = B), which contradicts the fact that A' is a generating subset of $\boldsymbol{A}$. Whence, a = a'. ■

Now, it is easy to see that there are nonisomorphic weakly free φ-algebras with the same basis. It is obvious because the pumped free algebras could have distinct cardinality. Further, the extension α: $A \to A$ of the mapping α' : A' $\to$ A given by α'a = a, where A is a weakly free φ-algebra with weak basis A', is not always unique. Namely, α could be a permutation on every set φ^{-1}b, for b $\in$ B. It is obvious that every weakly free φ-algebra „contains" a free φ-algebra with the same basis, and that every weakly free φ-algebra that is not free „contains" at least two free φ-algebras with the same basis.

In the case when the type is trivial everything said still holds, except for the condition $\varphi^{-1}\varphi$a = a, for a $\in$ A'. In this case distinct elements of the basis could have same image under φ.

REFERENCES

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Резиме

ЕДНО ОБОПШТУВАЊЕ НА ПОИМОТ АЛГЕБРА

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Во овој труд воведен е поимот на φ-алгебра кој е обопштување на поимот на алгебра. Како и во [2], постојат два вида подструктури, обични и јаки, и еден вид хомоморфизми. Како последица добиваме два вида слободни објекти (слободни и слабо слободни φ-алгебри) во класата на φ-алгебрите од даден тип. Двата вида подструктури имаат некои од познатите својства кај вообичаените алгебри, но не сите, и се чини дека секој од нив ги има оние својства кои другиот вид ги нема. На пример, хомоморфна слика на φ-подалгебра е φ-подалгебра, но истото не важи секогаш за јаките φ-подалгебри; пресек на јаки φ-подалгебри е јака φ-подалгебра, но истото не важи и за φ-подалгебрите. Во трудот е даден комплетен опис на слободните објекти од двата вида. Слабо слободните φ-алгебри имаат некои невообичаени својства. На пример, постојат неизморфни слабо слободни φ-алгебри со иста база. Повеќето резултати се слични на резултатите во [2], иако објектите што се разгледуваат се од различен тип.