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On (m+k,m)-groups for k < m

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The notion of $(m+k,m)$-groupoids, $(m+k,m)$-semigroups and $(m+k,m)$-groups, for m,k positive integers, generalizing the notion of (usual, binary) groupoids, semigroups and groups, and of n-groupoids, n-semigroups and n-groups, are introduced in [TC] and [C]. An expository paper on $(m+k,m)$-groupoids, semigroups and groups is [CCMD]. The behavior of the $(m+k,m)$-groups depends on the relation between m and k. They can be divided into three classes: for $k<m$, $k=m$ and $k>m$. In [D1] some existence conditions for $(n+1,n)$-groups were given, showing that certain finite sets do not admit an $(m+1,m)$-group structure. In [CD] it was shown that $(2m,m)$-groups behave like the usual groups, and examples of nontrivial finite $(2m,m)$-groups were given in [DI]. A big class of examples of $(2m+s,m)$-groups, for $s\geq 1$, including nontrivial finite ones, was given in [D2]. A combinatorial description of free $(m+1,m)$-groups was given in [D4], where it was shown that they are infinite. Free $(m+k,m)$-groups were described in [DJI]. In [D5] it was shown that nontrivial finite $(m+1,m)$-groups do not exist. The aim of this paper is to show that for $1\leq k<m$, nontrivial finite $(m+k,m)$-groups do not exist.

Let, m,k be two positive integers, $m,k\geq 1$, and let Q be a nonempty set.

For a positive integer t, Q^t denotes the t^{th} Cartesian power of Q. We will use the notation $\mathbf{x}=a_1a_2\dots a_t$ or just $\mathbf{x}=a_1^t$ instead of $\mathbf{x}=(a_1,a_2,\dots,a_t)$ for elements $\mathbf{x}\in Q^t$.

Definition 1. *A map $f:Q^n\to Q^m$, where $n=m+k$, $k\geq 1$, is called (n,m)-operation, and the pair (Q,f) is called (n,m)-groupoid. An (n,m)-groupoid (Q,f) is called (n,m)-semigroup if the operation f is associative, i.e. for every $1\leq i\leq k$ and every $x_1^{n+k}\in Q^{n+k}$,*

$$f(x_1^i f(x_{i+1}^{i+n})x_{i+n+1}^{n+k})=f(f(x_1^n)x_{n+1}^{n+k}). \tag{1}$$

An (n,m)-semigroup is called (n,m)-group if for each $\mathbf{a}\in Q^k$, $\mathbf{b}\in Q^m$, the equations

$$f(\mathbf{ax})=\mathbf{b}=f(\mathbf{ya}) \tag{2}$$

have solutions $\mathbf{x},\mathbf{y}\in Q^m$.

It is clear that the notions of $(n,1)$-semigroups and $(n,1)$-groups, are the same as the notions of n-semigroups and n-groups, and specially for $n=2$, are the same as the notions of semigroups and groups.

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The following theorem, shown in [CMD] is a generalization of the associative law for semigroups.

Theorem 1. *Let (Q, f) be an $(m+k, m)$-semigroup. For $s \geq 1$, let $f^s : Q^{m+sk} \to Q^m$ be defined inductively by:*

$$f^1 = f; \quad f^{s+1}(\mathbf{xy}) = f(f^s(\mathbf{x}), \mathbf{y}), \quad \mathbf{x} \in Q^{m+sk}, \quad \mathbf{y} \in Q^k. \tag{3}$$

Then:

(i) For each $s \geq 1$, (Q, f^s) is an $(m+sk, m)$-semigroup; and

(ii) For each $s, t \geq 1$, $\mathbf{x} \in Q^{m+sk}$, $\mathbf{yz} \in Q^{tk}$,

$$f^t(\mathbf{y} f^s(\mathbf{x})\mathbf{z}) = f^{s+t}(\mathbf{yxz}).$$

■

Because of the above theorem, when f is an associative (n, m)-operation, we will use the notation f instead of f^s , i.e. we will consider f as a map

$$f : \cup_{s \geq 1} Q^{m+sk} \to Q^m. \tag{4}$$

The following proposition is shown in [CCMD].

Proposition 2. *Let (Q, f) be an $(m+k, m)$-semigroup. Then the following conditions are equivalent.*

(i) (Q, f) is an $(m+k, m)$-group;

(ii) (Q, f) is an $(m+sk, m)$-group for some $s \geq 1$; and

(iii) (Q, f) is an $(m+sk, m)$-group for each $s \geq 1$. ■

For the rest of the paper let (Q, f) be a given $(m+k, m)$-semigroup, let p be the least non-negative integer, such that $m + p \equiv 0 (mod k)$, and let r be the least non-negative integer such that $k(r-1) < m \leq kr$ and $m + p = kr$. It is easy to check that $p \leq k$.

Definition 2. *If $p \geq 1$, for each $\mathbf{c} \in Q^p$ we define a binary operation $*$ (which depends on $\mathbf{c}$) on Q^m by:*

$$\mathbf{a} * \mathbf{b} = f(\mathbf{acb}), \tag{5}$$

and if $p = 0$, we define a binary operation $*$ on Q^m by:

$$\mathbf{a} * \mathbf{b} = f(\mathbf{ab}) \tag{6}$$

Proposition 3. *$(Q^m, *)$ is a semigroup. Moreover, if (Q, f) is an $(m+k, m)$-group, then $(Q^m, *)$ is a group.*

Proof. Let $\mathbf{a}, \mathbf{b}, \mathbf{d} \in Q^m$. Then:

$$(\mathbf{a} * \mathbf{b}) * \mathbf{d} = f(\mathbf{acb}) * \mathbf{d} = f(f(\mathbf{acb})\mathbf{cd}) = f(\mathbf{a}cf(\mathbf{bcd})) = \mathbf{a} * f(\mathbf{bcd}) = \mathbf{a} * (\mathbf{b} * \mathbf{d}),$$

implies that $(Q^m, *)$ is a semigroup. Next, let (Q, f) be an $(m+k, m)$-group. Let $\mathbf{a}, \mathbf{b} \in Q^m$. Then $\mathbf{ca}, \mathbf{ac} \in Q^{m+p} = Q^{kr}$. Proposition 2 implies that (Q, f) is an $(m+rk, m)$-group. So, the equations $f(\mathbf{acx}) = \mathbf{b} = f(\mathbf{yca})$ have solutions $\mathbf{x}, \mathbf{y} \in Q^m$, which implies that the equations $\mathbf{a} * \mathbf{x} = \mathbf{b} = \mathbf{y} * \mathbf{a}$ have solutions $\mathbf{x}, \mathbf{y} \in Q^m$. Hence $(Q, *)$ is a group. ∎

The above theorem implies the following proposition.

Proposition 4. *Let (Q, f) be an $(m+k, m)$-group. Then:*

(i) (Q, f) is cancellative, i.e.:

$$\textit{for each } \mathbf{a} \in Q^k \textit{ and } \mathbf{b}, \mathbf{c} \in Q^m, \ f(\mathbf{ab}) = f(\mathbf{ac}) \textit{ implies } \mathbf{b} = \mathbf{c} \textit{ in } Q^m.$$

(ii) For each $s \geq 1$, (Q, f^s) is cancellative, i.e.:

$$\textit{for each } \mathbf{a} \in Q^{sk} \textit{ and } \mathbf{b}, \mathbf{c} \in Q^m, \ f(\mathbf{ab}) = f(\mathbf{ac}) \textit{ implies } \mathbf{b} = \mathbf{c} \textit{ in } Q^m.$$

(iii) If for $\mathbf{x}, \mathbf{y} \in Q^{m+k-t}$, $\mathbf{a} \in Q^t$, $f(\mathbf{ax}) = f(\mathbf{ay})$, where $1 \leq t < m+k$, then for each $\mathbf{b} \in Q^t$, $f(\mathbf{bx}) = f(\mathbf{by})$ and $f(\mathbf{xb}) = f(\mathbf{yb})$.

Proof. (i) If $\mathbf{x} \in Q^{(r-1)k}$, then $\mathbf{xa} \in Q^{rk} = Q^{m+p}$. So, let $\mathbf{xa} = \mathbf{yd}$, where $\mathbf{y} \in Q^m$, $\mathbf{d} \in Q^p$ Then $f(\mathbf{ab}) = f(\mathbf{ac})$ implies that:

$$\mathbf{y} * \mathbf{b} = f(\mathbf{ydb}) = f(\mathbf{xab}) = f(\mathbf{x}f(\mathbf{ab})) = f(\mathbf{x}f(\mathbf{ac})) = f(\mathbf{xac}) = f(\mathbf{ydc}) = \mathbf{y} * \mathbf{c},$$

in the group $(Q^m, *)$ defined by (5) via $\mathbf{d}$. Since $(Q^m, *)$ is cancellative, it follows that $\mathbf{b} = \mathbf{c}$.

(ii) This follows from Proposition 2 and (i).

(iii) Let $ks \geq t$, and let $\mathbf{u} \in Q^{ks-t}$. Then for each $\mathbf{b} \in Q^t$,

$$f(\mathbf{ua}f(\mathbf{xb})) = f(\mathbf{u}f(\mathbf{ax})\mathbf{b}) = f(\mathbf{u}f(\mathbf{ay})\mathbf{b}) = f(\mathbf{ua}f(\mathbf{yb})),$$

together with (ii) implies that $f(\mathbf{xb}) = f(\mathbf{yb})$, and

$$f(f(\mathbf{bx})\mathbf{bu}) = f(\mathbf{b}f(\mathbf{xb})\mathbf{u}) = f(\mathbf{b}f(\mathbf{yb})\mathbf{u}) = f(f(\mathbf{by})\mathbf{bu}),$$

together with (ii) implies that $f(\mathbf{bx}) = f(\mathbf{by})$. ∎

We say that $(Q^m, *)$ is a derived semigroup for (Q, f) .

It is obvious that $(Q^m, *)$ is unique for $p = 0$. If (Q, f) is an $(m+k, m)$-group, then for $p \geq 1$ it seems that the group $(Q^m, *)$ defined by (5) depends on $\mathbf{c}$. The following proposition shows that this is not the case.

Proposition 5. *Let (Q, f) be an $(m+k, m)$-group. If $\mathbf{c}, \mathbf{d} \in Q^p$, then the groups $(Q^m, *)$ and $(Q^m, \circ)$ defined by (5) via $\mathbf{c}$ and $\mathbf{d}$ respectively, are isomorphic.*

Proof. Let $\mathbf{e}$ be the neutral element in $(Q^m, \circ)$. Then, for the element $\mathbf{e} * \mathbf{e} = \mathbf{w} \in Q^m$ we have $\mathbf{e} * \mathbf{e} = \mathbf{w} = \mathbf{e} \circ \mathbf{w}$, i.e. $f(\mathbf{ece}) = f(\mathbf{edw})$. Now, Proposition 4 (iii) implies that for each $\mathbf{x} \in Q^m$, $f(\mathbf{cex}) = f(\mathbf{dwx})$ and $f(\mathbf{xce}) = f(\mathbf{xdw})$, i.e. $\mathbf{x} * \mathbf{e} = \mathbf{x} \circ \mathbf{w}$. Let $\rho : Q^m \to Q^m$ be defined by $\rho(\mathbf{z}) = \mathbf{w} \circ \mathbf{z} = f(\mathbf{wdz})$. It is easy to check that ρ is a bijection. Let $\mathbf{x}, \mathbf{y} \in Q^m$. Then:

$$\mathbf{x} \circ \mathbf{w} \circ \mathbf{y} = (\mathbf{x} * \mathbf{e}) \circ \mathbf{y} = f(f(\mathbf{xce})\mathbf{dy}) = f(\mathbf{x}cf(\mathbf{edy})) = f(\mathbf{xc}(\mathbf{e} \circ \mathbf{y})) = \mathbf{x} * (\mathbf{e} \circ \mathbf{y}) = \mathbf{x} * \mathbf{y},$$

implies that $\rho(\mathbf{x}) \circ \rho(\mathbf{y}) = \mathbf{w} \circ \mathbf{x} \circ \mathbf{w} \circ \mathbf{y} = \mathbf{w} \circ (\mathbf{x} * \mathbf{y}) = \rho(\mathbf{x} * \mathbf{y})$. Hence ρ is a homomorphism, i.e. is an isomorphism. ∎

Because of the above Proposition, if (Q, f) is an $(m+k, m)$-group, we will say that $(Q^m, *)$ is its derived group, or that $(Q^m, *)$ is the derived group for (Q, f).

Proposition 6. *(i) If $p = 0$ then an $(m+k, m)$-semigroup (Q, f) is an $(m+k, m)$-group iff its derived semigroup $(Q^m, *)$ is a group.*

(ii) If $r > 1$, where $m + p = rk$, then an $(m+k, m)$-semigroup (Q, f) is an $(m+k, m)$-group iff there is $\mathbf{c} \in Q^p$ such that the semigroup $(Q^m, *)$ defined via $\mathbf{c}$ is a group.

(iii) If $r = 1$ and $p \neq 0$, where $m + p = rk$, then an $(m+k, m)$-semigroup (Q, f) is an $(m+k, m)$-group iff for each $\mathbf{c} \in Q^p$, the semigroup $(Q^m, *)$ defined via $\mathbf{c}$ is a group.

Proof. Proposition 3 implies that if (Q, f) is an $(m+k, m)$-group, then for $p = 0$, $(Q^m, *)$ is a group, and for $p \geq 1$, $(Q^m, *)$ is a group for each $\mathbf{c} \in Q^p$. For the converse, we consider the cases separately.

(i) Let $(Q^m, *)$ be a group. It is enough to show that the equations (2) have solutions in Q^m. Let $\mathbf{a} \in Q^k$, $\mathbf{b} \in Q^m$. Since $p = 0$, it follows that $m = rk$. Let $\mathbf{c} \in Q^{(r-1)k}$ and let $\mathbf{u}, \mathbf{v} \in Q^m$ be the solutions for the equations:

$$\mathbf{ac} * \mathbf{u} = \mathbf{b} = \mathbf{v} * \mathbf{ca}.$$

Then $f(\mathbf{a}f(\mathbf{cu})) = \mathbf{b} = f(f(\mathbf{vc})\mathbf{a})$, i.e. $\mathbf{x} = f(\mathbf{cu})$ and $\mathbf{y} = f(\mathbf{vc})$ are solutions for the equations $f(\mathbf{ax}) = \mathbf{b} = f(\mathbf{ya})$.

(ii) Let for some $\mathbf{c} \in Q^p$, $(Q, *)$ defined via $\mathbf{c}$ be a group. Since $r > 1$, it follows that $(r-1)k - p \geq 0$. Take an arbitrary element $\mathbf{d} \in Q^{(r-1)k-p}$. Then $\mathbf{ad}, \mathbf{da} \in Q^{rk-p} = Q^m$. So, the equations $\mathbf{ad} * \mathbf{u} = \mathbf{b} = \mathbf{v} * \mathbf{da}$ have solutions $\mathbf{u}, \mathbf{v} \in Q^m$. This implies that:

$$f(\mathbf{adcu}) = f(\mathbf{a}f(\mathbf{dcu})) = \mathbf{b}, \text{ and } f(f(\mathbf{vcd})\mathbf{a}) = f(\mathbf{vcda}) = \mathbf{b},$$

i.e. $\mathbf{x} = f(\mathbf{dcu})$ and $\mathbf{y} = f(\mathbf{vcd})$ are solutions for the equations:

$$f(\mathbf{ax}) = \mathbf{b} = f(\mathbf{ya}).$$

(iii) Since $r = 1$, $p \neq 0$ and $m + p = rk = k$, it follows that $m < k$. Let $\mathbf{a} \in Q^k$ and let $\mathbf{b} \in Q^m$. Then $\mathbf{a} = \mathbf{dc}$, for some $\mathbf{d} \in Q^m$ and $\mathbf{c} \in Q^p$. Since $(Q^m, *)$ defined via $\mathbf{c}$ is a group, it

follows that the equation $\mathbf{d} * \mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in Q^m$. This implies that $f(\mathbf{dcu}) = \mathbf{b}$, i.e. the equation $f(\mathbf{ax}) = \mathbf{b}$ has a solution un Q^m . Symmetrically, the equation $f(\mathbf{ya}) = \mathbf{b}$ has a solution $\mathbf{y} \in Q^m$. ∎

Next, let (Q, f) be an $(m+1, m)$-group. Then, since $k = 1$ and $m = mk$, it follows that $p = 0$. So, $(Q^m, *)$ defined by $\mathbf{x} * \mathbf{y} = f(\mathbf{xy})$ is a group. In [D1] it is shown that if $\mathbf{e} \in Q^m$ is the neutral element in $(Q^m, *)$, then there is an element $e \in Q$, such that $\mathbf{e} = ee \ldots ee$. Moreover, it is easy to check that for $S = \{f(x\mathbf{e}) \mid x \in Q\} \subseteq Q^m$, the product map $\varphi : S^m \to Q^m$, defined by $\varphi(x_1, x_2, \ldots, x_m) = x_1 * x_2 * \ldots * x_m$, is a bijection.

Because of the above fact, the notion of a group with unique product structure was introduced in [D5] . We recall its definition.

Definition 3. *We say that a group G has a unique m-element product structure, where m is a positive integer, if there is a subset $S \subseteq G$, such that the product map $\varphi : S^m \to G$, defined by:*

$$\varphi(x_1, x_2, \ldots, x_m) = x_1 x_2 \ldots x_m,$$

is a bijection.

It is obvious that every group has a unique 1-element product structure, so we say that a group G has a unique product structure if it has a unique m-element product structure for some $m > 1$.

The above discussion shows that if (Q, f) is an $(m+1, m)$-group, for $m \geq 1$, then the group $(Q^m, *)$ has a unique product structure. In [D5] it was shown that if $(G, *)$ is a finite group with unique product structure, then it is the trivial group, i.e. $\mid G \mid = 1$, where $\mid G \mid$ is the number of elements in G, which implies that there are no nontrivial finite $(m+1, m)$-groups. The existence of nontrivial $(m+1, m)$-groups, was shown in [D4] , which by Proposition 3 implies the existence of nontrivial $(m+s, m)$-groups for every $s \geq 1$. But all these $(m+s, m)$-groups are infinite.

The basic fact used to prove that there are no finite nontrivial groups with unique product structure is the following theorem, shown in [D5] .

Theorem 7. *Let G be a finite group, and $S \subseteq G$. If the product map $\varphi : S \times S \to G$ is $t-to-1$, for $t \geq 1$, (which means that the number of elements of $\varphi^{-1}(g)$ is t, i.e. $\mid \varphi^{-1}(g) \mid = t$, for every $g \in G$), then $S = G$.* ∎

Next we will show that there are no nontrivial finite $(m+k, m)$-groups for $k < m$.

Theorem 8. *Let $k < m$ and (Q, f) be a finite $(m+k, m)$-group. Then $\mid Q \mid = 1$, i.e. (Q, f) is a trivial $(m+k, m)$-group.*

Proof. Let Q have n elements, i.e. let $\mid Q \mid = n$. Let $m + p = rk$, $(r-1)k < m \leq rk$, be as above. Since $k < m$, it follows that $r \geq 2$. If $p = 0$, then $m = rk$, and $(Q^m, *)$ defined by (6) is a group, and if $p \neq 0$, then we choose an element $\mathbf{c} \in Q^p$, and again $(Q^m, *)$ defined by (5), via $\mathbf{c}$, is a group. Let $\mathbf{e} \in Q^m$ be the neutral element in $(Q^m, *)$. We consider two cases, when r is even and when r is odd.

Case 1. Let $r = 2t$, where $t \geq 1$. Let $S = \{f(\mathbf{xe}) \mid \mathbf{x} \in Q^{tk}\}$. Then $S \subseteq Q^m$. Since $m > k$, it follows that $\mathbf{e} = \mathbf{ab}$, where $\mathbf{a} \in Q^{tk-p}$ and $\mathbf{b} \in Q^{tk}$.

(a) Because (Q, f) is an $(m+k, m)$-group, it is cancellative. This implies that if $f(\mathbf{xe}) = f(\mathbf{ye})$, i.e. $f(\mathbf{xab}) = f(\mathbf{yab})$, then $\mathbf{xa} = \mathbf{ya}$, i.e. $\mathbf{x} = \mathbf{y}$. Hence S has n^{tk} elements, i.e. $\mid S \mid = n^{tk}$.

(b) Next we prove that the product map $\varphi : S^2 \to Q^m$ is n^p-to-1. If $f(\mathbf{xe}), f(\mathbf{ye}) \in S$, then, for $p = 0$,

$$\varphi(f(\mathbf{xe}), f(\mathbf{ye})) = f(\mathbf{xe}) * f(\mathbf{ye}) = f(\mathbf{xeye}) = f(\mathbf{xye}) = \mathbf{xy}, \tag{7}$$

and for $p \geq 1$

$$\varphi(f(\mathbf{xe}), f(\mathbf{ye})) = f(\mathbf{xe}) * f(\mathbf{ye}) = f(\mathbf{xecye}) = f(\mathbf{xye}). \tag{8}$$

The equalities (7) show that for $p = 0$, the map φ is 1-to-1, i.e. n^0-to-1.

Next, let $p \geq 1$, and $\mathbf{u} \in Q^m$. Then, for each $\mathbf{v} \in Q^p$, we have that $\mathbf{ve} \in Q^{2tk}$. Since (Q, f) is an $(m+k, m)$-group, it follows that it is an $(m+rk, m)$-group. So, for $\mathbf{ve}$ and $\mathbf{u}$, there is $\mathbf{z} \in Q^m$, such that $f(\mathbf{zve}) = \mathbf{u}$, and moreover this $\mathbf{z}$ is unique. But $\mathbf{zv} \in Q^{m+p}$, i.e. $\mathbf{zv} \in Q^{2tk}$. This implies that $\mathbf{zv} = \mathbf{xy}$, for some $\mathbf{x}, \mathbf{y} \in Q^{tk}$, and these $\mathbf{x}, \mathbf{y}$ are unique for the chosen $\mathbf{v}$. Hence:

$$\mathbf{u} = f(\mathbf{zve}) = f(\mathbf{xye}) = \varphi(f(\mathbf{xe}), f(\mathbf{ye})).$$

Since there are n^p choices for $\mathbf{v}$, the above discussion shows that $\varphi^{-1}(\mathbf{u})$ has n^p elements.

The above discussion shows that the map φ is n^p-to-1.

(c) Now, (b) and Theorem 7, imply that $S = Q^m$. Hence, by (a), $n^{tk} = \mid S \mid = \mid Q \mid^m = n^m$. If $n \geq 2$, this equality implies that $m = tk$, i.e. $tk = p$, which contradicts the fact that $p < k$. At the end, we conclude that $n = 1$.

Case 2. Although the proof of this case is similar to the proof of Case 1, for completeness, we give all details. Let $r = (2t+1)k$, where $t \geq 1$. Let $S = \{f(\mathbf{xe}) \mid \mathbf{x} \in Q^{(t+1)k}$. Then $S \subseteq Q^m$. Since $m > k$, it follows that $\mathbf{e} = \mathbf{ab}$, where $\mathbf{a} \in Q^{tk-p}$ and $\mathbf{b} \in Q^{(t+1)k}$.

(a) Because (Q, f) is an $(m+k, m)$-group, it is cancellative. This implies that if $f(\mathbf{xe}) = f(\mathbf{ye})$, i.e. $f(\mathbf{xab}) = f(\mathbf{yab})$, then $\mathbf{xa} = \mathbf{ya}$, i.e. $\mathbf{x} = \mathbf{y}$. Hence S has $n^{(t+1)k}$ elements, i.e. $\mid S \mid = n^{(t+1)k}$.

(b) Next we prove that the product map $\varphi : S^2 \to Q^m$ is n^{k+p}-to-1. If $f(\mathbf{xe}), f(\mathbf{ye}) \in S$, then:

$$\varphi(f(\mathbf{xe}), f(\mathbf{ye})) = f(\mathbf{xe}) * f(\mathbf{ye}) = f(\mathbf{xye}), \tag{9}$$

because for $p = 0$ we have $f(\mathbf{xe}) * f(\mathbf{ye}) = f(\mathbf{xeye}) = f(\mathbf{xye})$, and for $p \geq 1$ we have $f(\mathbf{xe}) * f(\mathbf{ye}) = f(\mathbf{xecye}) = f(\mathbf{xye})$

Now, $\mathbf{u} \in Q^m$. Then, for each $\mathbf{v} \in Q^p$, (if $p = 0$ then $\mathbf{v}$ is the empty word) and each $\mathbf{w} \in Q^k$ we have that $\mathbf{wve} \in Q^{2(t+1)k}$. Since (Q, f) is an $(m+k, m)$-group, it follows that it is an $(m+(r+1)k, m)$-group. So, for $\mathbf{wve}$ and $\mathbf{u}$, there is $\mathbf{z} \in Q^m$, such that $f(\mathbf{zwve}) = \mathbf{u}$, and moreover this $\mathbf{z}$ is unique. But $\mathbf{zwv} \in Q^{m+k+p}$, i.e. $\mathbf{zwv} \in Q^{2(t+1)k}$. This implies that $\mathbf{zwv} = \mathbf{xy}$, for some $\mathbf{x}, \mathbf{y} \in Q^{(t+1)k}$, and these $\mathbf{x}, \mathbf{y}$ are unique for the chosen $\mathbf{v}$ and $\mathbf{w}$. Hence:

$$\mathbf{u} = f(\mathbf{zwve}) = f(\mathbf{xye}) = \varphi(f(\mathbf{xe}), f(\mathbf{ye})).$$

Since there are n^p choices for $\mathbf{v}$, and n^k choices for $\mathbf{w}$, the above discussion shows that $\varphi^{-1}(\mathbf{u})$ has n^{p+k} elements, i.e. that the map φ is n^{p+k}-to-1.

(c) Now, (b) and Theorem 7, imply that $S = Q^m$. Hence, by (a), $n^{(t+1)k} = |\, S \,| = |\, Q \,|^m = n^m$. If $n \geq 2$, this equality implies that $m = (t+1)k$, i.e. $tk = p$, which contradicts the fact that $p < k$. At the end, we conclude that $n = 1$. ∎

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