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# CYCLIC SUBGROUPOIDS OF AN ABSOLUTELY FREE GROUPOID

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**Abstract.** Subgroupoids of an absolutely free groupoid $\boldsymbol{F}=(F,\cdot)$ with a free basis $B$ that are generated by one element (called cyclic subgroupoids of $\boldsymbol{F}$) are considered. It is shown that: two cyclic subgroupoids of $\boldsymbol{F}$ have common elements if and only if one of them is contained in the other; $\boldsymbol{F}$ has maximal cyclic subgroupoids and if $|B|\geq 2$, every cyclic subgroupoid is contained in a maximal one; any two maximal cyclic subgroupoids of $\boldsymbol{F}$ are either disjoint or equal. Also, a characterization of maximal cyclic subgroupoids of $\boldsymbol{F}$ by means of primitive elements in $F$ is given. This statements are also true for an absolutely free groupoid with one-element basis (with modified definition of maximal cyclic subgroupoid.

**Key words:** groupoid, subgroupoid, generating element, cyclic subgroupoid, free groupoid.

## 1. PRELIMINARIES

A pair $\boldsymbol{G}=(G,\cdot)$, where $G$ is a nonempty set and $\cdot:(x,y)\mapsto xy$ a mapping from $G\times G$ into $G$, is called a *groupoid*.

An element $a\in G$ is *prime* in $\boldsymbol{G}$ iff[1] $a\neq xy$, for all $x,y\in G$.

Throughout the paper we denote by $\boldsymbol{F}=(F,\cdot)$ an absolutely free groupoid (a.f.g.), i.e. free groupoid in the class of all groupoids with a free basis $B$. Recall that the following theorem characterizes free groupoids (see [1]; L.1.5).

**Theorem**. (Bruck) *A groupoid $\boldsymbol{F}=(F,\cdot)$ is an a.f.g. iff*

($i$) $\boldsymbol{F}$ *is injective, i.e.* $(\forall a,b,c,d\in F)\quad ab=cd\Rightarrow a=c, b=d$.

($ii$) *The set of primes in $\boldsymbol{F}$ is nonempty and generates $\boldsymbol{F}$.*

(*In that case $B$ is the unique free basis of $\boldsymbol{F}$.*) □

As a corollary (see[1]; T.1.4) of Bruck Theorem we have:

**Proposition 1.1.** *Every subgroupoid*[2] $\boldsymbol{Q}$ *of an a.f.g. $\boldsymbol{F}$ is free, with free basis the set of primes in $\boldsymbol{Q}$.* □

[1] "iff" means "if and only if"

The elements of $\boldsymbol{F}$ will be denoted by $t,u,v,w,x,y,...$ For any $v \in F$ we define the *length* $|v|$ of $v$ and the *set of parts* $P(v)$ of $v$ in the following way:

$$|b| = 1, \quad |tu| = |t| + |u|, \tag{1.1}$$

$$P(b) = \{b\}, \quad P(tu) = \{tu\} \cup P(t) \cup P(u) \tag{1.2}$$

for any $b \in B$, $t,u \in F$.

We will denote by $\boldsymbol{E} = (E, \cdot)$ an absolutely free groupoid with one-element basis $\{e\}$. The elements of $E$ will be denoted by $f, g, h, ...$ and called *groupoid powers* ([3]).

If $\boldsymbol{G} = (G, \cdot)$ is a groupoid, then each $f \in E$ induces a transformation $f^{\boldsymbol{G}}$ on $\boldsymbol{G}$ (called the *interpretation* of $f$ in $\boldsymbol{G}$) defined by:

$$f^{\boldsymbol{G}}(x) = \varphi_x(f),$$

where $\varphi_x : E \to G$ is the homomorphism from $\boldsymbol{E}$ into $\boldsymbol{G}$ such that $\varphi_x(e) = x$. In other words

$$e^{\boldsymbol{G}}(x) = x, \quad (fh)^{\boldsymbol{G}}(x) = f^{\boldsymbol{G}}(x)h^{\boldsymbol{G}}(x) \tag{1.3}$$

for any $f, h \in E$, $x \in G$.

(We will usually write $f(x)$ instead of $f^{\boldsymbol{G}}(x)$, when we work with a fixed groupoid $\boldsymbol{G}$, $f(t)$ instead of $f^{\boldsymbol{F}}(t)$ and $f(g)$ instead of $f^{\boldsymbol{E}}(g)$, in the cases when $\boldsymbol{G} = \boldsymbol{F}$ or $\boldsymbol{G} = \boldsymbol{E}$, respectively.)

The following statements are shown in [3].

**Proposition 1.2.** *If* $f, g \in E$, $t,u \in F$, *then*

a) $|f(t)| = |f| \cdot |t|$.

b) $f(t) = g(u)$ & $(|t| = |u| \vee |f| = |g|) \Rightarrow (f = g$ & $t = u)$.

c) $f(t) = g(u)$ & $|t| \geq |u| \Leftrightarrow (\exists!\, h \in E)(t = h(u)$ & $g = f(h))$. □

We define an other operation $\circ$ on $\boldsymbol{E}$ by:

$$f \circ g = f(g). \tag{1.4}$$

We obtain an algebra $(E, \circ, \cdot)$ with two operations, $\circ$ and $\cdot$, such that

$$e \circ g = g \circ e = g, \quad (f_1 f_2) \circ g = (f_1 \circ g)(f_2 \circ g) \tag{1.5}$$

for any $g, f_1, f_2 \in E$.

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[2] The notions as subgroupoid, subgroupoid generated by a set $A$, homomorphism, variety, …have usual meanings.

It is shown in [3] that $(E,\circ,\cdot)$ is a cancellative monoid.

An element $f\in E$ is said to be *irreducibile* in $(E,\circ,e)$ iff

$$f\neq e \quad \& \quad (f=g\circ h \Rightarrow g=e \text{ or } h=e) \tag{1.6}$$

## 2. SOME PROPERTIES OF CYCLIC SUBGROUPOIDS OF AN A.F.G.

Let $\boldsymbol{G}=(G,\cdot)$ be a groupoid and $A\subseteq G$. If $A=\{a_1,a_2,...,a_m\}$, then we denote by $\langle a_1,a_2,...,a_m\rangle$ the subgroupoid generated by $A$ and call it $m$*-generated subgroupoid* of $\boldsymbol{G}$ (by the analogy with the same notion for the semigroups (see [2], Part 2, IV.2.4).

A subgroupoid $\boldsymbol{C}$ of $\boldsymbol{G}$ is said to be *cyclic* (or 1-*generated*) iff there is $a\in G$, such that $\boldsymbol{C}$ is generated by $\{a\}$, i.e.

$$(\exists a\in G)\quad \boldsymbol{C}=\langle a\rangle.$$

(In that case we say that $a$ is a *generating element* (or a *generator*) of $\boldsymbol{C}$.)

Cyclic subgroupoids of $\boldsymbol{G}$ can be characterized in the following way (see [4], Prop.1.2.).

**Proposition 2.1.** If $\boldsymbol{G}=(G,\cdot)$ is a groupoid, then

$$(\forall a\in G)\quad \langle a\rangle=\{f^{\boldsymbol{G}}(a): f\in E\}.$$

**Proof.** We will show that the set $C=\{f^{\boldsymbol{G}}(a): f\in E\}$ is a subgroupoid of $\boldsymbol{G}$ and if $\boldsymbol{H}$ is a subgroupoid of $\boldsymbol{G}$ such that $\{a\}\subseteq H$, then $C\subseteq H$, i.e. $\boldsymbol{C}$ is the intersection of all subgroupoids of $\boldsymbol{G}$ that contain the element $a$.

Let $b,c\in C$. Then there are $g,h\in E$, such that $b=g^{\boldsymbol{G}}(a)$, $c=h^{\boldsymbol{G}}(a)$ and $bc=g^{\boldsymbol{G}}(a)h^{\boldsymbol{G}}(a)=f^{\boldsymbol{G}}(a)\in C$, where $f=gh$. Thus, $\boldsymbol{C}$ is a subgroupoid of $\boldsymbol{G}$.

Let $\boldsymbol{H}$ be a subgroupoid of $\boldsymbol{G}$ such that $a\in H$. By (1.3), $e^{\boldsymbol{G}}(a)=a$, so $e^{\boldsymbol{G}}(a)\in H$. Suppose that $g^{\boldsymbol{G}}(a)\in H$, for any $g\in E$, such that $|\,g\,|\leq k$. Let $f\in E$ be such that $|\,f\,|=k+1$. Then $f=f_1f_2$, where $|\,f_1\,|,|\,f_2|\leq k$, and so $f_1^{\boldsymbol{G}}(a), f_2^{\boldsymbol{G}}(a)\in H$. As $\boldsymbol{H}$ is a subgroupoid, $f^{\boldsymbol{G}}(a)=f_1^{\boldsymbol{G}}(a)f_2^{\boldsymbol{G}}(a)\in H$, i.e. $\{f^{\boldsymbol{G}}(a): f\in E\}\subseteq H$. Therefore, $\boldsymbol{C}$ is the intersection of all subgroupoids of $\boldsymbol{G}$ that contain the element $a$. □

The next results concern the cyclic subgroupoids of an a.f.g. $\boldsymbol{F}$. According to Prop.2.1, the subgroupoid of $\boldsymbol{F}$ generated by an element $t\in F$, i.e. a cyclic subgroupoid of $\boldsymbol{F}$, can be presented as

$$\langle t\rangle=\{f(t): f\in E\}. \tag{2.1}$$

It is clear that the only prime element in the subgroupoid $\langle t \rangle$ is $t$. Therefore, as an immediate consequence of Prop.1.1 we have:

**Proposition 2.2.** *For any $t \in F$, the subgroupoid $\langle t \rangle$ of $\boldsymbol{F}$ is free with the free basis $\{t\}$.* □

**Proposition 2.3.** *Let $\boldsymbol{F}$ be an a.f.g. with a free basis $B$.*

a) *$\boldsymbol{F}$ is cyclic iff $|B| = 1$.*

b) *Let $t \in F$ and $\boldsymbol{Q} = \langle t \rangle$. If $|B| \geq 2$, then $Q \subset F$. If $|B| = 1$ and $t \notin B$, then $Q \subset F$.*

**Proof.** a) If $\boldsymbol{F}$ is cyclic, then $F = \langle t \rangle$ for some $t \in F$. By Prop.2.2, $\{t\}$ is the free basis of $\langle t \rangle$, so $\{t\} = B$, i.e. $|B| = 1$. Conversely, if $|B| = 1$, for example $B = \{b\}$, then $F = \langle b \rangle$ (since $B$ is a generating set for $\boldsymbol{F}$), i.e. $\boldsymbol{F}$ is cyclic.

b) If $t \in F$, $|B| \geq 2$ and $\boldsymbol{Q} = \langle t \rangle$, then it is not possible $Q = F$, because $\boldsymbol{Q}$ (Prop.1.1) has one-element free basis and $\boldsymbol{F}$ has a basis with more than one elements. If $|B| = 1$ and $t \notin B = \{a\}$, then $a \notin \langle t \rangle = Q$, so $Q \neq F$. So, in both cases, $\boldsymbol{Q}$ is a proper subgroupoid of $\boldsymbol{F}$. □

A cyclic subgroupoid of a given groupoid $\boldsymbol{G}$ may have more than one different generators. For example, the groupoid $(\mathbb{Z}_{12}, +)$ has a cyclic subgroupoid $H = \{0,3,6,9\}$, such that $\langle 3 \rangle = H = \langle 9 \rangle$.

However, for the cyclic subgroupoids of an a.f.g. $\boldsymbol{F}$ the following proposition holds.

**Proposition 2.4.** *Every cyclic subgroupoid of an a.f.g. $\boldsymbol{F}$ has one and only one generator, i.e.*

$$(\forall t, u \in F)\ (\langle t \rangle = \langle u \rangle \Leftrightarrow t = u).$$

**Proof.** Let $\boldsymbol{H}$ be a cyclic subgroupoid of $\boldsymbol{F}$ such that $\boldsymbol{H} = \langle t \rangle$ and $\boldsymbol{H} = \langle u \rangle$. $\langle t \rangle = \langle u \rangle$ implies that $t \in \langle u \rangle$ and $u \in \langle t \rangle$, so $t = f(u)$ and $u = g(t)$, for some $f, g \in E$. Then $|t| = |f| \cdot |u|$ and $|u| = |g| \cdot |t|$, so $|t| = |f| \cdot |g| \cdot |t|$, which implies that $|f| = |g| = 1$, i.e. $f = g = e$. Therefore, $t = f(u) = e(u) = u$. □

Note that

$$(\forall t, u \in F)\ (\langle t \rangle \subseteq \langle u \rangle \Leftrightarrow t \in \langle u \rangle) \tag{2.2}$$

is also true. Namely, it is clear that: $\langle t\rangle \subseteq \langle u\rangle \Rightarrow t \in \langle u\rangle$. Conversely, if $t \in \langle u\rangle$, then $t = g(u)$, for some $g \in E$. Therefore,
$\langle t\rangle = \{f(t): f \in E\} = \{f(g(u)): f \in E\} =$
$= \{(f \circ g)(u): f \in E\} \subseteq \{h(u): h \in E\} = \langle u\rangle$.

**Theorem 2.1.** *Any two cyclic subgroupoids of* $\boldsymbol{F}$ *are either disjoint or one of them is contained in the other, i.e.*

$$(\forall t, u \in F)\ [\langle t\rangle \cap \langle u\rangle \neq \varnothing \Leftrightarrow \langle t\rangle \subseteq \langle u\rangle \vee \langle u\rangle \subseteq \langle t\rangle].$$

**Proof.** Let $\langle t\rangle \cap \langle u\rangle \neq \varnothing$. Then, there is $x \in F$ such that $x \in \langle t\rangle$ and $x \in \langle u\rangle$, i.e. $f(t) = x = g(u)$, for some $f, g \in E$. If $|f| = |g|$, then $|t| = |u|$. By Prop.1.2 b), it follows that $f = g$, $t = u$, i.e. $\langle t\rangle = \langle u\rangle$. If $|f| < |g|$, then $|t| > |u|$ and by Prop.1.2 c), we obtain that $t = h(u)$, for some $h \in E \setminus \{e\}$, i.e. $t \in \langle u\rangle$. As (2.2) holds, it follows that $\langle t\rangle \subseteq \langle u\rangle$. If $|f| > |g|$, then, by analogy, $\langle u\rangle \subseteq \langle t\rangle$. The converse is obvious. □

Note that Theorem 2.1 is not true if $\boldsymbol{F}$ is not an a.f.g. For example, in $(\mathbb{N}, +)$, where $\mathbb{N}$ is the set of positive integers, $12 \in \langle 4\rangle \cap \langle 6\rangle$. However, none of the subgroupoids $\langle 4\rangle$ and $\langle 6\rangle$ are contained in each other.

## 3. MAXIMAL CYCLIC SUBGROUPOIDS OF AN A.F.G.

Let $\boldsymbol{F}$ be an a.f.g. with a free basis $B$, $|B| \geq 2$. A cyclic subgroupoid $\boldsymbol{M}$ of $\boldsymbol{F}$ is said to be *maximal* in the class of all cyclic subgroupoids of $\boldsymbol{F}$ iff $\boldsymbol{M}$ is not a proper subgroupoid of a cyclic subgroupoid of $\boldsymbol{F}$.

**Theorem 3.1.** *Let* $\boldsymbol{F}$ *be an a.f.g. with* $|B| \geq 2$.

a) $\boldsymbol{F}$ *has maximal cyclic subgroupoids.*

b) *Every cyclic subgroupoid of* $\boldsymbol{F}$ *is contained in a maximal cyclic one.*

**Proof.** a) Let $b \in B$. Then the cyclic subgroupoid $\langle b\rangle$ is a maximal one. (Namely, if $\langle b\rangle \subset \langle t\rangle$, for some $t \in F$, then $b \in \langle t\rangle$ and $b \neq t$. Thus $b = f(t)$, for some $f \in E \setminus \{e\}$, but this contradicts the fact that $b$ is prime in $\boldsymbol{F}$.)

b) Let $t_0 \in F$. If $\langle t_0\rangle$ is not a maximal subgroupoid, then there is a cyclic subgroupoid $\langle t_1\rangle$, such that $\langle t_0\rangle \subset \langle t_1\rangle$. If $\langle t_1\rangle$ is not maximal one, then there is a cyclic subgroupoid $\langle t_2\rangle$, such that $\langle t_1\rangle \subset \langle t_2\rangle$ e.t.c.:

$$\langle t_0\rangle \subset \langle t_1\rangle \subset \langle t_2\rangle \subset \ldots \subset \langle t_k\rangle.$$

Suppose that the sequence $\langle t_0\rangle, \langle t_1\rangle, \langle t_2\rangle, ..., \langle t_k\rangle, ...$ is infinite, i.e. there is no maximal cyclic subgroupoid that contains $\langle t_0\rangle$. By (2.2) and (2.1), we obtain that $t_0 = f_1(t_1)$, $t_1 = f_2(t_2)$, …, $t_{k-1} = f_k(t_k)$,… where $f_i \neq e$, i.e. $|f_i| \geq 2$, for every $i \geq 0$. As $|t_0| = |f_1| \cdot |t_1|$,… (Prop.1.2 a)), it follows that $|t_0| > |t_1| > |t_2| > ...$ However, $|t_0|$ is a finite number, so the descending sequence of positive integers $|t_0|, |t_1|, |t_2|, ...$ must "stop", i.e. there is a $k \geq 0$, such that $|t_k| = |t_{k+1}|$. Using the fact that $t_k = f_k(t_{k+1})$, it follows that $t_k = t_{k+1}$, and that contradicts the supposition that $\langle t_k\rangle \subset \langle t_{k+1}\rangle$, for any $k \geq 0$. □

Let $\boldsymbol{G}$ be a groupoid. An element $c \in G$ is said to be *primitive* in $\boldsymbol{G}$ iff

$$(\forall a \in G)(\forall f \in E \setminus \{e\}) \quad c \neq f(a).$$

An element $c \in G$ is said to be *non-primitive* in $\boldsymbol{G}$ iff

$$(\exists a \in G)(\exists f \in E \setminus \{e\}) \quad c = f(a).$$

As an immediate consequence of the definition of primitive element in $\boldsymbol{G}$, when $\boldsymbol{G} = \boldsymbol{F}$, we obtain the following

**Proposition 3.1.** *The following conditions are equivalent:*

a) $v$ *is primitive in* $\boldsymbol{F}$;

b) $(\forall u \in F)(\forall f \in E)\ (v = f(u) \Rightarrow f = e)$;

c) $(\forall u \in F)(\forall f \in E)\ (v = f(u) \Rightarrow v = u)$. □

**Lema 3.1.** *For any non-primitive element* $v$ *in* $\boldsymbol{F}$ *there is a uniquely determined primitive element* $u \in F$ *and uniquely determined* $f \in E \setminus \{e\}$ *such that* $v = f(u)$.

In that case we say that $u$ is a *base* of $v$ (and denote it by $\underline{v} = u$) and $f$ is a *power* of $v$.

**Proof.** Existence. If $v$ is a non-primitive element in $\boldsymbol{F}$, then there are $u \in F$ and $f \in E \setminus \{e\}$ such that $v = f(u)$. If $u$ is primitive in $\boldsymbol{F}$, then the statement is shown. Suppose that $u$ is a non-primitive element in $\boldsymbol{F}$. By Prop.1.2 a), it follows that $|v| = |f(u)| = |f| \cdot |u|$. Since $|f| \geq 2$, we have $|v| > |u|$. From the definition of non-primitive element, it follows directly that there are $u_1 \in F$ and $f_1 \in E \setminus \{e\}$, such that $u = f_1(u_1)$, so $v = f(f_1(u_1)) = (f \circ f_1)(u_1)$. Continuing this procedure, we obtain a descending sequence $(|u_i|)$ of positive integers. This sequence must end, i.e. there are $u_n \in F$ and $f_n \in E \setminus \{e\}$, such that $v = (f \circ f_1 \circ f_2 \circ ... \circ f_n)(u_n)$ and $u_n$ is a primitive element in $\boldsymbol{F}$.

Uniqueness. Let $v \in F$ and suppose that $v = f(u) = g(t)$, where $u$ and $t$ are primitive in $\boldsymbol{F}$. Clearly, $|t| = |u|$ (because in the opposite case, there would be $h \in E$ such that $t = h(u)$ (or $u = h(t)$), and $t$ (or $u$) would not be primitive). By Prop.1.2 it follows that $u = t$ and $f = g$. □

The following theorem characterizes maximal cyclic subgroupoids of $\boldsymbol{F}$.

**Theorem 3.2.** *The subgroupoid $\langle t \rangle$ of $\boldsymbol{F}$ (with $|B| \geq 2$) is maximal one iff $t$ is a primitive element in $\boldsymbol{F}$.*

**Proof.** Let $\langle t \rangle$ be a maximal subgroupoid of $\boldsymbol{F}$. Suppose that $t$ is not a primitive element in $\boldsymbol{F}$. Then, by Lemma 3.1, $t = f(u)$, for some $u \in F$ and $f \in E \setminus \{e\}$, so we obtain that $\langle t \rangle \subset \langle u \rangle$, i.e. $\langle t \rangle$ is not maximal. Thus $t$ is a primitive element in $\boldsymbol{F}$. Conversely, let $t$ be a primitive element in $\boldsymbol{F}$ and suppose that $\langle t \rangle$ is not a maximal subgroupoid of $\boldsymbol{F}$. Then, there is an element $u \in F$, such that $\langle t \rangle \subset \langle u \rangle$. Therefore $t = f(u)$, for some $u \in F$ and $f \in E \setminus \{e\}$, i.e. $t$ is not a primitive element in $\boldsymbol{F}$. □

As a consequence of Theorem 3.2 and Lemma 3.1 we obtain the following

**Proposition 3.2.** Let $\boldsymbol{F}$ be an a.f.g. with $|B| \geq 2$. The following conditions are equivalent:

a) $t$ is a primitive element in $\boldsymbol{F}$;
b) $(\forall u \in F)(\forall f \in E)\ (t = f(u) \Rightarrow t = u)$;
c) $(\forall u \in F)(\forall f \in E)\ (t = f(u) \Rightarrow f = e)$;
d) $\langle t \rangle$ is a maximal cyclic subgroupoid of $\boldsymbol{F}$. □

**Theorem 3.3.** *If $\langle u \rangle$ and $\langle v \rangle$ are maximal cyclic subgroupoids of an a.f.g. $\boldsymbol{F}$ (with $|B| \geq 2$), then either $\langle u \rangle \cap \langle v \rangle = \varnothing$ or $\langle u \rangle = \langle v \rangle$.*

**Proof.** Let $\langle t \rangle \cap \langle u \rangle \neq \varnothing$. Then, by Theorem 2.1, it follows that $\langle u \rangle \subseteq \langle v \rangle$ (or $\langle v \rangle \subseteq \langle u \rangle$). It is not possible to be $\langle u \rangle \subset \langle v \rangle$ (or $\langle v \rangle \subset \langle u \rangle$) because this would contradict the supposition that the subgroupoids $\langle u \rangle$ and $\langle v \rangle$ are maximal. Therefore $\langle u \rangle = \langle v \rangle$. □

As a consequence of Theorem 3.3 we obtain that different maximal cyclic subgroupoids of $\boldsymbol{F}$ are disjoint, i.e. the class of maximal cyclic subgroupoids of $\boldsymbol{F}$ consists of (pairwise) disjoint subgroupoids.

Bellow $\boldsymbol{E}=(E,\cdot)$ denotes an a.f.g. with one-element basis $\{e\}$. Clearly, the results of Section 2 are true in the case $\boldsymbol{F}=\boldsymbol{E}$ and we will repeat some of them in the following proposition.

**Proposition 3.3.** The following statements are true in $\boldsymbol{E}$ for any $f,g,h\in E$.

a) $\langle f\rangle\subseteq\langle g\rangle \Leftrightarrow (\exists! h\in E)\ \ f=h(g)$;

b) $\langle f\rangle\subset\langle g\rangle \Leftrightarrow (\exists! h\in E\setminus\{e\})\ \ f=h(g)$;

c) $\langle f\rangle\cap\langle g\rangle\neq\varnothing \Leftrightarrow \langle f\rangle\subseteq\langle g\rangle \vee \langle g\rangle\subseteq\langle f\rangle$;

d) $\langle f\rangle=\langle g\rangle \Leftrightarrow f=g$;

e) $\langle e\rangle$ is the largest cyclic subgroupoid of $E$ and $\langle e\rangle=E$. □

Now we will modify the definition of a maximal cyclic subgroupoid for $\boldsymbol{E}$.

A cyclic subgroupoid $\boldsymbol{M}$ of $\boldsymbol{E}$ is said to be *maximal* in the class of all cyclic subgroupoids of $\boldsymbol{E}$ iff there is no proper cyclic subgroupoid of $\boldsymbol{E}$ that contains $\boldsymbol{M}$.

**Proposition 3.4.** *The subgroupoid $\langle f\rangle$ is a proper maximal cyclic subgroupoid of* $\boldsymbol{E}$ *iff $f$ is irreducibile element in the monoid* $(E,\circ,e)$.

**Proof.** Let $\langle f\rangle$ be a proper maximal cyclic subgroupoid of $\boldsymbol{E}$. If $f$ is not irreducible, i.e. $f=h(g)=h\circ g$, $(h\neq e, g\neq e)$, then (for example) $\langle f\rangle\subset\langle g\rangle\subset E$. This contradicts the supposed of maximality of $\langle f\rangle$.

Conversely, let $f$ be irreducibile and let $\langle f\rangle\subseteq\langle g\rangle$. By Prop.3.3 a), there is a unique $h\in E$, such that $f=h(g)=h\circ g$. The choice of $f$ implies that $h=e$, so $f=g$, i.e. $\langle f\rangle=\langle g\rangle$. Therefore, there is no cyclic subgroupoid $\langle g\rangle$ of $\boldsymbol{E}$, such that $\langle f\rangle\subset\langle g\rangle$, i.e. $\langle f\rangle$ is a proper maximal cyclic subgroupoid of $\boldsymbol{E}$. □

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