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A NOTE ON ORTHOGONALITY OF THE QUASIGROUPS ARISING FROM SOME PERFECT m-CYCLE SYSTEMS

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Abstract

In this paper we show that the quasigroups obtained from $\{k, k+1, k+2\}$-perfect m-cycle system by using the k-construction and the $(k+1)$-construction are mutually orthogonal.

An m-cycle system of order n is a pair (S, C), where C is a collection of edge disjoint m-cycles which partitions the edge set of the complete undirected graph K_n with vertex set S, $|S| = n$. Let (S, C) be an m-cycle system and for each m-cycle $c \in C$ denote by $c(k)$ the distance k graph of c (that is, the graph formed on the same set of vertices as c by joining two vertices if and only if they are connected by a path of length k in c). If the edges belonging to $c(k)$, all $c \in C$, partition the edge set of K_n, the m-cycle system (S, C) is said to be k-perfect. If K is a finite set of positive integers, the m-cycle system is said to be K-perfect provided it is k-perfect for each $k \in K$.

Let (S, C) be an m-cycle system of order n and $1 \leq h < m/2$. The h-construction is a binary operation $\circ$ on S, defined by $a \circ a = a$ for all $a \in S$ and $a \circ b = u$ and $b \circ a = v$ if and only if $(a, b, \ldots, u, \ldots, v, \ldots) \in C$, the distance from b to u is h and the distance from a to v is h. It is easy to see that $(S, \circ)$ is a quasigroup precisely when (S, C) is both h and $h+1$ perfect.

The aim of this paper is to show that the quasigroups obtained in this way are orthogonal, in some cases.

Two finite groupoids $(G, \cdot)$ and $(G, *)$ defined on the same set G is said to be orthogonal if the pair of equations $x \cdot y = a$ and $x * y = b$ (where a and b are any two given elements of G) are satisfied simultaneously by an unique pair of elements x and y from G.

Theorem 1. *Let (S, C) be an m-cycle system which is $\{k, k+1, k+2\}$-perfect for some integer $k \geq 1$. Then the quasigroups arising from (S, C) by using the k-construction and the $(k+1)$-construction are orthogonal.*

Proof. Proof Denote by $(S, \circ)$ and $(S, *)$ the quasigroups obtained by using the k-construction and the $(k+1)$-construction respectively.

Let a and b be arbitrary elements of S. Note that $x \circ y = a$ and $x * y = a$ if and only if $x = y = a$. So, let $a \neq b$ and $c \in C$ be the cycle which contains the edge $\{a, b\}$. Denote c by $(x_1, x_2, \ldots, x_{k+1}, a, b, x_{k+2}, \ldots, x_{m-2})$. Then $x_1 \circ x_2 = a$ and $x_1 * x_2 = b$. The assumption of the existence of elements $x_1', x_2' \in S$, such that $(x_1, x_2) \neq (x_1', x_2')$ and $x_1' \circ x_2' = a$, $x_1' * x_2' = b$, implies that there is a cycle $c' \neq c$ which can be denoted by $(x_1', x_2', \ldots, x_{k+1}', a, b, x_{k+2}', \ldots, x_{m-2}')$, contradicting the uniqueness of the cycle containing the edge $\{a, b\}$.

Hence, $(\forall a, b \in S)(\exists!(x, y) \in S \times S)\ x \circ y = a,\ x * y = b$, i.e. $(S, \circ)$ and $(S, *)$ are orthogonal quasigroups. □

It is clear that every m-cycle system (S, C) is 1-perfect, since $c(1) = c$ for all $c \in C$. It is also known that $\{2, 3\}$-perfect m-cycle systems can be equationally defined for $m = 5, 7, 8, 9$ and 11 only ([2],[3],[4],[5]). Therefore, every finite quasigroup belonging to the corresponding varieties has an orthogonal mate.

Note that the condition of being a quasigroup, required for the groupoids arising from m-cycle systems, plays no role in the proof of the theorem. So, we can generalize the statement in the following way.

Theorem 2. *Let (S, C) be an m-cycle system. Then the groupoids arising from (S, C) by using the k-construction and the $(k+1)$-construction are orthogonal.* □

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ОРТОГОНАЛНОСТ НА КВАЗИГРУПИТЕ КОИ ПРОИЗЛЕГУВААТ ОД ОДРЕДЕНИ ПЕРФЕКТНИ m-ЦИКЛИЧНИ СИСТЕМИ

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Р е з и м е

Во овој труд е покажано дека квазигрупите добиени од $\{k, k+1, k+2\}$-перфектни m-циклични системи со користење на k-конструкцијата и $(k+1)$-конструкцијата се заемно ортогонални.

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