ON THE FUNCTIONAL EQUATION $A(x, B(x, y)) = y$ IN THE VARIETY OF GROUPOIDS

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Dedicated to Professor Slaviša B. Prešić in occasion of his 65^{th} anniversary

ABSTRACT. We focus on finding general solutions of the functional equation $A(x, B(x, y)) = y$ in the class of groupoids where A, B are unknown groupoid operations over the same set. We also consider functional equations symmetric to the mentioned one, as well as systems of such functional equations.

1. Preliminaries

In what follows we present description of the general solutions of the functional equation $A(x, B(x, y)) = y$, as well as symmetric ones: $A(x, B(y, x)) = y$, $A(B(y, x), x) = y$, $A(B(x, y), x) = y$, where A and B are unknown groupoid operations over a same set. We characterize the solutions in the class of finite groupoids and in the class of all groupoids. The main motivation to consider such functional equations arose from the groupoid identity $A(x, A(x, y)) = y$ [5]. Clearly, if this identity holds in a groupoid, then the pair (A, A) is a solution to our equation, where A denotes the operation of the groupoid. Further back, consideration of such an identity was motivated by the fact that groupoids that satisfy it have orthogonal complements which are right zero (or left unit) groupoids. So, we thought that finding the general solutions of these functional equations might be of interest. In the sequel we frequently use the following notions. A right zero groupoid is a groupoid satisfying the law $xy = y$. Every right zero groupoid is a semigroup where each element is a right zero and a left unit. A groupoid $(G; A)$ is said to be left (right) cancellative groupoid if

$$A(x, y) = A(x, z) \implies y = z \qquad (A(y, x) = A(z, x) \implies y = z)$$

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for all $x, y, z \in G$. A groupoid that is both left and right cancellative will be called cancellative. $(G; A)$ is called left (right) solvable groupoid if the equation $A(x, a) = b$ $(A(a, y) = b)$ has a solution x (y), for every $a, b \in G$. If $(G; A)$ is both left and right solvable then it is solvable. $(G; A)$ is said to be left (right) quasigroup if it is left (right) solvable and right (left) cancellative. The solution x (y) of the equation $A(x, a) = b$ $(A(a, y) = b)$ in a left (right) quasigroup is unique, and vice versa. $(G; A)$ is a quasigroup if it is left and right quasigroup. Sometimes we shall call the operation A (left, right) cancellative, solvable, quasigroup if the groupoid $(G; A)$ has the mentioned property. The following simple facts hold.

PROPOSITION 1.

(i) *Every finite left (right) cancellative groupoid is a right (left) quasigroup.*
(ii) *Every finite left (right) solvable groupoid is left (right) quasigroup.*

If $(G; A)$ is a quasigroup, then the quasigroup operations A^{-1}, ${}^{-1}A$, A^*, defined on G by $A^{-1}(x, z) = y$, ${}^{-1}A(z, y) = x$, $A^*(y, x) = z$ if $A(x, y) = z$, are called parastrophes (or conjugates) of A. If $(G; A)$ is left (right) quasigroup, then $(G; A^*)$ is right (left) quasigroup, the left parastrophe ${}^{-1}A$ (the right parastrophe A^{-1}) is defined and $(G; {}^{-1}A)$ $((G; A^{-1}))$ is left (right) quasigroup as well.

2. Functional equation $A(x, B(x, y)) = y$ in the variety of groupoids

We focus our attention on the functional equation $A(x, B(x, y)) = y$ where A and B are unknown groupoid operations on a same set.

PROPOSITION 2. *If $A(x, B(x, y)) = y$ is satisfied by some groupoid operations A and B, then:*

(i) *B is left cancellative.*
(ii) $B(x, y) = z \implies A(x, z) = y$.
(iii) *A is right solvable.*
(iv) $A(x, z) = y \wedge B(x, y) = t \implies A(x, z) = A(x, t) = y$.
(v) *If B is right solvable, then A is right quasigroup.*
(vi) *If A is left cancellative, then B is right quasigroup.*
(vii) *A is right quasigroup if and only if B is right quasigroup, and when A and B are right quasigroups then $B(x, y) = z \iff A(x, z) = y$ i.e. $A = B^{-1}$ $(B = A^{-1})$.*

PROOF. (i), (ii) and (iii) are obvious.

(iv) When $A(x, z) = y$ and $B(x, y) = t$ we have $A(x, t) = A(x, B(x, y)) = y = A(x, z)$.
(v) Let $A(x, z) = A(x, t)$. There exist y_1, y_2 such that $B(x, y_1) = z$, $B(x, y_2) = t$ and we have $y_1 = A(x, B(x, y_1)) = A(x, z) = A(x, t) = A(x, B(x, y_2)) = y_2$, hence $z = t$. By (iii), A is right quasigroup.
(vi) Let A be left cancellative. By (i), it is enough to prove that B is right solvable. Let x, z be given and $A(x, z) = y$, $B(x, y) = t$. Then by (iv) we have $A(x, z) = A(x, t) = y$ and hence $z = t$. So, $B(x, y) = z$.

(vii) The first part is a consequence of (*i*), (*iii*), (*v*) and (*vi*). Let A and B be right quasigroups. $B(x,y) = z \implies A(x,z) = y$ holds by (ii). Let $A(x,z) = y$ and $B(x,y) = t$. Then $A(x,t) = y = A(x,z)$ by (iv), hence $z = t$.

□

THEOREM 1. *The solution of the equation $A(x, B(x,y)) = y$ over the class of all finite groupoids consists of arbitrary mutually right-inverse right quasigroups i.e., right quasigroups satisfying $A = B^{-1}$.*

PROOF. If A and B form a solution of the equation and are over a finite set then, by Proposition 2(i), B is left cancellative, and the statement follows by Proposition 1(i) and Proposition 2(vii). On the other hand, if A and B are mutually right-inverse right quasigroups and if $B(x,y) = z$, then $A(x,z) = y$ i.e. A and B form a solution of the equation. □

EXAMPLE 2.1. The following operations A and B defined on the set $\mathbb{N}$ of positive integers by:

$$A(x,y) = div(x,y) = \left[\frac{y}{x}\right], \ B(x,y) = xy$$

are solutions of the functional equation. Namely, $A(x, B(x,y)) = div(x, xy) = y$. Note that A is neither left nor right cancellative and B is neither left nor right solvable.

THEOREM 2. *The solution of the functional equation*

$$A(x, B(x,y)) = y$$

over the class of all groupoid operations consists of any left cancellative operation B and a corresponding right solvable operation A that satisfies the condition $B(x,y) = z \implies A(x,z) = y$.

PROOF. If A and B form a solution then it satisfies the conditions by Propositions 2 (i), (ii), (iii). The other direction of the statement follows directly from the assumptions. □

In the special case of the equation $A(x, A(x,y)) = y$ we get that the solution is a right quasigroup that is self-right-inverse. Out of symmetry similar results hold for the functional equations

$$A(x, B(y,x)) = y, \quad A(B(y,x),x) = y, \quad A(B(x,y),x) = y.$$

So, we have the following properties.

THEOREM 3. *The solution of the functional equation $A(x, B(y,x)) = y$ over the class of all groupoids consists of a right cancellative operation B and a right solvable operation A satisfying the condition $B(y,x) = z \implies A(x,z) = y$. In the finite case, the solution consists of a right quasigroup A and a left quasigroup B satisfying $A = ({}^{-1}B)^*$.*

THEOREM 4. *The solution of the functional equation $A(B(y,x),x) = y$ over the class of all groupoids consists of a right cancellative operation B and a left solvable operation A satisfying the condition $B(y,x) = z \implies A(z,x) = y$. In the finite case, the solution consists of arbitrary finite mutually left-inverse left quasigroups i.e., left quasigroups satisfying $A = {}^{-1}B$.*

THEOREM 5. *The solution of the functional equation $A(B(x,y),x) = y$ over the class of all groupoids consists of a left cancellative operation B and a left solvable operation A satisfying the condition $B(x,y) = z \implies A(z,x) = y$. In the finite case, the solution consists of a right quasigroup B and a left quasigroup A satisfying $A = (B^{-1})^*$.*

3. Systems of equations

Here we note some consequences of the results in previous section, considering systems of functional equations consisting of pairs of equations of the mentioned types. By Theorem 2 we get the following theorem.

THEOREM 6. *The solution of the system of functional equations*

$$A(x, B(x,y)) = y, \quad B(x, A(x,y)) = y$$

over the class of all groupoids consists of right quasigroups A and B satisfying the condition $A = B^{-1}$.

THEOREM 7. *The solution of the system of functional equations*

$$A(x, B(y,x)) = y, \quad B(x, A(y,x)) = y$$

over the class of all groupoids consists of quasigroups A and B such that $B = A = {}^{-1}(A^{-1}) = ({}^{-1}A)^{-1}$, $A^* = A^{-1} = {}^{-1}A$.

PROOF. If A and B are solutions then, by *Theorem 3*, we have that they are both right solvable and right cancellative. Then also $B(y,x) = z \implies A(x,z) = y \implies B(z,y) = x \implies A(y,x) = z \implies B(x,z) = y$. Hence, $A = B$ and $A(x,y_1) = A(x,y_2) = z \implies y_1 = A(x,z) = y_2$, i.e. A is right quasigroup. Moreover, $x = A(a,b)$ is the solution of the equation $A(x,a) = b$, therefore A is left quasigroup, i.e. A is quasigroup. By $A(z,y) = x \iff A(y,x) = z$ we have $A^{-1} = {}^{-1}A$. Hence, A and B are quasigroups such that $B = A$, $A^{-1} = {}^{-1}A$, and we only have to note that $A = (A^{-1})^{-1} = {}^{-1}({}^{-1}A)$, $A^* = {}^{-1}(({}^{-1}A)^{-1})$. □

The next two theorems are obtained in the same manner as above.

THEOREM 8. *The solution of the system of functional equations*

$$A(B(y,x),x) = y, \quad B(A(y,x),x) = y$$

over the class of all groupoids consists of left quasigroups A and B satisfying $A = {}^{-1}B$.

THEOREM 9. *The solution of the system of functional equations*

$$A(B(x,y),x) = y, \quad B(A(x,y),x) = y$$

over the class of all groupoids consists of quasigroups A and B satisfying the conditions $B = A = {}^{-1}(A^{-1}) = ({}^{-1}A)^{-1}$, $A^* = A^{-1} = {}^{-1}A$.

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