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(4,2)-HOMOLOGY GROUPS

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A b s t r a c t: In this paper we generalize the well-known notion of chain complexes of abelian groups to the notions of weak and strong (4,2)-chain complexes of commutative (4,2)-groups. We also define several homology groups for these (4,2)-chain complexes.

Key words: homology group, (4,2)-chain complexes, commutative (4,2)-groups

§.1. PRELIMINARIES

The (n,m)-groups were introduced in [1]. Here we will define them and focus our attention only on commutative (4,2)-groups. Some of results about (n,m)-groups can be found for example in: [1], [2], [3], [4], [6], [7], [8], [9], [10], [11], [12].

Let G be a nonempty set and let n,m be positive integers, such that $n - m = k > 0$.

A map $[\ ] : G^n \to G^m$ is said to be an *(n,m)-operation* on G, and the pair $(G,[\ ])$ is said to be an *(n,m)-groupoid*. An (n,m)-groupoid $(G,[\ ])$ is said to be an *(n,m)-semigroup*, if for each $\mathbf{x},\mathbf{u} \in G^n$, $\mathbf{y} \in G^k$, $\mathbf{v} \in G^r$, $\mathbf{w} \in G^s$, where $r > 0$, $s \geq 0$, $r + s = k$ and $\mathbf{xy} = \mathbf{vuw}$ in G^{n+k}, the following *associativity* condition $[[\mathbf{x}]\mathbf{y}] = [\mathbf{v}[\mathbf{u}]\mathbf{w}]$ is satisfied. An (n,m)-semigroup is said to be an *(n,m)-group* if for each $\mathbf{a} \in G^k$, $\mathbf{b} \in G^m$, there exist $\mathbf{x},\mathbf{y} \in G^m$, satisfying $[\mathbf{ax}] = \mathbf{b} = [\mathbf{ya}]$.

It follows directly from the definition that the notion of a (2,1)-group is the usual notion of a group.

Next, let $n = 4$, and $m = 2 = k$. For a better clarification we will restate the definition of (4,2)-groups. A (4,2)-group is a pair $(G,[\ ])$, where $[\ ] : G^4 \to G^2$ is a map satisfying the following two coditions:

(i) $[[xyzt]uv] = [x[yztu]v] = [xy[ztuv]]$ for each $x,y,z,t,u,v \in G$ (associativity condition); and

(ii) for each $\mathbf{a}, \mathbf{b} \in G^2$, there exist $\mathbf{x}, \mathbf{y} \in G^2$ satisfying $[\mathbf{ax}] = \mathbf{b} = [\mathbf{ya}]$.

Proposition 1.1. ([2]) A (4,2)-semigroup $(G,[\ ])$ is a (4,2)-group if and only if $(G^2, \circ)$ is a group, where "$\circ$" is the binary operation on G^2 defined by $\mathbf{x} \circ \mathbf{y} = [\mathbf{xy}]$. Moreover:

a) the neutral element $\mathbf{e}$ in the group $(G^2, \circ)$ is of the form (e,e), for an $e \in G$, and

b) $[xeey] = (x,y)$ for each $x,y \in G$. □

We say that the element $e \in G$ is the *neutral element* in the (4,2)-group $(G,[\ ])$.

For a given group $(G,*)$, the pair $(G,[\ ])$, where $[\ ] : G^4 \to G^2$ is defined by:

$$[xyzt] = (x*z,\ y*t)$$

is a (4,2)-group, called *trivial* (4,2)-*group*. A trivial (4,2)-group is the direct product of a group with itself.

Let $(G,[\ ])$ and $(K,[\ ])$ be two given (4,2)-groups. A map $f : G \to K$ is called a (4,2)-*homomorphism* if $f^2([xyzt]) = [f(x)\,f(y)\,f(z)\,f(t)]$ where $f^2 : G^2 \to K^2$ is the product map of f, i.e. $f^2(u,v) = (f(u), f(v))$. It is obbious that f is a (4,2)-homomorphism if and only if $f^2 : (G^2,\circ) \to (K^2,\circ)$ is a group homomorphism.

The (4,2)-groups as objects and the (4,2)-homomorpisms as morfisms form a category, denoted by (4,2)-*Gp*.

Let $f : (G,[\ ]) \to (K,[\ ])$ be a (4,2)-homomorphism, e the neutral element in $(G,[\ ])$, k the neutral element in $(K,[\ ])$ and $H = ker(f) = \{x \mid x \in G, f(x) = k\}$. Let us examine same properties of H. First of all H^2 is a normal subgroup of $(G^2,\circ)$. Moreover, H satisfies the following conditions:

$$[x_1x_2\,H^2] = [x_1H^2\,x_2] = [H^2\,x_1x_2]; \tag{1.1}$$

and

$$[x_1x_2\,H^2] = [y_1y_2\,H^2] \Leftrightarrow [x_ix_iH^2] = [y_iy_i\,H^2] \quad \text{for } i = 1,2. \tag{1.2}$$

Above, $[xyH^2]$ denotes the set $\{[xyuv] \mid u,v \in H\}$, $[xH^2y]$ denotes the set $\{[xuvy] \mid u,v \in H\}$, and $[H^2xy]$ denotes the set $\{[uvxy] \mid u,v \in H\}$.

We say that a subset H of a given (4,2)-group $(G,[\ ])$ is a (4,2)-*subgroup* if H^2 is a subgroup of $(G^2,\circ)$. A (4,2)-subgroup H of $(G,[\ ])$ is called a *normal* (4,2)-*subgroup* if it satisfies the condition (1.2) and H^2 is a normal subgroup of $(G^2,\circ)$.

Hence, *ker(f)* is a normal (4,2)-subgroup of a given (4,2)-group $(G,[\ ])$ for any (4,2)-homomorphism f from $(G,[\ ])$ to some (4,2)-group $(K,[\ ])$.

Let $(H,[\ ])$ be a normal (4,2)-subgroup of $(G,[\ ])$. We define a relation "~" on G by: $x \sim y \Leftrightarrow [xxH^2] = [yyH^2]$. It is easy to check that this relation is an equivalence relation on G. We denote the factor set $G/\sim$ by G/H and its elements $x^{\sim}$ by xH. Next we define the (4,2)-map [ ] on G/H by:

$$[(xH)(yH)(zH)(tH)] = ((uH),(vH)) \Leftrightarrow [xyzt] = (u,v).$$

It is shown in [2] that $(G/H,[\ ])$ is a (4,2)-group (called the *factor group* of G by H, and that the natural map $\pi : G \rightarrow G/H$ defined by $\pi(x) = xH$ is a (4,2)-homomorphism such that $ker(\pi) = H$.

We say that a (4,2)-group is *commutative* if the associated group $(G^2,\circ)$ is commutative ([10]).

Proposition 1.2 ([10]) A (4,2)-semigroup $(G,[\ ])$ is a commutative (4,2)-group if and only if there exist $e \in G$ and a map $g : G \rightarrow G$ such that:

(a) $[eexy] = (x,y)$ for each $x,y \in G$;

(b) $[xxg(x)g(x)] = (e,e)$ for each $x \in G$; and

(c) $[xyzt] = [xtzy]$ for each $x,y,z,t \in G$

In a commutative (4,2)-group $(G,[\ ])$

(1) the neutral element will be called *zero* and denoted by 0,

(2) for each $x \in G$ the element $g(x)$ will be denoted by $-x$. □

Remark: A (4,2)-subgroup of a commutative (4,2)-group does not have to be a normal (4,2)-subgroup, which is the case for the abelian groups, but a factor group of a commutative (4,2)-group by a normal (4,2)-subgroup is a commutative (4,2)-group.

Proposition 1.3. ([10]) Let $(G,[\ ])$ be a commutative (4,2)-group. Then, for each $x,y,z,t \in G$:

(a) $[00xy] = [x00y] = [xy00] = (x,y)$;

(b) $[xyzt] = [xtzy] = [zyxt] = [ztxy]$;

(c) $[xyzt] = (u,v) \Leftrightarrow [yxtz] = (v,u)$;

(d) $[0xy0] = (y,x)$;

(e) $[xxyy] = (u,v) \Rightarrow u = v$;

(f) $[xx(-x)(-x)] = [x(-x)x(-x)] = (0,0)$;

(g) $[xx(-x)(-x)] = (0,0)$;

(h) $D = \{(x,x) \mid x \in G\} \subseteq G^2$ is a subgroup of the group $(G^2, \circ)$;

(i) $[x(-x)y(-y)] = (u,v) \Rightarrow v = -u$; and

(j) $K = \{(x,(-x)) \mid x \in G\} \subseteq G^2$ is a subgroup of the group $(G^2, \circ)$. □

The commutative (4,2)-groups together with the (4,2)-homomorphism, form a category denoted by (4,2)-*Ab*. It is obvious that (4,2)-*Ab* is a subcategory of (4,2)-*Gp*.

Now we will define three functors denoted by Φ_2, Φ_+ and Φ_* from the category (4,2)-*Ab* to the category *Ab* of abelian (commutative) groups. Let $\underline{G} = (G,[\ ])$ be a given commutative (4,2)-group.

Let $\Phi_2(\underline{G})$ be the group $(G^2, \circ)$, defined above.

Let $\Phi_+(\underline{G}) = (G,+)$ where $x + y = a \Leftrightarrow [xxyy] = (a,a)$. Proposition 1.3 implies that the operation $+\colon G^2 \to G$ is well defined that $(G, +)$ is a commutative group whose zero is 0, and that $x + (-x) = 0$ for each $x \in G$:

Let $\Phi_*(\underline{G}) = (G,*)$, where $x*y = a \Leftrightarrow [x(-x)y(-y)] = (a,(-a))$. Proposition 1.3 implies that the operation $*: G^2 \to G$ is well defined, and that $(G,*)$ is a commutative group whose zero is 0, and that $x*(-x) = 0$ for each $x \in G$.

A (4,2)-homomorphism is mapped to itself by Φ_2, Φ_+ and Φ_*.

It is easy to check that Φ_2, Φ_+ and Φ_* are covariant functors from (4,2)-*Ab* to *Ab*.

Example 1.3.4 Let $(\mathbb{Z}_6,+)$ be the cyclic group of order 6, i.e. the aditive group of integers modulo 6. For the elements of $\mathbb{Z}_6$ we define: $\overline{0} = 0 = \overline{3}$; $\overline{1} = 1 = \overline{4}$; $\overline{2} = 2 = \overline{5}$.

Let $[\ ] : (\mathbb{Z}_6)^4 \to (\mathbb{Z}_6)^2$ be defined by:

$$[x\,y\,z\,t] = (x+z-\overline{y}-\overline{t}+\overline{y+t}\ ,\ y+t-\overline{x}-\overline{z}+\overline{x+z})$$

Then ($\mathbb{Z}_6$,[ ]) is a commutative (4,2)-group, which is not a trivial (4,2)-group. For example [1 0 5 0] = (1+5, $-\overline{1}-\overline{5}+\overline{1+5}$) = (0,–1–2) = (0, 5+4) = (0, 3) ≠ (0, 0).

§.2. (4,2)-CHAIN COMPLEXES

Next we give a generalization of the well-known notion of chain complexes of abelian groups to chain complexes of commutative (4,2)-groups.

Definition 2.1. A *weak (4,2)-chain complex*, denoted by *wK*, is a sequence

$$0 \xleftarrow{\partial_0} (K_0,[]) \xleftarrow{\partial_1} \dots (K_{n-1},[]) \xleftarrow{\partial_n} (K_n,[]) \xleftarrow{\partial_{n+1}} (K_{n+1},[]) \dots \quad (2.1)$$

where $(K_n,[\,])$ are commutative (4,2)-groups, $n \geq 0$ and

$$\partial_n : (K_n,[\,]) \to (K_{n-1},[\,])$$

are (4,2)-homomorphisms such that $\partial_n \partial_{n+1} = 0$ is the zero homomorphism, i.e., such that for each $x \in K_{n+1}$, $\partial_n \partial_{n+1}(x) = 0$.

If *wK* and *wK'* are two weak (4,2)-chain complexes, we define a *(4,2)-chain map* $f : wK \to wK'$ as a sequence of (4,2)-homomorphisms

$$f_n : (K_n,[\,]) \to (K'_n,[\,]), \qquad n \geq 0,$$

such that $\partial'_n f_n = f_{n-1}\partial_n$, i.e., such that the following diagram is commutative:

$$\begin{array}{ccccccccccc}
0 \xleftarrow{\partial_0} & (K_0,[\,]) & \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} & (K_{n-1},[\,]) & \xleftarrow{\partial_n} & (K_n,[\,]) & \xleftarrow{\partial_{n+1}} \cdots \\
 & \downarrow f_0 & & \downarrow f_{n-1} & & \downarrow f_n & \\
0 \xleftarrow{\partial'_0} & (K'_0,[\,]) & \xleftarrow{\partial'_1} \cdots \xleftarrow{\partial'_{n-1}} & (K'_{n-1},[\,]) & \xleftarrow{\partial'_n} & (K'_n,[\,]) & \xleftarrow{\partial'_{n+1}} \cdots
\end{array}$$

If $f : wK \to wK'$ and $f' : wK' \to wK''$ are two (4,2)-chain maps, their composition $f' \circ f : wK \to wK''$ is defined by:

$$(f' \circ f)_n = f'_n \circ f_n, \qquad n \geq 0.$$

The equalities

$$\partial''_n(f'f)_n = \partial''_n f'_n f_n = f'_{n-1}\partial'_n f_n = f'_{n-1} f_{n-1}\partial_n = (f'f)_{n-1}\partial_n$$

show that the composition of (4,2)-chain maps is a (4,2)-chain map, i.e., show that the following diagram is commutative

$$\begin{array}{ccccccccc}
0\xleftarrow{\partial_0}(K_0,[\,]) & \xleftarrow{\partial_1}\cdots\xleftarrow{\partial_{n-1}} & (K_{n-1},[\,]) & \xleftarrow{\partial_n} & (K_n,[\,]) & \xleftarrow{\partial_{n+1}}\cdots \\
\downarrow f_0 & & & & \downarrow f_n & \downarrow f_{n+1}: \\
0\xleftarrow{\partial'_0}(K'_0,[\,]) & \xleftarrow{\partial'_1}\cdots\xleftarrow{\partial'_{n-1}} & (K'_{n-1},[\,]) & \xleftarrow{\partial'_n} & (K'_n,[\,]) & \xleftarrow{\partial'_{n+1}}\cdots \\
\downarrow f'_0 & & & & \downarrow f'_n & \downarrow f'_{n+1} \\
0\xleftarrow{\partial''_0}(K''_0,[\,]) & \xleftarrow{\partial''_1}\cdots\xleftarrow{\partial''_{n-1}} & (K''_{n-1},[\,]) & \xleftarrow{\partial''_n} & (K''_n,[\,]) & \xleftarrow{\partial''_{n+1}}\cdots
\end{array}$$

Considering weak (4,2)-chain complexes as objects and (4,2)-chain maps as morphisms we obtain the category, which we denote by (4,2)-$w\partial K$, and we call it the *category of weak (4,2)-chain complexes.*

Definition 2.2. A *strong (4,2)-chain* complex, denoted by sK, is a sequence

$$0\xleftarrow{\partial_0}(K_0,[\,])\xleftarrow{\partial_1}\ldots(K_{n-1},[\,])\xleftarrow{\partial_n}(K_n,[\,])\xleftarrow{\partial_{n+1}}(K_{n+1},[\,])\ldots(2.2)$$

where $(K_n,[\,])$ are commutative (4,2)-groups, and $\partial_n:(K_n,[\,])\to(K_{n-1},[\,])$, $n\geq 0$, are (4,2)-homomorphisms such that $\mathrm{Im}\partial_{n+1}$ is a normal (4,2)-subgroup of $ker\partial_n$.

Since $\mathrm{Im}\partial_{n+1}$ is a (4,2)-normal subgroup of $ker\partial_n$, it follows that $\partial_n\partial_{n+1}=0$, $n\geq 0$. Then directly from the definitions it follows that a strong (4,2)-chain complex is also a weak (4,2)-chain complex.

A (4,2)-*chain map* between two strong (4,2)-chain complexes is a (4,2)-chain map considered as weak (4,2)-chain complexes.

Considering strong (4,2)-chain complexes as objects and the (4,2)-chain maps as morphisms we obtain the category, which we denote by (4,2)-$s\partial K$ and call the *category of strong (4,2)-chain complexes.* It follows directly from the definition that (4,2)-$s\partial K$ is a subcategory of (4,2)-$w\partial K$.

Next, we will define three functors (covariant) from the category (4,2)-$w\partial K$ to the usual category ∂K of chain complexes of abelian groups.

The first functor F_2 : (4,2)-$w\partial K \rightarrow \partial K$ is defined as follows. For a weak (4,2)-chain complex wK given by (2.1), $F_2(wK)$ is the chain complex

$$0 \xleftarrow{\partial_0^2} (K_0^2, \circ) \xleftarrow{\partial_1^2} \dots \xleftarrow{\partial_{n-1}^2} (K_{n-1}^2, \circ) \xleftarrow{\partial_n^2} (K_n^2, \circ) \xleftarrow{\partial_{n+1}^2} \dots$$

where $(K_n^2, \circ) = \Phi_2(K_n, [\ ])$ for the functor Φ_2 : (4,2)-$Ab \rightarrow Ab$ defined in §.1, and the homomorphisms $\partial_n^2 : (K_n^2, \circ) \rightarrow (K_{n-1}^2, \circ)$ are defined by $\partial_n^2(x,y) = (\partial_n(x), \partial_n(y))$. Since $(K_n^2, \circ)$ are abelian groups and $\partial_n^2 \circ \partial_{n+1}^2(x,y) = (\partial_n\partial_{n+1}(x), \partial_n\partial_{n+1}(y)) = (0,0)$, it follows that $F_2(wK)$ is a chain complex.

Let f : (4,2)-$wK \rightarrow$ (4,2)-wK' be a (4,2)–chain map. We define $F_2(f)$, denoted also by f^2, to be the sequence of homomorphisms f_n^2 : $F_2(wK) \rightarrow F_2(wK')$ defined by $f_n^2(x,y) = (f_n(x), f_n(y))$. Since

$$\partial_n^{'2} \circ f_n^2(x,y) = (\partial' f_n(x), \partial' f_n(y)) = (f_{n-1}\partial_n(x), f_{n-1}\partial_n(y)) = f_{n-1}^2 \circ \partial_n^2(x,y),$$

it follows that $F_2(f) = f^2$: $F_2(wK) \rightarrow F_2(wK')$ is a chain map.

Let f : $wK \rightarrow wK'$ and f': $wK' \rightarrow wK''$ be two (4,2)-chain maps. Then

$$(F_2(f' \circ f))_n(x,y) = (f' \circ f)_n^2(x,y) = (f_n' \circ f_n(x), f_n' \circ f_n(y))$$

$$= (f_n'(f_n(x), f_n'(f_n(y)) = f_n^{'2}(f_n(x), f_n(y)) = f_n^{'2}(f_n^2(x,y))$$

$$= F_2(f')_n(F_2(f)_n(x,y)) = (F_2(f')_n \circ F_2(f)_n)(x,y)$$

implies that $F_2(f' \circ f) = F_2(f') \circ F_2(f)$.

It is obvious that $F_2(id) = id^2$. Hence F_2 is a covariant functor.

The second functor $F_+ : (4,2) - w\partial K \rightarrow \partial K$ is defined as follows. For a weak (4,2)-chain complex wK, given by (2.1), $F_+(wK)$ is the chain complex

$$0 \xleftarrow{\partial_0^+} (K_0^+, +) \dots \xleftarrow{\partial_{n-1}^+} (K_{n-1}^+, +) \xleftarrow{\partial_n^+} (K_n^+, +) \xleftarrow{\partial_{n+1}^+} \dots$$

where (K_n^+, +) = $\Phi_+(K_n,[\ ])$ for the functor $\Phi_+ : (4,2)\text{-}Ab \to Ab$ defined in §.1 and the homomorphisms $\partial_n^+ : (K_n^+,+) \to (K_{n-1}^+,+)$ are defined as $\Phi_+(\partial_n)$. Since $\Phi_+ : (4,2)\text{-}Ab \to Ab$ is a covariant functor and $\partial_n \partial_{n+1} = 0$, it follows that (K_n^+, +) are commutative groups and that $\partial_n^+ \partial_{n+1}^+ = 0$. Hence, $F_+((4,2)\text{-}wK)$ is a chain complex.

Let $f : wK \to wK'$ be a (4,2)–chain map. We define $F_+(f)$, denoted also by f_n^+, to be the sequence of homomorphisms $f_n^+ : F_2(wK) \to F_2(wK')$ defined by $f_n^+(x) = f_n(x)$. This $F_+(f)$ is a chain map.

Let $f : wK \to wK'$ and $f' : wK' \to wK''$ be two (4,2)-chain maps. Then

$$F_+(f' \circ f)_n(x) = (f' \circ f)_n^+(x) = (f' \circ f)_n(x) = f_n' \circ f_n\ (x) = f_n^{+'} \circ f_n^+(x)$$
$$= f_n^{+'}(f_n^+(x)) = F_+(f_n')(F_+(f_n(x))) = F_+(f')_n \circ F_+(f)_n(x)$$

implies that $F_+(f' \circ f) = F_+(f') \circ F_+(f)$.

It is obvious that $F_+(id) = id^+$. Hence F_+ is a covariant functor.

The third functor $F_* : (4,2) - w\partial K \to \partial K$, is defined as follows. For a weak (4,2)-chain complexes wK, given by (2.1), $F_*(wK)$ is the chain complex

$$0 \xleftarrow{\partial_0^+} (K_0^*,*) \xleftarrow{\partial_0^+} \dots \xleftarrow{\partial_{n-1}^*} (K_{n-1}^*,*) \xleftarrow{\partial_n^+} (K_n^*,*) \xleftarrow{\partial_{n+1}^*} \dots$$

where (K_n^*, $*$) = $\Phi_*[K_n,[\ ])$ for the functor $\Phi_* : (4,2)\text{-}Ab \to Ab$ defined in §.1, and the homomorphisms $\partial_n^* : (K_n,*) \to (K_{n-1},*)$ are defined as $\Phi_*(\partial_n)$. Since $\Phi_* : (4,2)\text{-}Ab \to Ab$ is a covariant functor and $\partial_n \partial_{n+1} = 0$, it follows that (K_n^*, $*$) are commutative groups and $\partial_n^* \partial_{n+1}^* = 0$. Hence, $F_*((4,2)\text{-}wK)$ is a chain complex.

Let $f : wK \to wK'$ be a (4,2)–chain map. We define $F_*(f)$, denoted also by f_n^*, to be the sequence of homomorphisms $f_n^* : F_*(wK) \to F_*(wK')$ defined by $f_n^*(x) = f_n(x)$. This $F_*(f)$ is a chain map.

Let $f : wK \to wK'$ and $f' : wK' \to wK''$ be two (4,2)-chain maps. Then

$$F_*(f^{'} \circ f)_n(x) = (f^{'} \circ f)_n^*(x) = (f^{'} \circ f)_n(x) = f_n^{'} \circ f_n(x) = f_n^{'*} \circ f_n^*(x) =$$
$$f_n^{'*}(f_n^*(x)) = F_*(f_n^{'})(F_*(f_n(x))) = F_*(f^{'})_n \circ F_*(f)_n(x)$$

implies that $F_*(f^{'} \circ f) = F_*(f^{'}) \circ F_*(f)$.

It is obvious that $F_*(id) = id^*$. Hence, F_* is a covariant functor.

§.3. HOMOLOGY GROUPS OF (4,2)-CHAIN COMPLEXES

In this paragraph we define several homology groups for the categories (4,2)-$s\partial K$ and (4,2)-$w\partial K$.

Let sK be a strong (4,2)-chain complex given by (2.2).

For $n \geq 0$, we denote $ker\partial_n$ by (4,2)-$Z_n(sK)$ and $im\partial_{n+1}$ by (4,2)-$B_n(sK)$. Since ((4,2)-$B_n(sK)$, [ ]) is a normal (4,2)-subgroup of ((4,2)-$Z_n(sK)$, [ ]), we have the commutative factor (4,2)-group ((4,2)-$Z_n(sK)$/(4,2)-$B_n(sK)$,[ ]), denoted by (4,2)-$H_n(sK)$ and called the *n-th* (4,2)- *homology group* for the strong (4,2)-chain complex sK.

For a (4,2)-chain map $f: sK \to sK^{'}$, the equation $f_{n-1}\partial_n = \partial_n^{'} f_n$ implies that $f_n(ker\partial_n) \subseteq ker\partial_n^{'}$ and $f_n(im\partial_{n+1}) \subseteq im\partial_{n+1}^{'}$. So, for each $n \geq 0$, the (4,2)-chain map f, induces three (4,2)-homomorphisms:

$$(4,2)-Z_nf : ((4,2)-Z_n(sK),[\,]) \to ((4,2)-Z_n(sK^{'}),[\,])$$

defined by: $(4,2)-Z_nf(x) = f_n(x)$;

$$(4,2)-B_nf : ((4,2)-B_n(sK),[\,]) \to ((4,2)-B_n(sK^{'}),[\,])$$

defined by: $(4,2)-B_nf(x) = f_n(x)$; and

$$(4,2)-H_nf : ((4,2)-H_n(sK),[\,]) \to ((4,2)-H_n(sK^{'}),[\,])$$

defined by $(4,2)-H_nf(x^{\sim}) = (f_n(x))^{\sim}$,

where ~ is the equivalence defined by $z \sim y \Leftrightarrow [xx(im\partial_{n+1})^2] = [yy(im\partial_{n+1})^2]$.

The map (4,2)-H_nf is well defined because

$$x \sim y \Rightarrow [xx(im\partial_{n+1})^2] = [yy(im\partial_{n+1})^2]$$

$$\Rightarrow f_n^2[xx(im\partial_{n+1})^2] = f_n^2[yy(im\partial_{n+1})^2]$$

$$\Rightarrow [f_n(x)f_n(x)f_n^2(im\partial_{n+1})^2] = [f_n(y)f_n(y)f_n^2(im\partial_{n+1})^2]$$

$$\Rightarrow [f_n(x)f_n(x)(m\partial'_{n+1})^2] = [f_n(y)f_n(y)(im\partial'_{n+1})^2]$$

$$\Rightarrow f_n(x) \sim f_n(y).$$

It is easy to check that the above, defined (4,2)-Z_n and (4,2)-B_n, are covariant functors from the category (4,2)-$s\partial K$ to the category (4,2)-Ab.

Next we prove that (4,2)-H_n : (4,2)-$s\partial K \to$ (4,2)-Ab is also a covariant functor.

1) Since $(4,2)-H_n id(x^{\sim}) = (id(x))^{\sim} = x^{\sim}$, it follows that

$$(4,2)\text{-}H_n((4,2)\text{-}id) = id_{(4,2)-H_n(sK)}.$$

2) Let $f : sK \to sK'$ and $f' : sK' \to sK''$ be two (4,2)-chain maps. Then

$$(4,2)\text{-}H_n f' \circ (4,2)\text{-}H_n f(x^{\sim}) = (4,2)\text{-}H_n f'((4,2)\text{-}H_n f(x^{\sim}) = (4,2)\text{-}H_n f'((f_n(x))^{\sim})$$
$$= (f'(f(x)))^{\sim} = (f' \circ f)(x^{\sim}) = (4,2)\text{-}H_n(f' \circ f)(x^{\sim}).$$

implies that $(4,2)-H_n(f') \circ (4,2)-H_n(f) = (4,2)-H_n(f' \circ f)$.

Next we will define several covariant functors from the category (4,2)-$w\partial K$ to the category of abelian groups Ab.

Let $Z_n(K) = ker\partial_n$, $B_n(K) = im\partial_{n+1}$ and $H_n(K) = Z_n(K)/B_{n+1}(K)$ be the usual covariant functors from the category of chain complexes ∂K to the category Ab.

The compositions of these three functors with the functors F_2 : (4,2)-$w\partial K \to \partial K$, F_+:(4,2)-$w\partial K \to \partial K$ and F_* : (4,2)-$w\partial K \to \partial K$ produce nine covariant functors from the category of weak (4,2)-chain complexes (4,2)-$w\partial K$ to the category Ab:

$H_n \circ F_2$: (4,2)-$w\partial K \to Ab$ denoted by $H_{n,2}$

$Z_n \circ F_2$: (4,2)-$w\partial K \to Ab$ denoted by $Z_{n,2}$;

$B_n \circ F_2$: (4,2)-$w\partial K \to Ab$ denoted by $B_{n,2}$;

$H_n \circ F_+$: (4,2)-$w\partial K \to Ab$ denoted by $H_{n,+}$;

$Z_n \circ F_+$: (4,2)-$w\partial K \to Ab$ denoted by $Z_{n,+}$;

$\mathrm{B_n \circ F_+}$: (4,2)-$\mathrm{w\partial K \to Ab}$ denoted by $\mathrm{B_{n,+}}$;

$H_n \circ F_*$: (4,2)-$w\partial K \to Ab$ denoted by $H_{n,*}$;

$Z_n \circ F_*$: (4,2)-$w\partial K \to Ab$ denoted by $Z_{n,*}$; and

$B_n \circ F_*$: (4,2)-$w\partial K \to Ab$ denoted by $B_{n,*}$

Since (4,2)-$s\partial K$ is a subcategory of (4,2)-$w\partial K$, it follows that the above nine functors are also nine functors from the category (4,2)-$s\partial K$ to Ab. On the other hand, we have the nine functors produced by the composition of the three functors (4,2)-Z_n, (4,2)-B_n and (4,2)-H_n with the functors Φ_2, Φ_+ and Φ_*:

$\Phi_2 \circ$ (4,2)-Z_n : (4,2)-$s\partial K \to$ (4,2)-Ab denoted by (4,2)-$Z_{n,2}$;

$\Phi_2 \circ$ (4,2)-B_n : (4,2)-$s\partial K \to$ (4,2)-Ab denoted by (4,2)-$B_{n,2}$;

$\Phi_2 \circ$ (4,2)-H_n : (4,2)-$s\partial K \to$ (4,2)-Ab denoted by (4,2)-$H_{n,2}$;

$\Phi_+ \circ$ (4,2)-Z_n : (4,2)-$s\partial K \to$ (4,2)-Ab denoted by (4,2)-$Z_{n,+}$;

$\Phi_+ \circ$ (4,2)-B_n : (4,2)-$s\partial K \to$ (4,2)-Ab denoted by (4,2)-$B_{n,+}$;

$\Phi_+ \circ$ (4,2)-H_n : (4,2)-$\mathrm{s}\partial K \to$ (4,2)-Ab denoted by (4,2)-$H_{n,+}$;

$\Phi_* \circ$ (4,2)-Z_n : (4,2)-$s\partial K \to$ (4,2)-Ab denoted by (4,2)-$Z_{n,*}$;

$\Phi_* \circ$ (4,2)-B_n : (4,2)-$s\partial K \to$ (4,2)-Ab denoted by (4,2)-$B_{n,*}$; and

$\Phi_* \circ$ (4,2)-H_n : (4,2)-$s\partial K \to$ (4,2)-Ab denoted by (4,2)-$H_{n,*}$.

Theorem 3.2. For each $n \geq 0$:

a) (4,2)-$Z_{n,2} = Z_{n,2}$; (4,2)-$B_{n,2} = B_{n,2}$;

b) (4,2)-$\mathrm{Z_{n,+}} = \mathrm{Z_{n,+}}$; (4,2)-$B_{n,+} = B_{n,+}$;

c) (4,2)-$Z_{n,*} = Z_{n,*}$; (4,2)-$B_{n,*} = B_{n,*}$;

d) (4,2)-$H_{n,+} = H_{n,+}$; (4,2)-$H_{n,*} = H_{n,*}$; and

e) For each strong (4,2) chain complex sK, (4,2)-$H_{n,2}(sK) \cong H_{n,2}(sK)$. i.e. the following diagram:

$$\begin{array}{ccc} (4,2)\text{-}\, s\partial K & \xrightarrow{(4,2)\text{-}\Im_n} & (4,2)\text{-}Ab \\ \Big\downarrow F_\omega & & \Big\downarrow \Phi_\omega \\ \partial K & \xrightarrow{\Im_n} & Ab \end{array}$$

is commutative for $\Im_n = Z_n, B_n, \omega = 2, +, *$ and $\Im_n = H_n, \omega = +, *$, and is commutative up to the isomorphism for $\Im_n = H_n, \omega = 2$.

Proof. Let sK be a strong (4,2)-chain complex.

a) (4,2)-$Z_{n,2}(sK) = ((ker\partial_n)^2, \circ) = (ker(\partial_n)^2, \circ) = Z_{n,2}(sK)$;

(4,2)-$B_{n,2}(sK) = ((im\partial_{n+1})^2, \circ) = (im(\partial_{n+1})^2, \circ) = B_{n,2}(sK)$;

b) (4,2)-$Z_{n,+}(sK) = (ker\partial_n, +) = Z_{n,+}(sK)$;

(4,2)-$B_{n,+}(sK) = (im\partial_{n+1}, +) = B_{n,+}(sK)$;

c) (4,2)-$B_{n,*}(sK) = (ker\partial_n, *) = Z_{n,*}(sK)$;

(4,2)-$B_{n,*}(sK) = (im\partial_{n+1}, *) = B_{n,*}(sK)$;

d) Let $x,y \in ker\partial_n$. Since $im\partial_{n+1}$ is a normal (4,2)-subgroup of $ker\partial_n$, it follows that for the equivalent relation ~, we have:

$$x \sim y \Leftrightarrow [xx(im\partial_{n+1})^2] = [yy(im\partial_{n+1})^2] \Leftrightarrow x + im\partial_{n+1} = y + im\partial_{n+1}$$
$$\Leftrightarrow [(-x)(-x)(im\partial_{n+1})^2] = [(-y)(-y)(im\partial_{n+1})^2]$$
$$\Leftrightarrow [x(-x)(im\partial_{n+1})^2] = [y(-y)(im\partial_{n+1})^2] \Leftrightarrow x * im\partial_{n+1} = y * im\partial_{n+1}.$$

This imples that (4,2)-$H_{n,+}(sK) = H_{n,+}(sK)$ and (4,2)-$H_{n,*}(sK) = H_{n,*}(sK)$.

e) First, we have that: (4,2)-$H_{n,2}(sK) = (\{(x^{\sim}, y^{\sim}) \mid x,y \in ker\partial_n\}, \circ)$, and $H_{n,2}(sK) = ker(\partial_n)^2 / im(\partial_{n+1})^2$.

We define $\varphi: H_{n,2}(sK) \to (4,2)\text{-}H_{n,2}(sK)$ by $\varphi((x,y)im(\partial_{n+1})^2) = (x^{\sim},y^{\sim})$.

1) φ is well defined.

Let $(x,y)im(\partial_{n+1})^2 = (u,v)im(\partial_{n+1})^2$. Then, $[xy(im\partial_{n+1})^2] = [uv(im\partial_{n+1})^2]$ implies that $[xx(im\partial_{n+1})^2] = [uu(im\partial_{n+1})^2]$ and $[yy(im\partial_{n+1})^2] = [vv(im\partial_{n+1})^2]$, i.e. $(x^{\sim},y^{\sim}) = (u^{\sim},v^{\sim})$.

2) The definition of φ implies that it is a homomorphism and a surjection, i.e., it is an epimorphism.

3) φ is a monomorphism.

Let $\varphi((x,y)im(\partial_{n+1})^2) = \varphi((u,v)im(\partial_{n+1})^2)$, i.e., $(x^{\sim},y^{\sim}) = (u^{\sim},v^{\sim})$. Then

$$[xx(im\partial_{n+1})^2] = [uu(im\partial_{n+1})^2] \text{ and } [yy(im\partial_{n+1})^2] = [vv(im\partial_{n+1})^2].$$

Since $im\partial_{n+1}$ is a normal (4,2)-subgroup of $ker\partial_n$, this implies that

$$[xy(im\partial_{n+1})^2] = [uv(im\partial_{n+1})^2], \text{ i.e., that } (x,y)im(\partial_{n+1})^2 = (u,v)im(\partial_{n+1})^2. \ \square$$

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Р е з и м е

(4,2)- ГРУПИ НА ХОМОЛОГИЈА

Во оваа работа е дадено обопштување на добро познатиот поим за верижен комплекс од абелови групи до поимите слаби и јаки (4,2)-верижни комплекси од комутативни (4,2)-групи, а потоа се дефинирани неколку групи на хомологија за такви (4,2)-верижни комплекси.

Клучни зборови: групи на хомологија, (4,2)-верижни комплекси, комутативни (4,2)-групи

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